Toothpick Squares: An Introduction to Formulas

O V E R V I E W

Rows of squares are formed with toothpicks. The relationship between the number of squares in a row and the number of toothpicks needed to form them is investigated, leading to the introduction of algebraic notation and the use of formulas.

Prerequisite Activity

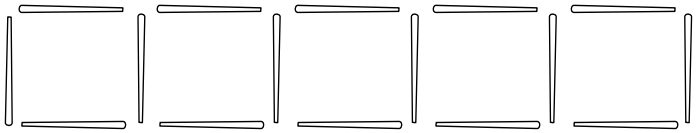
None, although experience in visual patterning (e.g., Unit 1, Activity 2, *Cube Patterns*, or Unit 1, Activity 3, *Pattern Block Trains and Perimeters*) is helpful.

Materials

Flat toothpicks.

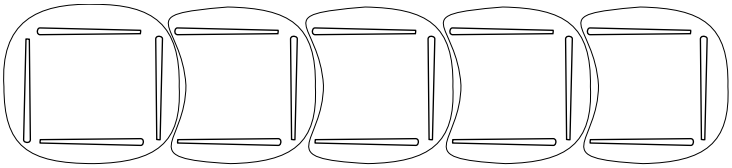
Actions

1. Distribute about 25 toothpicks to each student. Have the students form 5 squares in a row as shown here.



2. Ask the students if they can see ways, in addition to one-by-one counting, to determine the total number of toothpicks in the 5 squares. Discuss different ways of "seeing the total of 16."

(a)



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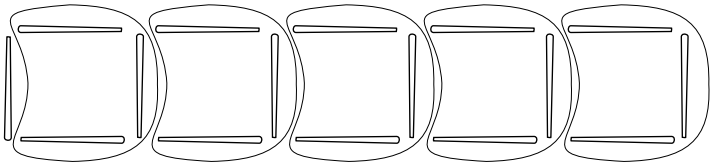

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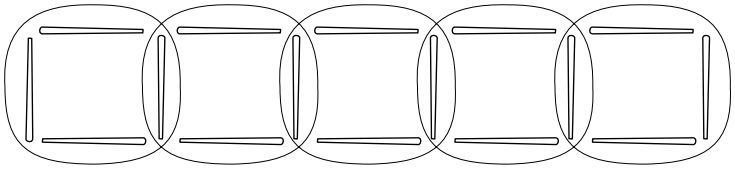

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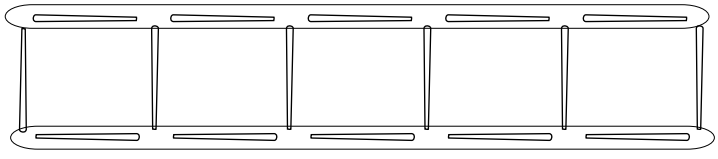

Comments

1. The pattern of squares can be drawn on the chalkboard or formed by placing toothpicks on an overhead projector.

2. Below are some ways of viewing the number of toothpicks. The students may find others. A master for a transparency which can be used to illustrate different methods of counting the toothpicks is attached (Master 1).

(a) One square of 4 toothpicks and 4 groups of 3: $4 + 4(3) = 16$.

(b) One toothpick at the left and 5 groups of 3: $1 + 5(3) = 16$.

(c) Five squares of 4 toothpicks with 4 toothpicks counted twice: $5(4) - 4 = 16$.

(d) Two rows of 5 toothpicks and 6 vertical toothpicks: $2(5) + 6 = 16$.

Actions	Comments
3. Ask the students to imagine extending the row of 5 squares to 12 squares and then predict the total number of toothpicks needed to build the 12 squares. Discuss the methods used to predict the total.	3. Twelve squares require 37 toothpicks. Here are ways of determining this, corresponding to the methods described in Action 2: (a) 4 + 11(3) = 37 (1 square of 4 toothpicks and 11 groups of 3), (b) 1 + 12(3) = 37 (1 toothpick on the left and 12 groups of 3), (c) 12(4) – 11 = 37 (12 squares of 4 toothpicks with 11 toothpicks counted twice), (d) 2(12) + 13 = 37 (2 rows of 12 toothpicks and 13 vertical toothpicks).
4. Have the students determine the number of toothpicks if the row of squares is extended to (a) 20 squares, (b) 43 squares, (c) 100 squares. Discuss.	4. In determining their answers, a student is likely to use one of the methods discussed in Action 3. You can ask them to verify their work by using one of the other methods suggested.
5. Tell the students to suppose you made a row of toothpick squares and to suppose you have told them how many squares are in your row. Working in groups of 3 or 4, have the students devise various ways to determine the number of toothpicks from this information.	5. Having students discuss with one another their ideas for determining the number of toothpicks may help them clarify their thoughts. A student may suggest a method that works for a specific number of squares, say 45. If this happens, you can ask the student how their method would work no matter what the number of squares is.
6. For each method a group has devised, ask them to write verbal directions for using that method. Suggest they begin each set of directions with the phrase, “To determine the number of toothpicks, …”. Encourage the groups to review their written directions for clarity and correctness.	6. You may have to explain to the students that “verbal directions” means directions expressed in words, without using symbols.

