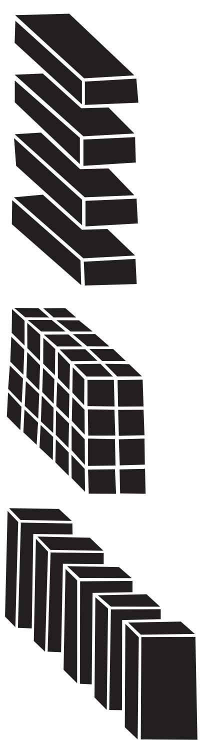
# Variables & Formulas

Lesson 3

## THE BIG IDEA

**When students invent formulas as ways of representing what they do, see, and think related to their actions with physical objects, algebraic notation develops naturally. Variables provide a vehicle for describing students' generalizations about patterns they see. Parentheses are used to convey the order of their thoughts and actions.**



## CONNECTOR

OVERVIEW

Students use cubes to illustrate their views of the meanings of the four basic operations and to explore order of operations and the commutative, associative, and distributive properties.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Cubes, 30 per student.
- ✔ Connector Master A, 1 copy per student and 1 transparency.

## FOCUS

OVERVIEW

Rows of squares are formed with toothpicks. The relationship between the number of squares in a row and the number of toothpicks needed to form the row is investigated, leading to the introduction of algebraic notation and formulas.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Flat toothpicks, 25 per student and 25 for use at the overhead.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 transparency.
- ✔ Focus Master C, 1 transparency per group.
- ✔ Overhead pens, 1 or more per group.
- ✔ Scissors, 1 pair per group.
- ✔ Butcher paper (optional), 1 sheet per group.
- ✔ Glue or tape and a marking pen (optional) for each group.

## FOLLOW-UP

OVERVIEW

Students use variables and formulas to express relationships between numbers of toothpicks and numbers of polygons in toothpick figures.

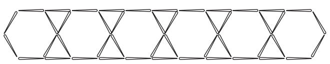
MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 3.1, 1 copy per student.

## LESSON IDEAS

JOURNALS

To capture a "snapshot" of students' early work with writing formulas and using algebraic notation, after completing this lesson you might have students sketch (or tape) several copies of the following toothpick pattern in their journals and then complete the thought starters listed below:



*Here are several different ways of "seeing" and counting (other than one by one) the number of toothpicks in this row of toothpick arrangements...* (students loop diagrams and describe their thinking).

*Assuming any extended row of the above arrangements of toothpicks always begins and ends with a hexagon, here are formulas that generalize the methods I described above...*

See *Starting Points* for ideas about implementing journals.

TIMING

In general, it is important to time lessons so students have opportunities to build both deep and broad understanding. Because the curriculum is structured with the view that learning about an idea is an ongoing process, ideas generally resurface in varied contexts throughout the *Visual Mathematics* courses. Hence, although it is important to allow plenty of exploration time during a lesson, it is equally important to move through a range of lesson topics, allowing students to see connections among ideas. (See *Starting Points* for other lesson timing ideas.)

LOOKING AHEAD

This Connector activity reviews models and ideas that were explored throughout *Course I* and are used in the Focus. However, it isn't expected that students have "mastery" of the ideas before exploring the Focus activity, since there are many opportunities both in the Focus and in later lessons to revisit those ideas. Similarly, the Focus activity is an introduction to writing formulas involving variables, which is also emphasized in Lessons 4, 9, 10, 26, and 27 of this course.

## SELECTED ANSWERS

1. a) One method is to multiply the number of triangles by 3; add 1 for each triangle to account for the vertical toothpicks; multiply the number of spaces between the triangles by 2; and add these amounts to get $(3 \times 5) + 5 + (2 \times 4) = 28$.

   b) Using the method in part a), the number of toothpicks is 898: $3 \times 150 + 150 + 2(149) = 898$.

   c) Here is one possibility, where $N$ is the number of toothpicks and $T$ is the number of triangles: $N = 3T + T + 2(T - 1)$. There are others.

   d) 97 triangles.

2. a) One possibility, where $N$ is the number of toothpicks and $S$ is the number of squares, is $N = 4S + 3(2S) + 1$.

   b) 1381

   c) 146

# Connector Teacher Activity

OVERVIEW & PURPOSE

*Students use cubes to illustrate their views of the meanings of the four basic operations and to explore order of operations and the commutative, associative, and distributive properties.*

MATERIALS

✔ Cubes, 30 per student.

✔ Connector Master A, 1 copy per student and 1 transparency.

## ACTIONS

1 Arrange the students in pairs or small groups and distribute cubes to each student. Write the computation shown in a) below on the overhead and ask the groups to use their cubes to show their views of the *meaning* of these symbols. Discuss the students' models. Then repeat for computations b)-d). Encourage students to, where appropriate, illustrate more than one way of viewing the meaning of each computation.

