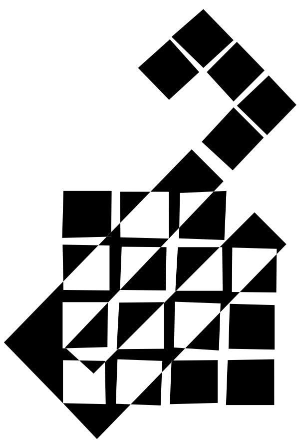
Visual Reasoning

Lesson 5

THE BIG IDEA

Patterns are found throughout mathematics and can be used to help students understand concepts and solve problems. Studying patterns helps students to develop intuitions about algebra, in particular, about the concept of a variable.



CONNECTOR

OVERVIEW

Students explore ways of viewing and recording the number of tile in an arrangement of tile.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per group of students.
- ✔ Tile for use at the overhead.
- ✔ Connector Master A, 1 copy per group and 1 transparency.

FOCUS

OVERVIEW

Students use visual reasoning to predict the size and shape of the 4th, 5th, 10th, and other figures in sequences of arrangements of tile.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Tile for use at the overhead.
- ✔ Focus Master A (optional), 1 transparency or 1 copy per group.

FOLLOW-UP

OVERVIEW

Students make observations, predictions, and generalizations about visual patterns.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 5.1, 1 copy per student.
- ✔ 2-cm grid paper (optional, see *Blackline Masters*), 2 sheets per student.

LESSON IDEAS

JOURNALS
Have students describe in their journals experiences with Problem 6 on the Follow-up.

PORTFOLIOS
This Follow-up could go in students' Growth-folios (see *Starting Points*) as baseline information about their abilities to generalize patterns and explain their thinking. Later, students can compare this to related activities and look for evidence of growth.

TIMING
It isn't necessary to explore every problem in this Lesson or Follow-up. Because visual patterning comes up in several lessons, you may want to move on after 3-4 hours of exploration.

QUOTE
Identifying patterns is a powerful problem-solving strategy. It is also the essence of inductive reasoning. As students explore problem situations appropriate to their grade level, they can often consider or generate a set of specific instances, organize them, and look for a pattern. These, in turn, can lead to conjectures about the problem. Students should be encouraged to validate these conjectures by constructing supporting arguments, which can be at many levels of sophistication.

NCTM Standards

FOLLOW-UP
In general, Follow-ups are intended to help students clarify their understanding and questions. Following are some suggestions for using Follow-ups:
• Assign 1 or 2 Follow-up problems as homework (or let students pick 2) after each day of the lesson.
• Instead of going over all problems in class, have students ask the class for clues about certain problems (without giving away answers). Sometimes give other clues during class, like: "Mary's observation may be helpful on Problem 3." "This activity could give you insights about Problem 5," etc.
• To encourage more detailed responses, have students write responses on other paper.
• Don't collect Follow-ups until the Lesson is completed. On other days have students turn in a daily report (see *Starting Points*) telling what they did and what they have questions about.
• Encourage students to refine and revise their work as they gain insights.
• Have students use a Follow-up Assessment Guide (see *Starting Points*) before turning in a completed Follow-up.

SELECTED ANSWERS

Each solution given is one of several possibilities:

1. a) The first figure has 1 tile; the 2nd has one tile in the center with 1 tile attached to each of its sides; the 3rd has a center tile and lines of 2 tile attached to each of its sides. So the 50th figure has a center tile with lines of 49 tile attached to each of its sides.

   b) 49 + 49 + 49 + 49 + 1

   c) (4 × 50) – 3 (4 "arms" of 50 tile; subtract 3 since the center tile is counted 4 times)

2. Method 1: A row of 36 tile across the bottom, with a column of 2 × 35 tile sitting on top of the far left tile; 36 + (2 × 35) = 36 + 70 = 106 tile.

   Method 2: An "L" formed by 1 corner tile plus 3 groups of 35 (1 extending to the right and 2 extending up); 1 + (3 × 35) = 106 tile.

3. a) The 70th figure has a 70 × 70 square with rows of 72 tile attached to the top and bottom sides of the square; this figure contains 4900 + 144 = 5044 tile.

   b) The 70th figure is a 70 × 72 rectangle with 1 tile added on each corner; 70 × 72 + 4 = 5044 tile.

   c) The 20th figure.

