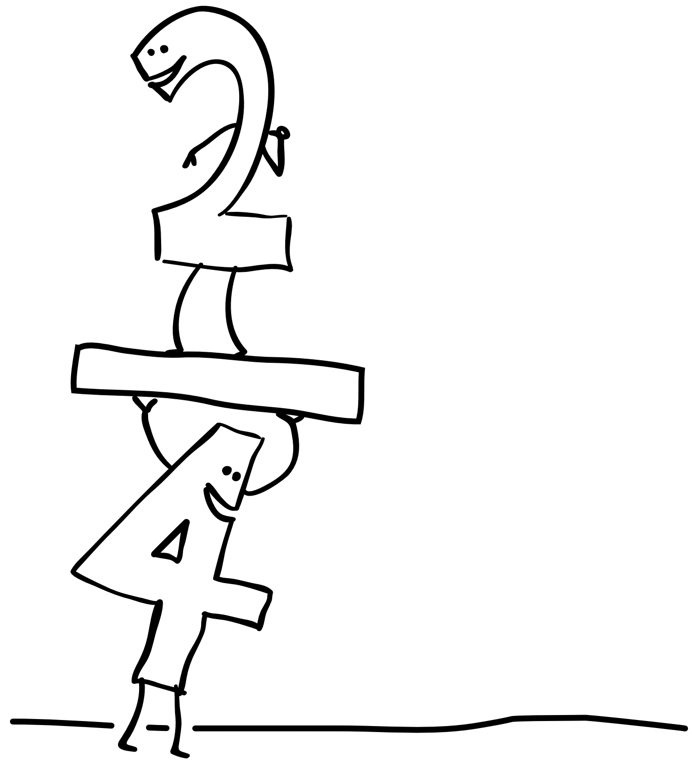


FRACTIONS

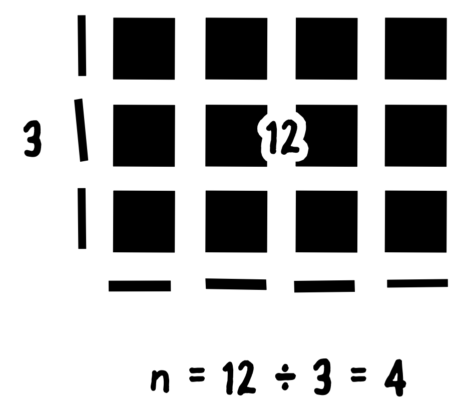
The ninth scene in a series of articles
on elementary mathematics.

written by Eugene Maier
designed and illustrated by Tyson Smith

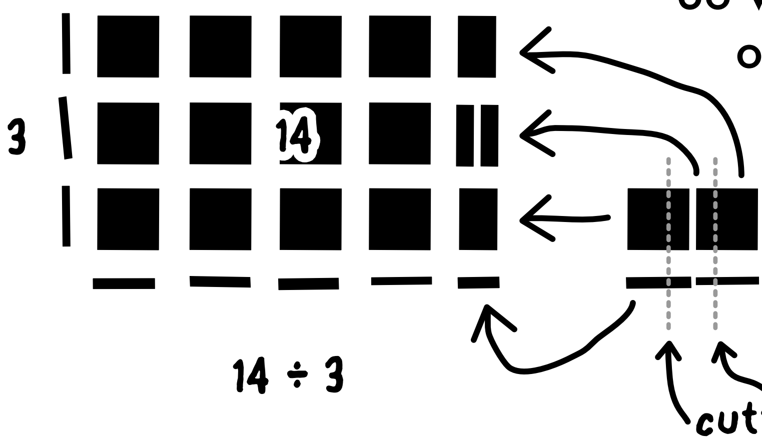
We began the ***Story of Numbers*** with the counting numbers and their arithmetic. However, within the counting numbers equations such as $7 + \boldsymbol{n} = 4$ aren't solvable—there is no counting number $\boldsymbol{n}$ which can be added to 7 to get 4. The integers were introduced to overcome this deficiency. In the integers, the equation $7 + \boldsymbol{n} = 4$ is solvable—there is an integer $\boldsymbol{n}$ such that $7 + \boldsymbol{n} = 4$, namely -3, which is the difference $4 - 7$. Indeed, in the integers, every equation of this form is solvable: for any two integers $\boldsymbol{a}$ and $\boldsymbol{b}$ there is an integer $\boldsymbol{n}$ such that $\boldsymbol{a} + \boldsymbol{n} = \boldsymbol{b}$, namely the difference $\boldsymbol{b} - \boldsymbol{a}$.

However, there are other equations that aren't solvable within the integers. For example, there is no integer $\boldsymbol{n}$ for which $3\boldsymbol{n} = 14$. In order to solve this equation and others like it, we must expand our collection of numbers.