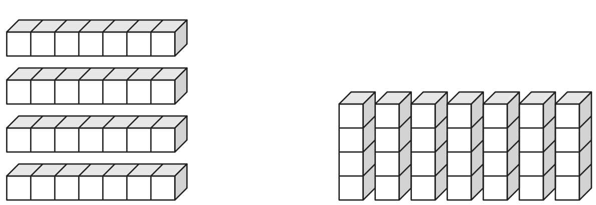
a) 11 + 6

b) 14 – 8

c) $4 \times 7$

d) $17 \div 5$

Repeated Addition Method

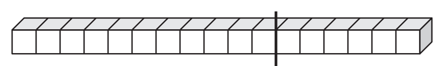


$4 \times 7$ $4 \times 7$

## COMMENTS

1 Models that illustrate the meanings of the operations were first explored in Lesson 2 of *Visual Mathematics, Course I*. Throughout this course, these models, together with students' knowledge of the properties discussed in Action 2, will provide the basis for inventing symbolic statements to represent students' thoughts and actions, and a basis for interpreting the meanings of symbolic statements made by others.

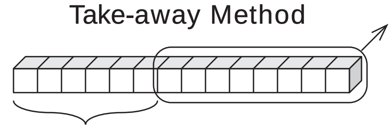
a) Most students will probably refer to addition as the *joining together* of 2 sets of objects and, hence, view 11 + 6 = 17 as the result of combining 11 cubes with 6 cubes.



11 + 6

b) Some students may view 14 – 8 according to the *take-away* method of subtraction and, hence, remove 8 cubes from a collection of 14 cubes. 14 – 8 is the number of cubes remaining. This is illustrated below.

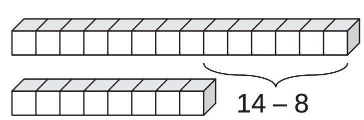
Take-away Method



14 – 8

Or, they may view 14 – 8 according to the *difference* method. Here they may compare a set of 14 cubes with a set of 8 cubes to determine the difference between the numbers of cubes in the 2 sets, as shown below.

Difference Method



14 – 8

c) Some students may view $4 \times 7$ according to the *repeated addition* method and form 4 groups of 7 or 7 groups of 4 as shown at the left.

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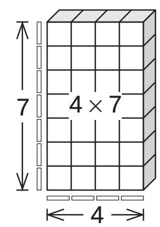
Connector Teacher Activity (cont.)

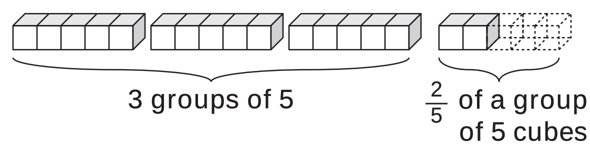
| ACTIONS | COMMENTS |
|---|---|

1 (continued.)
Others may use the *area* method and, hence, represent $4 \times 7$ as a rectanglular array with dimensions 4 by 7 and a total of $4 \times 7 = 28$ total cubes. (Note: in this view, the thickness of the cubes is ignored.)



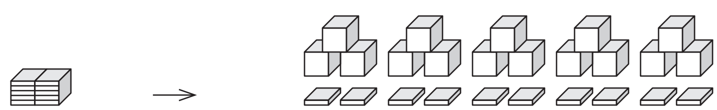

Area Method

d) $17 \div 5$ could be represented according to the *grouping* method of division as "How many groups of 5 cubes are contained in 17 cubes?"




There are $17 \div 5 = 3\frac{2}{5}$ groups of 5 cubes in 17 cubes.

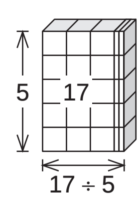
Or, using the *sharing* method $17 \div 5$ could be viewed as, "Arrange 17 cubes in 5 equal groups. How many cubes are in each group?"



Slice the 2 leftover cubes in 5 equal parts.

If 17 cubes are divided into 5 equal groups, there are $17 \div 5 = 3\frac{2}{5}$ cubes in each group.

The *area* method is a third way of interpreting the meaning of division. As shown below, $17 \div 5$ represents the missing dimension of a rectangle whose area is 17 square units, and one dimension is 5 linear units. (Note: in this view, the thickness of the cubes is irrelevant.)



Connector Teacher Activity (cont.)

## ACTIONS

2 Write the two mathematical expressions shown in a) below on the overhead. Ask each pair of students to use cubes to model these expressions and to show how the expressions are the same and how they are different. Ask the students to determine pairs of expressions involving other numbers of cubes and/or other operations that would have similar relationships. Have volunteers illustrate their ideas at the overhead. Then repeat for b) and c).

a) 13 + 9 and 9 + 13

b) (4 + 7) + 5 and 4 + (7 + 5)

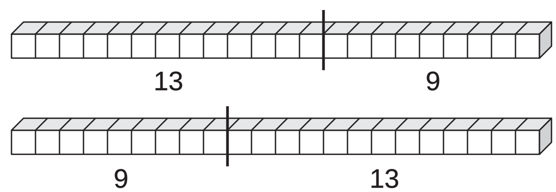
c) 3 × (5 + 2) and (3 × 5) + (3 × 2)

## COMMENTS

2 The intent here is to examine the commutative, associative, and distributive properties, and to establish for which operations these properties do and do not hold. If students don't recall names for the properties, you may wish to bring them up. In Lesson 3 of *Course I*, students were first introduced to these properties and to the use of parentheses to communicate the order of one's mathematical thought processes and actions with materials. Depending on the background and comfort of your students, you may wish to refer to that lesson for additional ideas.