Connector Teacher Activity

OVERVIEW & PURPOSE

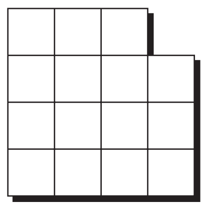
Students explore ways of viewing and recording the number of tile in an arrangement of tile. Discussion centers on recognizing a variety of counting methods and on writing number statements that reflect those methods.

MATERIALS

- ✔ Tile, 30 per group of students.
- ✔ Tile for use at the overhead.
- ✔ Connector Master A, 1 copy per group and 1 transparency.

ACTIONS

1 Place the students in pairs or small groups and distribute tile to each group. Form the following arrangement of tile at the overhead and ask the groups to form it at their tables.

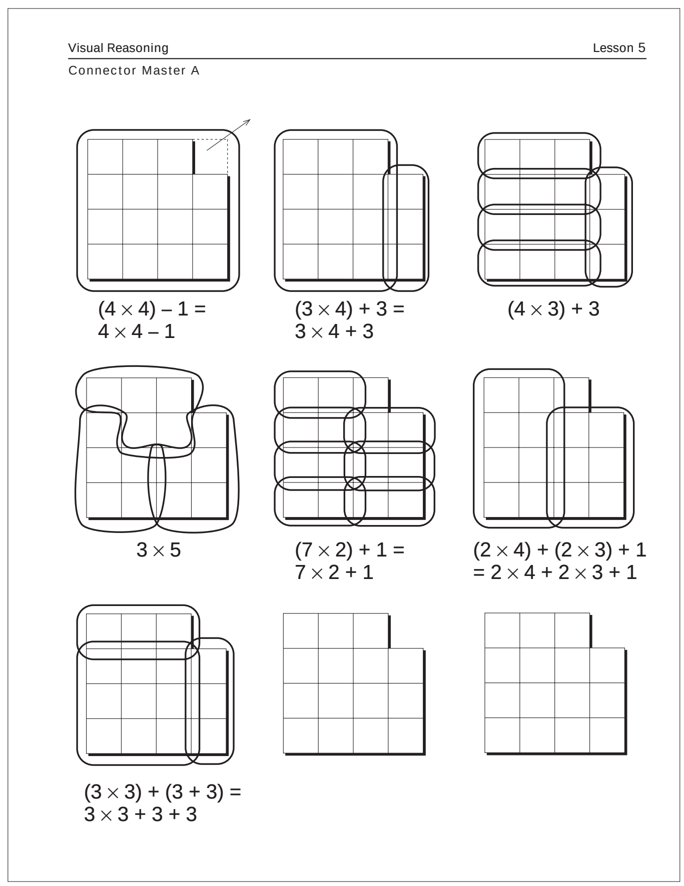


2 Point out that one way to determine the number of tile in the above arrangement is to count them 1 by 1. Ask the groups to find several other ways to "see" the total number of tiles. Give each group a copy of Connector Master A and ask them to record each of their methods and write a number statement to describe each method. Have volunteers show their methods on a transparency of Master A.

COMMENTS

1 To avoid influencing students' thinking by the order in which you place the tile on the overhead, turn the overhead projector off while you form the figure.

2 A few examples of ways students may view the arrangement are shown below on a copy of Master A. Students may emphasize the order of their counting methods by including more parentheses than is necessary (because of order of operations). This is okay.



Connector Teacher Activity (cont.)

ACTIONS

3 Ask the students to imagine expanding this arrangement of tile so that it is 10 tile high and 10 tile across and still has 1 tile missing in the corner. Have them determine the number of tile in this new arrangement. Repeat as appropriate for arrangements 20 tile high and 20 across, 100 by 100, etc.

COMMENTS

3 If students have difficulty with this, you might encourage them to try a variety of the methods which came up during Action 2. For example, using the first method in Comment 2, a similar arrangement with 10 tile high and 10 tile across would have $(10 \times 10) - 1 = 99$ tile.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use visual reasoning to predict the size and shape of the 4th, 5th, 10th, and other figures in sequences of arrangements of tile. They examine these visual patterns from many points of view and have the opportunity to extend their own and their classmates' ways of thinking about the patterns. This is the first of many Visual Mathematics patterning activities that lay important groundwork for the study of algebra.