Actions

7. Ask for a volunteer to read one set of directions from their group. Record them, as read, on the chalkboard or overhead. Discuss the directions with the students, revising as necessary, until agreement is reached that following them, as written, leads to a correct result. Repeat this action until directions for several different methods are displayed.

8. Have the students suggest symbols to stand for the phrases “the number of toothpicks” and “the number of squares”. Discuss their suggestions.

Comments

7. A master for an overhead transparency that can be used in recording the directions is attached (Master 2).

If a set of directions is suspected to be incorrect, you can suggest to the students that they test the directions for specific instances. For example, if the number of squares is 20, following the directions for finding the number of toothpicks should result in 61 toothpicks, as determined in Action 4.

Possible directions corresponding to the methods described in Action 2 are:

(a) “To determine the number of toothpicks, multiply one less than the number of squares by three and add this amount to four.”

(b) “To determine the number of toothpicks, add one to three times the number of squares.”

(c) “To determine the number of toothpicks, multiply the number of squares by four and then decrease this amount by one less than the number of squares.”

(d) “To determine the number of toothpicks, double the number of squares and then add to this amount one more than the number of squares.”

8. While the choice of symbols is a matter of personal preference, it is helpful to choose symbols which are easily recorded, not readily confused with other symbols in use, and are suggestive of what they represent. For example, “the number of squares” might be represented by *n* (the first letter of the word “number”), or by *S* (the first letter of the word “square”). The latter choice may be preferable since it is not as likely to be taken to mean “the number of toothpicks”.

Actions

9. From the suggestions made in Action 8, select symbols to represent the number of toothpicks and the number of squares. Have the students use these symbols and standard arithmetic symbols to write each set of directions in symbolic form. Point out to the students that a set of directions written in symbolic form is called an *algebraic formula*.

10. For each set of directions displayed in Action 7, ask for volunteers to show their formulas. Discuss.

11. Discuss symbols and their role in writing mathematics.

Comments

9. If the students work in groups, they can assist one another in writing appropriate formulas. If the issue is raised, you may want to suggest the use of "grouping" symbols, such as parentheses, to avoid ambiguities.

10. If the validity of a formula is in question, you can ask the students to test it to evaluate the number of toothpicks given a specified number of squares.

Following are formulas corresponding to the directions listed in Comment 7. In the formulas, T stands for the number of toothpicks and S stands for the number of squares.

(A symbol, such as S or T, that stands for a quantity that can have different values is called a *variable*.)

(a) $T = 4 + 3(S - 1)$,
(b) $T = 3S + 1$,
(c) $T = 4S - (S - 1)$,
(d) $T = 2S + (S + 1)$.

Some students may write "$3S - 1$" for "$3(S - 1)$" in formula (a). If this happens, you can comment on the need to distinguish between "subtracting 1 from 3 times the number of squares" and "subtracting 1 from the number of squares and then multiplying by 3". Parentheses are used to make this distinction.

Other ambiguities may arise. They can be discussed as they occur.

11. One way to begin the discussion is to ask the students what they perceive as advantages or disadvantages in using symbols rather than words.

The use of symbols enables one to write mathematical statements concisely and precisely. However, it can obscure meaning if the reader is unfamiliar with the symbols used or lacks practice in reading symbolic statements.

Actions

12. (Optional.) If the row of squares in Action 1 is extended until 142 toothpicks are used, how many squares will there in the row?

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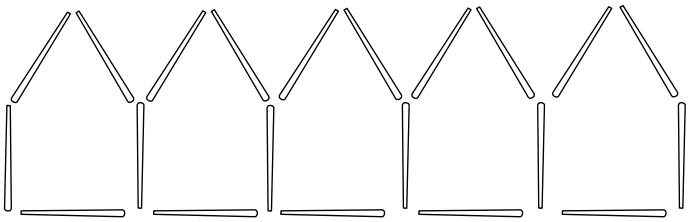
13. (Optional.) Form a row of pentagons with toothpicks as shown. Ask the students to write a formula relating the number of toothpicks used with the number of pentagons in the row.