There are many ways to model these expressions. Some examples are given below:

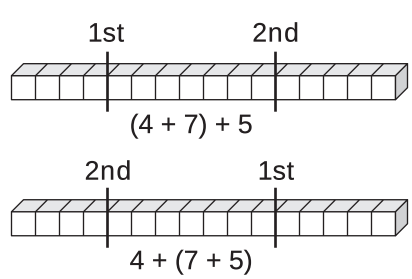
a) The following representations of 9 + 13 and 13 + 9 show that, although the order of the numbers is different, the total number of cubes is the same.




Commutative property: 13 + 9 = 9 + 13

The above model illustrates the *commutative property* for addition, which holds for any pair of numbers. Students can also use cubes to show that multiplication is commutative, but subtraction and division are not.

b) The "1st" and "2nd" in each model below suggest the order in which the sets of cubes were combined, and parentheses are used to record this order symbolically. The fact that 7 can be associated with 4 or 5 without changing the sum is an illustration of the *associative property* for addition.




Associative property for addition:
(4 + 7) + 5 = 4 + (7 + 5)

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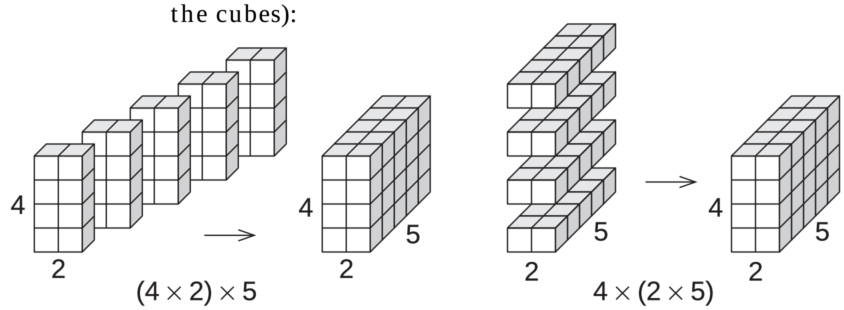
Connector Teacher Activity (cont.)

## ACTIONS

## COMMENTS

2 (continued.)
The following example illustrates why the associative property holds also for multiplication (once again parentheses are used to record the order of the actions with the cubes):



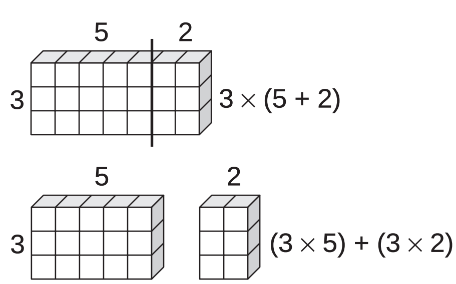

Associative property for multiplication:
$(4 \times 2) \times 5 = 4 \times (2 \times 5)$

Some students may illustrate the associative property for multiplication by simply rotating a rectangular solid, as shown:


The bottom layer of this solid contains $4 \times 2$ cubes. There are 5 identical layers, so the solid contains $(4 \times 2) \times 5 = 40$ cubes.

This rotated view of the same solid shows 4 identical layers of $2 \times 5$ cubes, or $4 \times (2 \times 5) = 40$ cubes.

Attempts to model the associative property for division or subtraction lead to contradictions. Thus, counterexamples can be used to show why division and subtraction are not associative.




Distributive property for multiplication over addition:
$3 \times (5 + 2) = (3 \times 5) + (3 \times 2)$

c) The fact that $3 \times (5 + 2) = (3 \times 5) + (3 \times 2)$ illustrates the *distributive property* for multiplication over addition, as shown at left.

To encourage further investigation and generalizations about the distributive property, you could pose pairs of expressions such as the following and have students determine whether they think multiplication distributes over subtraction, division over addition; division over subtraction, subtraction over addition, etc.

$3 \times (5 - 2)$ and $(3 \times 5) - (3 \times 2)$
$3 \div (5 + 2)$ and $(3 \div 5) + (3 \div 2)$
$3 \div (5 - 2)$ and $(3 \div 5) - (3 \div 2)$
$3 - (5 + 2)$ and $(3 - 5) + (3 - 2)$

Connector Teacher Activity (cont.)