MATERIALS

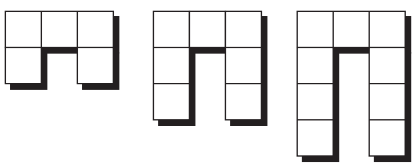
✔ Tile, 30 per student.

✔ Tile for use at the overhead.

✔ Focus Master A (optional), 1 transparency or 1 copy per group.

ACTIONS

1 Arrange the students in groups and give tile to each student. Form the 3 tile figures shown below on the overhead.

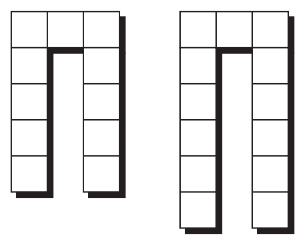


Tell the students the above figures are the first 3 figures in a sequence that is based on a pattern that you have in mind. Ask them to use their tile to form what they think are most likely to be the 4th and 5th figures in your sequence, based on what they observe about the first 3 figures. Have volunteers build their 4th and 5th figures at the overhead and describe how they decided their shape and size.

2 Ask the students to each imagine in their mind's eye what they believe the 10th figure in the sequence would look like, without building the intervening figures. Have volunteers build or sketch their 10th figure at the overhead and describe how they decided its shape and the number of tile it contains.

COMMENTS

1 Encourage students to think and work privately before sharing, thus allowing several ways of thinking about the pattern to emerge. The 4th and 5th figures most commonly formed by students are shown here:

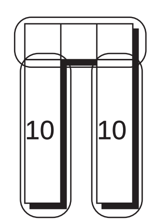


Asking students to guess what pattern you have in mind, allows you to acknowledge all ideas but focus on a specific pattern. You may want to extend an interesting alternate pattern suggested by a student.

2 Some students may find imagining larger figures difficult at first. However, this usually becomes easier after listening to others explain their reasoning. Thus, it is important to elicit a variety of approaches.

Here are some possible descriptions of the 10th figure, based on the pattern suggested in Comment 1:

"The 1st figure has 3 tile on top and 1 tile on each side; the 2nd has 3 on top and 2 on each side; the 3rd has 3 on top and 3 on each side. So the 10th has 3 tile on top and 10 on each side, or 10 + 10 + 3 = 23 tile."



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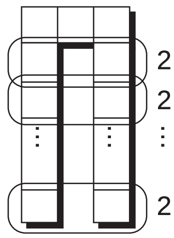
Focus Teacher Activity (cont.)

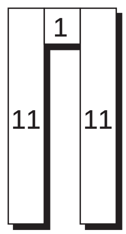
ACTIONS

COMMENTS

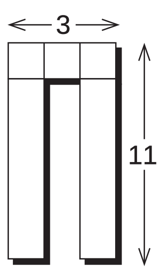
2 (continued.)



"The 1st figure has 5 tile; for each subsequent figure, you add 2 tile (1 on each side); so the 10th figure has 5 plus 9 groups of 2 tile, or $5 + 9 \times 2 = 5 + 18 = 23$ tile."



"The 1st figure has 2 on each side and 1 in the middle of the top row; the 2nd has 3 on each side and 1 in the middle; and the 3rd has 4 on each side and 1 in the middle. So the 10th has 11 on each side and 1 in the middle of the top row, or 11 + 11 + 1 tile."



"The 1st is a 3 by 2 rectangle with 1 tile missing tile; the 2nd is a 3 by 3 rectangle with 2 missing tile; and the 3rd is a 3 by 4 rectangle with 3 missing. So the 10th is a 3 by 11 rectangle with 10 missing tile. Hence, it has $3 \times 11 - 10$ tile."

If at first students have difficulty describing their thinking, you could share how you "see" the 10th figure and how you think about the number of tile it contains. It is important to keep in mind throughout this lesson that this is the first of many lessons involving visual patterning, and therefore, it is not intended that students show "mastery" of the topic. (See *Opening Eyes to Mathematics*, for some introductory patterning activities.)

3 Ask the students to imagine the 20th figure in the sequence, including its shape and the number of tile it contains. Ask for volunteers to describe their mental images of the 20th figure and their methods of determining the number of tile in it.