Comments

12. There are 47 squares.

Some students may arrive at the answer by a "guess-and-check" method. Other students may use their knowledge of how squares are formed: "After 1 toothpick is placed, there are 141 left and it takes 3 more to form each square. So 141 ÷ 3, or 47, squares are formed."

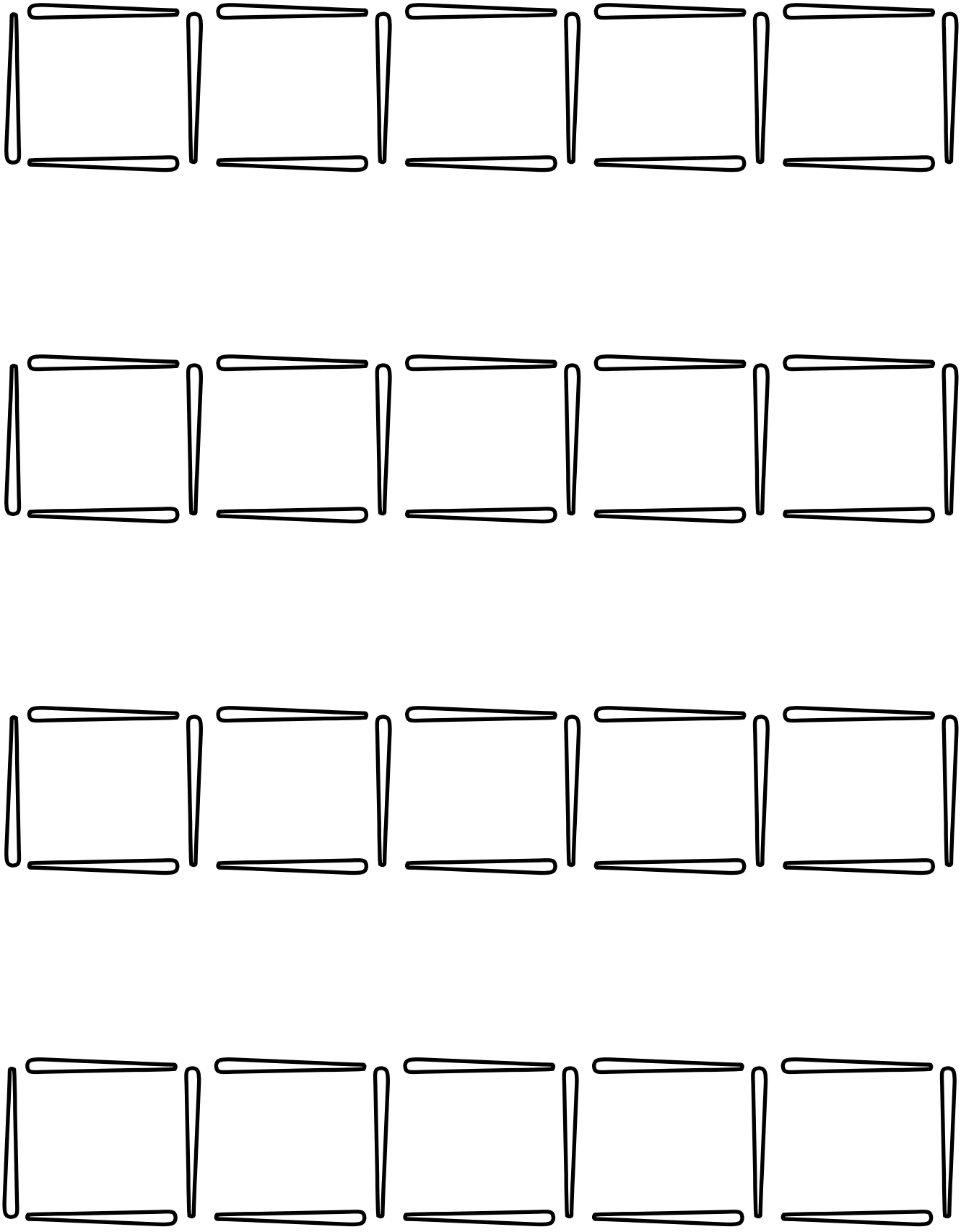
You may wish to point out to the students that an answer may also be arrived at by replacing T by 142 in the formula in Action 10 and determining what S must be to have equality. In arriving at an answer, they have determined the solution of the *equation*: $142 = 3S + 1$.

13. If T is the number of toothpicks used and P is the number of pentagons formed, then

$$T = 1 + 4P.$$

In giving a formula, it is necessary to give the meaning of symbols like T and P that do not have standard meanings.

This formula can be written in other forms. Also, students might choose symbols other than T and P to represent the number of toothpicks and the number of pentagons.



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