## ACTIONS

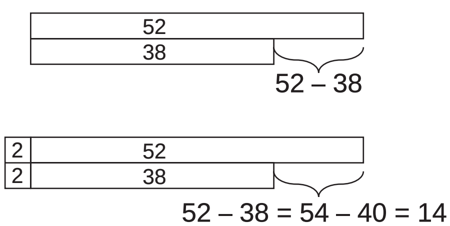
3 Write expression a) below on the overhead and ask each student to mentally compute the answer. Then, before discussing, ask each student to think of another method of mentally computing the correct answer. Discuss a variety of mental strategies used by students, including ways they used the properties discussed in Action 2 and/or knowledge of the rules for order of operations to facilitate their mental calculations. Encourage students to discuss visual models they use to help them “see” the answers. Repeat for one or more of b)-d).

a) $17 + 6 + 3 + 14$

b) $52 - 19 \times 2$

c) $6 \times (6 + 8)$

d) $8 \div 4 \times (3 + 5) - 7 + 9$




## COMMENTS

3 In Lesson 4 of *Course I*, students were introduced to the rules for order of operations and to the fact that these rules were established arbitrarily (i.e., other interpretations of the symbols could be equally logical) by mathematicians years ago to facilitate communication about numerical procedures and relationships. The rules are as follows: computations within parentheses should be carried out first; then exponents are evaluated; next products and quotients should be computed in the order they occur from left to right; and finally sums and differences should be computed in the order they occur from left to right.

Mental computation strategies were emphasized in Lessons 31 and 37 of *Course I* and will be explored again in Lesson 19 of this course. Note that some students may not relate their mental strategies to the properties by name and they may not remember names of particular mental methods. It isn’t necessary to emphasize these names, although you may wish to point them out as they are used.

a) Many students will probably use a combination of the associative and commutative properties to view this computation as $(17 + 3) + (6 + 14) = 20 + 20$. They may refer to this as choosing *compatible* numbers that are easier to add mentally.

b) Based on the correct order of operations, this is equivalent to the computation $52 - 38$. Some students will compute this difference by forming the *equal difference*, $54 - 40$, which is easier to compute mentally. The diagram at the left shows that adding equal amounts to both numbers does not change the difference between the numbers.

Note that although $(52 - 19) \times 2$ is a logical interpretation of the expression in b) it is not the standard interpretation. You might have students test their calculators on this computation to see whether their calculators have order of operations built in.

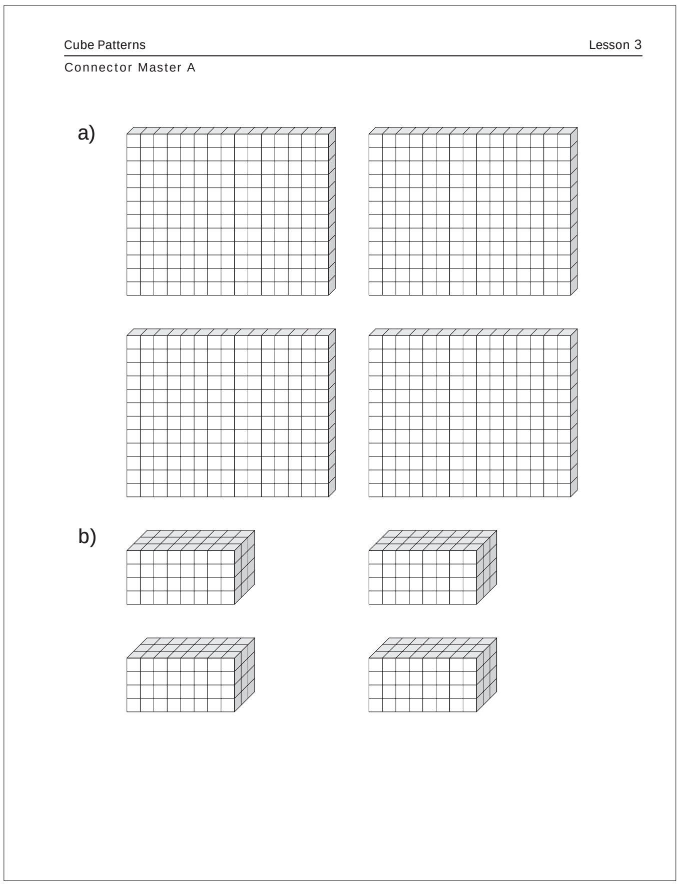
c) Some students may compute: $6 \times (6 + 8) = 6 \times 14 = 6 \times (15 - 1) = 90 - 6$. Others may compute $6 \times (6 + 8) = (6 \times 6) + (6 \times 8) = 36 + 48$.

d) According to the rules for the order of operations $8 \div 4 \times (3 + 5) - 7 + 9 = 2 \times 8 - 7 + 9 = 16 - 7 + 9 = 9 + 9 = 18$.