3 It is interesting to poll students to see how many adopted or adapted methods they heard others share in Action 2 to help them think about the 20th figure. This provides an opportunity to discuss the value of sharing ideas.

Focus Teacher Activity (cont.)

ACTIONS

4 Select one of your student's methods from Action 2 or 3. Ask the class to imagine the 50th (and/or 100th, 75th, etc.) figure using that student's method and to determine the number of tile the figure contains. Discuss. Repeat as appropriate.

5 (Optional) Pose one or more of the following questions for groups to investigate about the preceding sequence. Ask for volunteers to share their group's methods.

a) How many tile are in the figure that has 80 "empty spaces" (i.e., tile missing in the middle row)?

b) Which figure has 69 tile? 50 tile?

c) Two consecutive figures (one is right next to the other) have a total of 156 tile. Which figures are they?

COMMENTS

4 Choose the number of the figure according to the students' comfort level with the preceding actions.

This action engages students in generalizing methods and provides encouragement for trying more than one approach. Often when a student volunteers an especially interesting or useful method, the class attaches the student's name to that method. This can be very gratifying to the student who volunteered the method.

5 You may find it helpful to encourage students to use methods other than trial and error. While trial and error works, it doesn't promote the "algebraic thinking" that generally occurs when students base their thinking on visual relationships.

a) The 80th figure has 80 empty spaces and has $(2 \times 80) + 3 = 163$ tiles.

b) Notice that if the top row of 3 tile are removed from a figure, 2 columns are left and the number of tile in each column is the same as the number of the figure. Hence, the figure that contains 69 tile is the $(69 - 3)/2 = 33$rd figure.

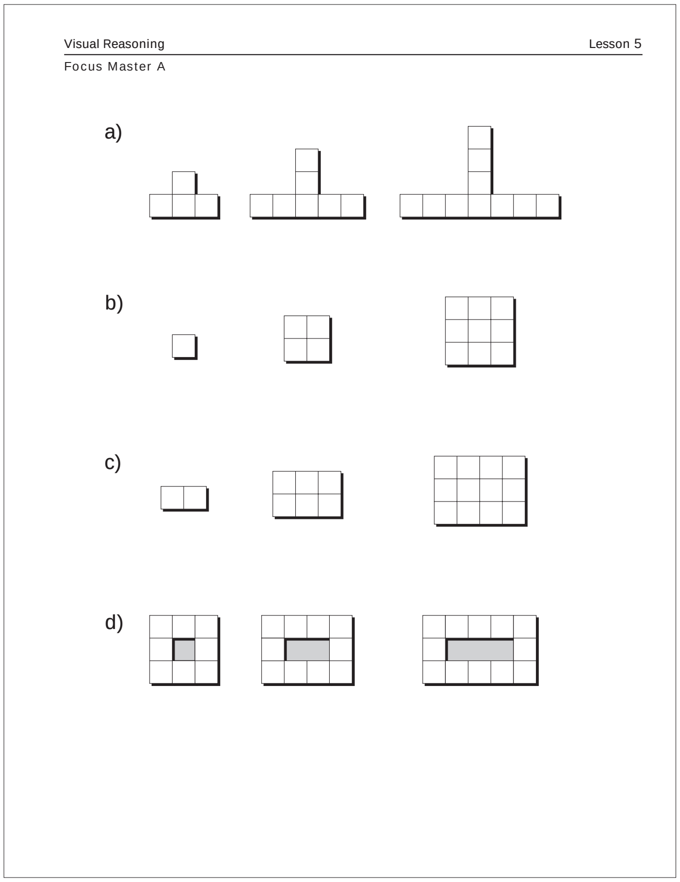
No figure has 50 tile. Notice that removing 3 tile leaves 47 tile, which can't be divided into 2 equal columns.

c) Here is one possible method: If you remove 2 tile from the larger figure, you are left with 2 copies of the smaller figure, each with $154 \div 2 = 77$ tile. Using methods such as those in a) and b) above, the 37th figure has 77 tile. Therefore, the 37th and 38th figures have a total of 156 tile.

Focus Teacher Activity (cont.)

ACTIONS

6 Have students investigate one or more of the tile patterns on Focus Master A. Discuss.



COMMENTS

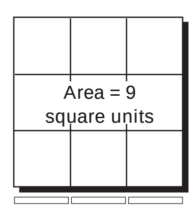
6 As an alternative to building the figures on the overhead, you could use a transparency of Focus Master A or give each group 1 copy.