## Connector Teacher Activity (cont.)

### ACTIONS

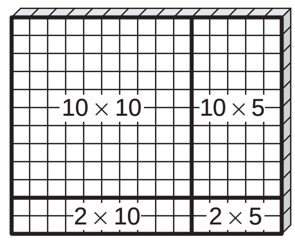
4 Give each student a copy of Connector Master A (see below) and ask them to determine different methods of seeing and counting the total number of cubes in the 12 × 15 rectangles shown in a), and then to use numbers and math symbols to record their thought processes. Have volunteers copy their symbolic statements on the overhead, and then invite other students to speculate about the counting methods associated with the recordings. Repeat, as needed, for the arrangements in part b) of Master A.


Cube Patterns Lesson 3
Connector Master A
a)
b)

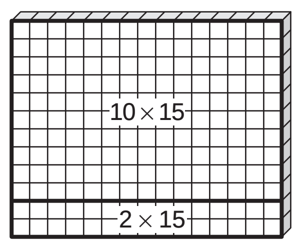
### COMMENTS

4 The intent here is for students to use parentheses and number properties to help communicate their thought processes so someone reading their recordings can “see” the cubes as they did.

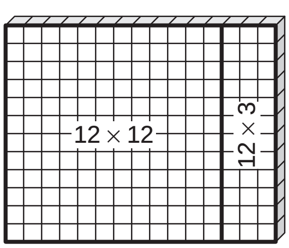
A transparency of Connector Master A is useful for recording and comparing different counting methods. There are many methods that may come up; here are four possibilities for a), although students’ recordings may not be as detailed as those shown below:



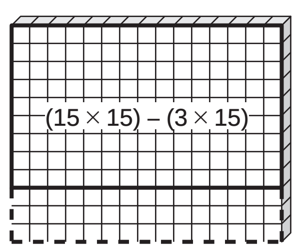

$$\begin{aligned}12 \times 15 &= (10 + 2) \times (10 + 5)\\ &= (10 \times 10) + (10 \times 5) + (2 \times 10) + (2 \times 5)\\ &= 100 + 50 + 20 + 10\end{aligned}$$




$$\begin{aligned}12 \times 15 &= (10 + 2) \times (15)\\ &= (10 \times 15) + (2 \times 15)\\ &= 150 + 30\end{aligned}$$




$$\begin{aligned}12 \times 15 &= 12 \times (12 + 3)\\ &= (12 \times 12) + (12 \times 3)\\ &= 144 + 36\end{aligned}$$




$$\begin{aligned}12 \times 15 &= (15 \times 15) - (3 \times 15)\\ &= 225 - 45\end{aligned}$$

Connector Teacher Activity (cont.)

## ACTIONS

5 (Optional) Ask the pairs to determine the surface areas of the arrangements of cubes on Connector Master A, and to use numbers and math symbols to communicate their methods of seeing and counting the area units. Have volunteers share their recordings and thought processes.

## COMMENTS

5 The surface area of the arrangement in a) is 414 square units and the surface area of the arrangement in b) is 136 square units.

**TEACHER NOTES:**

# Focus Teacher Activity

OVERVIEW & PURPOSE

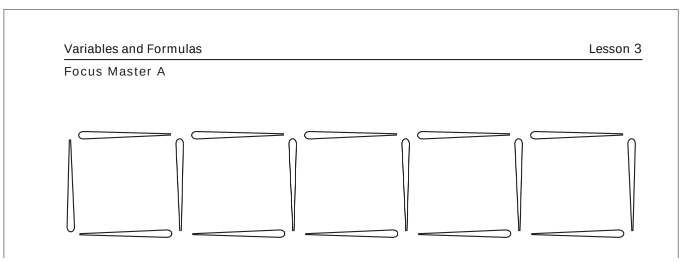
*Rows of squares are formed with toothpicks. The relationship between the number of squares in a row and the number of toothpicks needed to form the row is investigated, leading to the introduction of algebraic notation and formulas.*

MATERIALS

- ✔ Flat toothpicks, 25 per student and 25 for use at the overhead.
- ✔ Focus Masters A and B, 1 transparency of each.
- ✔ Focus Master C, 1 transparency per group.
- ✔ Overhead pens, 1 or more per group.
- ✔ Scissors, 1 pair per group.
- ✔ Butcher paper (optional), 1 sheet per group.
- ✔ Glue or tape and a marking pen (optional) for each group.

## ACTIONS

1 Arrange the students in groups and distribute about 25 toothpicks to each student. Place a transparency of Focus Master A on the overhead, covering all but the top row of toothpick squares. Have the students form an identical arrangement of toothpicks.


Variables and Formulas Lesson 3
Focus Master A

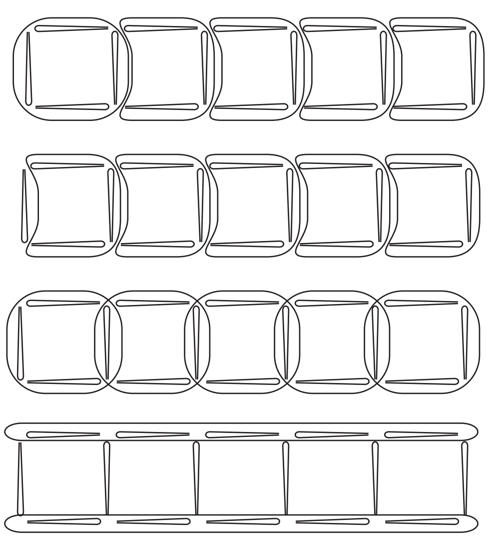
2 Ask the students to describe ways, in addition to one-by-one counting, to determine the total number of toothpicks in the 5 squares. Discuss different ways of "seeing" the total of 16.

a)

b)

c)

d)



## COMMENTS

1 The pattern of squares can also be displayed by placing toothpicks on the overhead projector.

2 Having students illustrate their methods by drawing loops around the toothpicks on the transparency of Focus Master A helps other students "see" different ways of thinking. Below are some ways of viewing the number of toothpicks. The students may find others.

a) One square of 4 toothpicks and 4 groups of 3:
$4 + 4(3) = 16$.

b) One toothpick at the left and 5 groups of 3:
$1 + 5(3) = 16$.

c) Five squares of 4 toothpicks with 4 toothpicks counted twice: $5(4) – 4 = 16$.

d) Two rows of 5 toothpicks and 6 vertical toothpicks:
$2(5) + 6 = 16$.

Focus Teacher Activity (cont.)

## ACTIONS

3 Ask the students to imagine extending the row of 5 squares to 12 squares and then predict the total number of toothpicks needed to build the 12 squares. Discuss the methods they use to predict the total.

## COMMENTS

3 Twelve squares require 37 toothpicks. Here are ways of determining this, corresponding to the methods described in Action 2:

a) 4 + 11(3) = 37 (1 square of 4 toothpicks and 11 groups of 3),

b) 1 + 12(3) = 37 (1 toothpick on the left and 12 groups of 3),

c) 12(4) – 11 = 37 (12 squares of 4 toothpicks with 11 toothpicks counted twice),

d) 2(12) + 13 = 37 (2 rows of 12 toothpicks and 13 vertical toothpicks).

## ACTIONS

4 Have the students determine the number of toothpicks if the row of squares is extended to:

a) 20 squares, b) 43 squares, c) 100 squares.

Discuss.

## COMMENTS

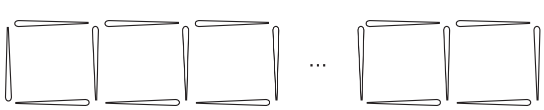
4 In determining an answer, a student is likely to use one of the methods discussed in Action 3. To encourage comfort with more than one strategy, you could ask students to verify their results by using one of the other methods suggested.

## ACTIONS

5 Place a transparency of Focus Master B on the overhead and give each group a transparency of Focus Master C, a pair of scissors, and an overhead pen. Allow time and provide encouragement for groups to review their written directions for clarity and correctness before sharing in Action 6.

Lesson 3 Variables and Formulas

Focus Master B



I have in mind a *secret* row of toothpick squares. If I told you how many squares are in my secret row, what are some ways you could use that information to determine the number of toothpicks I used? On Focus Master C, write verbal directions for each method your group devises.

## COMMENTS

5 Having students discuss with one another their ideas for determining the number of toothpicks may help them clarify their thoughts. You may have to explain to the students that “verbal directions” means directions expressed *only* in words, without using other symbols such as: numbers, letters, arithmetic notation, diagrams, and sketches.

As you circulate, you may notice groups recording methods that work for a specific number of squares, say 45. If this happens, you could ask the group how their method would work no matter what the number of squares.

Variables and Formulas Lesson 3

Focus Master C

To determine the number of toothpicks,

To determine the number of toothpicks,

Focus Teacher Activity (cont.)

## ACTIONS

6 Ask for a volunteer to show one set of directions at the overhead, with the understanding that the class will offer feedback regarding parts that are especially clear and parts that are unclear or need revisions. As a large group, have the class help the volunteer edit and adjust the directions until agreement is reached that following the directions, as written, leads to a correct result. Repeat this action until directions for several different methods are agreed upon.

## COMMENTS

6 One way to carry this out is to have the groups cut apart their sets of directions. One set can be placed on the overhead. If other groups have written directions for the same general method, their directions could also be placed on the overhead for comparison. A revised set of directions may be a composite of these and/or refinements of these. Possible directions corresponding to the methods described in Action 2 are:

a) *To determine the number of toothpicks, multiply one less than the number of squares by three and add this amount to four.*

b) *To determine the number of toothpicks, add one to three times the number of squares.*

c) *To determine the number of toothpicks, multiply the number of squares by four and then decrease this amount by one less than the number of squares.*

d) *To determine the number of toothpicks, double the number of squares and then add to this amount one more than the number of squares.*

If a set of directions is suspected to be incorrect, students could test the directions for specific instances. For example, if the number of squares is 20, following the directions for finding the number of toothpicks should result in 61 toothpicks, as determined in Action 4. If students do not notice ambiguity that exists in certain wordings, you might pose a wrong answer that results from a literal interpretation of the directions. Then ask the students to discuss how they think you interpreted the directions and have them clarify the directions so as to avoid such an interpretation. For example, if the wording for method a) above were "multiply 1 less than the number of squares by 3 plus 4" then one might determine incorrectly that $19 \times 7 = 133$ toothpicks are contained in 20 squares.