One way to carry out this Action is to have small groups record on poster paper several different ways of viewing selected figures (the 4th, 10th, 20th, 100th, etc.) in each pattern you assign. Have them also describe ways of determining the total number of tile each figure contains. As you circulate while groups work, lay groundwork for a rich discussion during the Lesson 6 Connector by pointing out evidence you notice that students' inner mathematicians are at work.

Let students know that you are most interested in the quality of the mathematical content of their posters (as compared to their artistic qualities). You could provide a scoring guide, or have students help create one (see *Starting Points*), to use as a basis for evaluating their posters. One way to give feedback to groups is to write two "I appreciate…" and two "I wish…" comments about each poster. Groups can also provide this information to each other.

If appropriate when discussing pattern b), point out that the numbers 1, 4, 9, 16, … which represent the numbers of tile in the figures of this pattern, are historically called *square* numbers. A square number of tile is any whole number of tile that can be arranged to form a perfect square with no overlaps or gaps. (To reinforce this idea you could have students select a handful of tile and, without counting the tile, give a "visual proof" that they do or do not have a square number of tile.)

Note that the length of each side of a square is called the *square root* of its area. For example, if the area of a square is 9 square units, then the length of its side is $\sqrt{9}$ (read "square root of 9") linear units, which also happens to be 3 linear units. Hence, the lengths of the edges of the squares in sequence b) are: $\sqrt{1} = 1$, $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{16} = 4$, $\sqrt{25} = 5$, ….

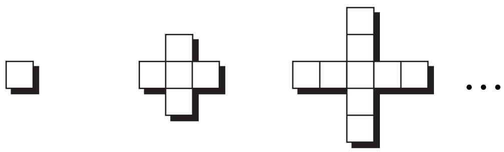


Length = $\sqrt{9}$ = 3
linear units

Follow-up Student Activity 5.1

NAME ______________________________ DATE ______________

1 The first 3 figures in a pattern are shown below. Cut out squares and form what you think is the 4th figure. Sketch your 4th figure below.

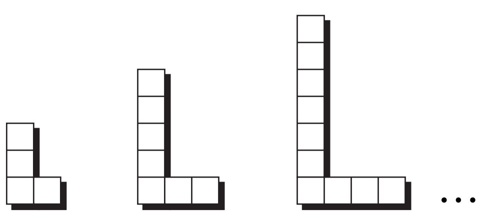


a) Assuming your pattern continues, explain how you think these 3 figures give you clues to what the 50th figure looks like.

b) Tell how (other than building the figure and counting tile) to find the total number of tile in the 50th figure.

c) Describe another method (other than building and counting) of finding the number of tile in the 50th figure of the pattern above.

2 The first 3 figures in another pattern are shown below. Form what you think is the 4th figure. Draw your 4th figure below.



Assuming your pattern continues, explain *two* different methods (other than building the figure and counting) of telling what the 35th figure looks like and how many tile it contains.

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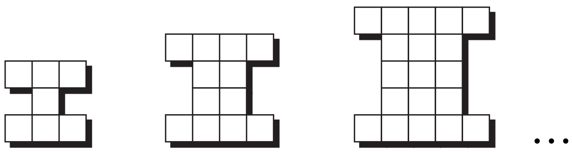
Follow-up Student Activity (cont.)

3 The first 3 figures in another pattern are shown below:



a) Tell how many tile you think are in the 70th figure and explain how you decided this number.

b) Tell another method (other than building and counting) of finding the number of tile in the 70th figure.

c) Suppose a certain figure in the above pattern has exactly 444 square tile in it. Which figure is it? Explain how you decided this.

4 Create the first 4 figures in an interesting pattern of tile figures. Sketch your 4 figures below.

5 Describe your pattern in Problem 4 and tell what the 20th figure in your pattern looks like.

6 Cut out squares from the attached grid paper and build the first 3 figures in the pattern in Problem 1. Get an adult to share with you how they “see” the 20th figure in the pattern. Repeat this process for Problems 2 and 3. If needed, help the adult by sharing ways you “see” each 20th figure. On another sheet, describe what happened.