## ACTIONS

7 Have the students suggest symbols to stand for the phrases "the number of toothpicks" and "the number of squares." Discuss their suggestions.

## COMMENTS

7 While the choice of symbols is a matter of personal preference, it is helpful to choose symbols which are easily recorded, not readily confused with other symbols in use, and are suggestive of what they represent. For example, "the number of squares" might be represented by $n$ (the first letter of the word "number"), or by $S$ (the first letter of the word "square"). The latter choice may be preferable since it is not as likely to be taken to mean "the number of toothpicks."

## Focus Teacher Activity (cont.)

### ACTIONS

8 From the suggestions made in Action 7, select symbols to represent the number of toothpicks and the number of squares. Have the groups use these symbols and standard arithmetic symbols to write, in symbolic form, each set of directions agreed upon in Action 6. Point out to the students that a set of directions written in symbolic form is called an *algebraic formula*. Ask for volunteers to show their formulas and encourage students to test one another's formulas.

### COMMENTS

8 For the sake of discussion and making comparisons, it is useful if everyone agrees to use the same symbols. If the issue is raised, you may want to suggest the use of "grouping" symbols, such as parentheses, to avoid ambiguities (see Connector activity).

If the validity of a formula is in question, you can ask the students to test it to evaluate the number of toothpicks given a specified number of squares.

Following are formulas corresponding to the directions listed in Comment 6. In the formulas, $T$ stands for the number of toothpicks and $S$ stands for the number of squares. (A symbol, such as $S$ or $T$, that stands for a quantity that can have different values is called a *variable*.)

a) $T = 4 + 3(S - 1)$,

b) $T = 3S + 1$,

c) $T = 4S - (S - 1)$,

d) $T = 2S + (S + 1)$.

Some students may write "$3S - 1$" for "$3(S - 1)$" in formula a). If this happens and students don't raise the issue, you might comment on the need to distinguish between "subtracting 1 from 3 times the number of squares" and "subtracting 1 from the number of squares and then multiplying by 3." Parentheses are used to make this distinction. Other ambiguities may arise. They can be discussed as they occur.

Keep in mind this lesson is many students' first experience with using variables to write formulas. It is not important to expect mastery at this point. Note that using variables and writing formulas are the major emphasis of Lessons 4, 9, 10, 26, and 27 in this course and many lessons in *Visual Mathematics, Courses III* and IV.

### ACTIONS

9 Discuss symbols and their role in writing mathematics.

### COMMENTS

9 One way to begin the discussion is to ask the students what they perceive as advantages and/or disadvantages in using symbols rather than words.

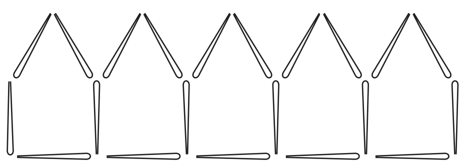
The use of symbols enables one to write mathematical statements concisely and precisely. However, it can obscure meaning if the reader is unfamiliar with the symbols used or lacks practice in reading symbolic statements.

Focus Teacher Activity (cont.)

## ACTIONS

**10** Pose the following for discussion by the groups: If the row of squares in Action 1 is extended until 142 toothpicks are used, how many squares will there be in the row? Discuss. Ask the students to determine other possible values for *T* and *S*.

**11** Form or sketch a row of toothpick pentagons as shown below and ask the students to also form this row.



Pose situations such as the following for groups to investigate regarding rows of pentagons. Have students write equations that represent mathematical observations about each situation.

a) There are 58 pentagons in a certain row of pentagons.

b) There are 293 toothpicks in another row of pentagons.

c) Leticia's row of pentagons has 3 times the number of pentagons that Manuel has in his row of pentagons, and together they used 114 toothpicks.

d) Chiu built two rows of pentagons so that one row has $\frac{1}{5}$ as many pentagons as the other row. One row contains 33 toothpicks.

## COMMENTS

**10** There are 47 squares.

Some students may arrive at the answer by a "guess-and-check" method. Other students may use their knowledge of how squares are formed: "After 1 toothpick is placed, there are 141 left and it takes 3 more to form each square. So 141 ÷ 3, or 47, squares are formed."

You may wish to point out to the students that an answer may also be arrived at by replacing *T* by 142 in the formula in Comment 8 and determining what *S* must be to have equality. In arriving at an answer, students have determined the solution of the *equation*: $142 = 3S + 1$.

Note: the point here is *not* to use traditional algebraic methods but rather to reason from the model.

**11** Encourage students to reason from the models and then use symbols to record their actions and thought processes. Here is one possible observation about each situation:

a) The given row contains 5 pentagons, and each pentagon uses 4 toothpicks. There is one extra toothpick on the end. So this row contains $1 + (4 \times 5) = 21$ toothpicks. Using this same line of reasoning, a row of 58 pentagons contains $1 + (4 \times 58) = 233$ toothpicks. If it hasn't come up previously, you might discuss the common practice of writing $4 \times 58$ as $4 \cdot 58$ or $4(58)$.

b) This row contains $(293 - 1) \div 4 = 73$ pentagons.

c) Leticia's row contains 21 pentagons.

d) There are 8 pentagons in one row and 40 in the other.

Focus Teacher Activity (cont.)

## ACTIONS

12 If it hasn't already come up, ask the students to write a formula relating the number of toothpicks used with the number of pentagons in the row shown in Action 11.

13 (Optional) Invite the groups to create rows of other toothpick arrangements and to pose to the class questions or situations such as those in Action 11 about their rows.

## COMMENTS

12 If $T$ is the number of toothpicks used and $P$ is the number of pentagons formed, then one formula could be: $T = 1 + 4P$.

This formula can be written in other forms. Also, students might choose symbols other than $T$ and $P$ to represent the number of toothpicks and the number of pentagons.

In giving a formula, it is necessary to give the meaning of symbols like $T$ and $P$ that do not have standard meanings.

Note that while some students may be able to tell the number of toothpicks for a row of 100 pentagons, they may have difficulty writing a formula using variables. In such cases it is important to note these students *are* generalizing. Use of variables will become more comfortable as they explore patterns in later lessons.

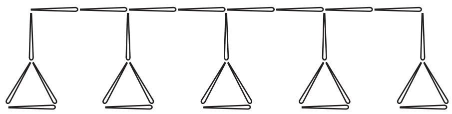
13 One way to carry this action out is to provide each group with a sheet of butcher paper on which they tape or glue their toothpick arrangements and use marking pens to write several questions or situations involving these arrangements. Groups then exchange posters and write (directly on the posters, or on "sticky notes" that are attached to the poster) mathematical observations and formulas regarding each other's toothpick arrangements, questions, and situations. Display these posters in the classroom and invite the students in the class to, over time, attach other mathematical observations and formulas to the posters.

# Follow-up Student Activity 3.1

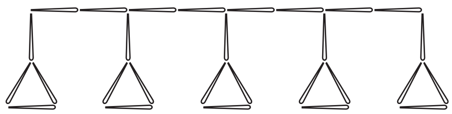
NAME ______________________ DATE __________

1 a) Describe in words two different methods, other than one-by-one counting, of “seeing” that the total number of toothpicks in the following figure is 28. “Loop” the toothpicks to show your thinking.

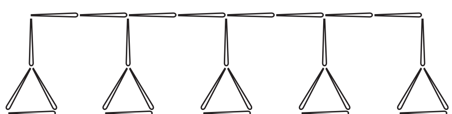
Method I



Method II



b) Suppose the row of toothpick arrangements above could be extended, and the row always contains a “hanging triangle” on each end. The row shown contains 5 triangles. Determine the number of toothpicks there will be if the figure is extended to a figure with 150 triangles. Explain your methods and loop the diagram to illustrate your thinking.



c) Write at least one formula for $N$, the total number of toothpicks in a row that contains $T$ triangles.

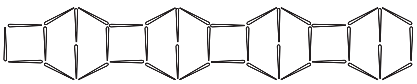
d) A certain row of the above toothpick arrangements contains 580 toothpicks. What is the number of triangles in the figure? Explain the methods you used to decide.

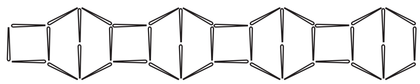
(Continued on back.)

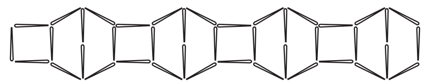
Follow-up Student Activity (cont.)

2 Suppose the row of toothpick arrangements shown below could be extended, and the row always begins with a square and ends with 2 trapezoids.

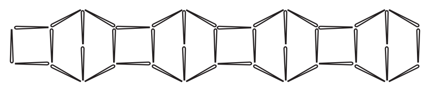
a) Write 3 different formulas for the total number of toothpicks in a row that contains 5 squares. Loop the diagrams below to show how your formulas work.







b) If a row is extended so that the total number of squares is 138, how many toothpicks will it have? Explain how you decided this. Mark the diagram to help show your methods.



c) If a row is extended so that it has a total of 731 toothpicks, how many trapezoids will it have? Explain your thought processes.

3 On another sheet, create a row of toothpick arrangements that can be extended. Write and answer several interesting questions about the row when it is extended.