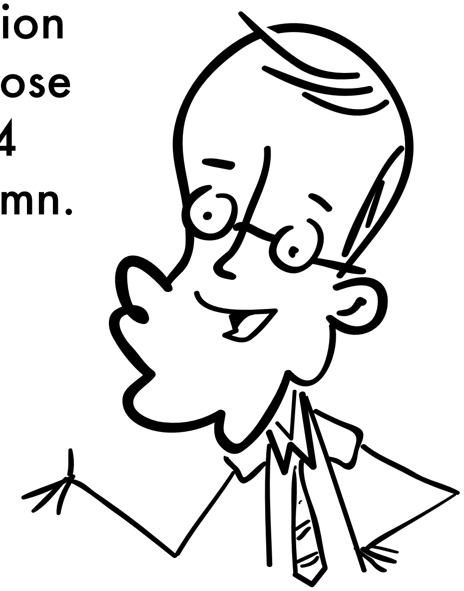
We begin by examining an equation, $3\boldsymbol{n} = 12$, which does have a solution in the integers. If one uses the array model for multiplication and division introduced in Scenes 4 and 5 to represent the relationship expressed by this equation, $\boldsymbol{n}$ is the value of the bottom edge of an array of 12 black tile whose left edge is composed of 3 black edge pieces, that is $\boldsymbol{n}$ is the quotient $12 \div 3$, or 4.



Suppose now we change 12 to 14 and attempt to solve the equation $3\boldsymbol{n} = 14$. To do this, we want to form an array of 14 black tile whose left edge contains 3 black edge pieces. We have enough tile for 4 columns, but the 2 remaining tile are not enough for another column.

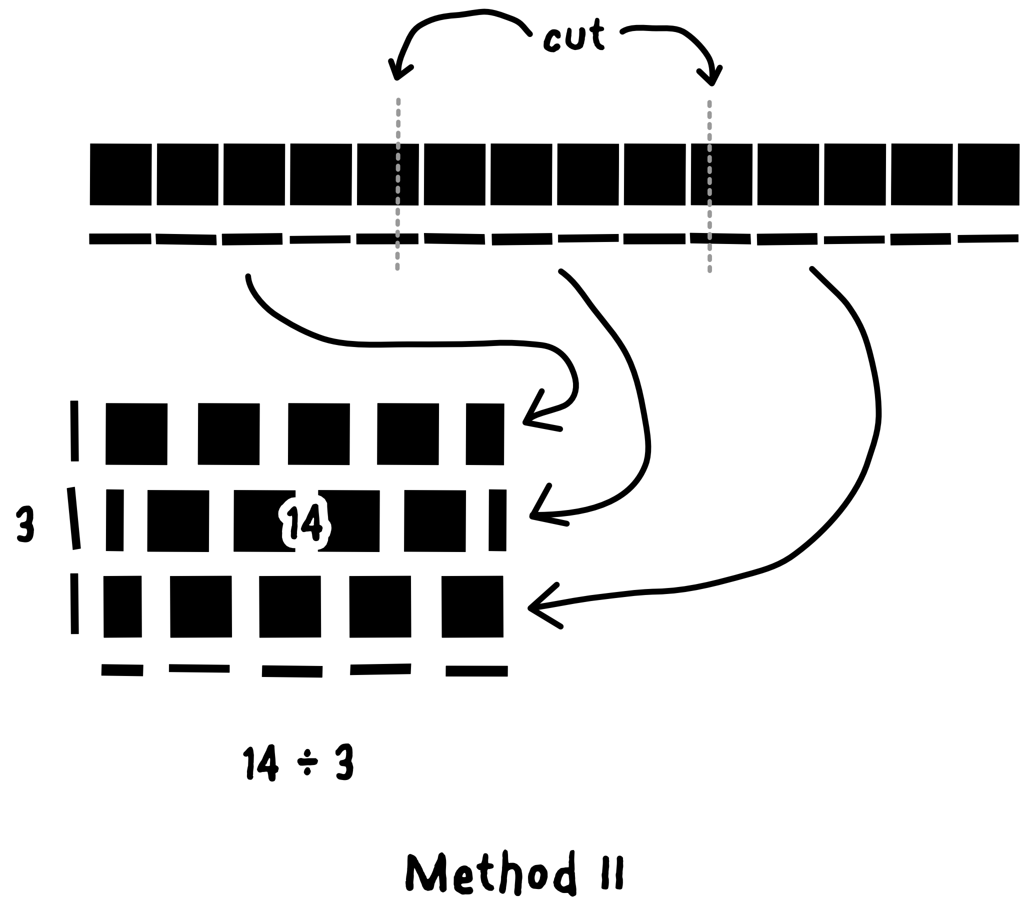


Method 1

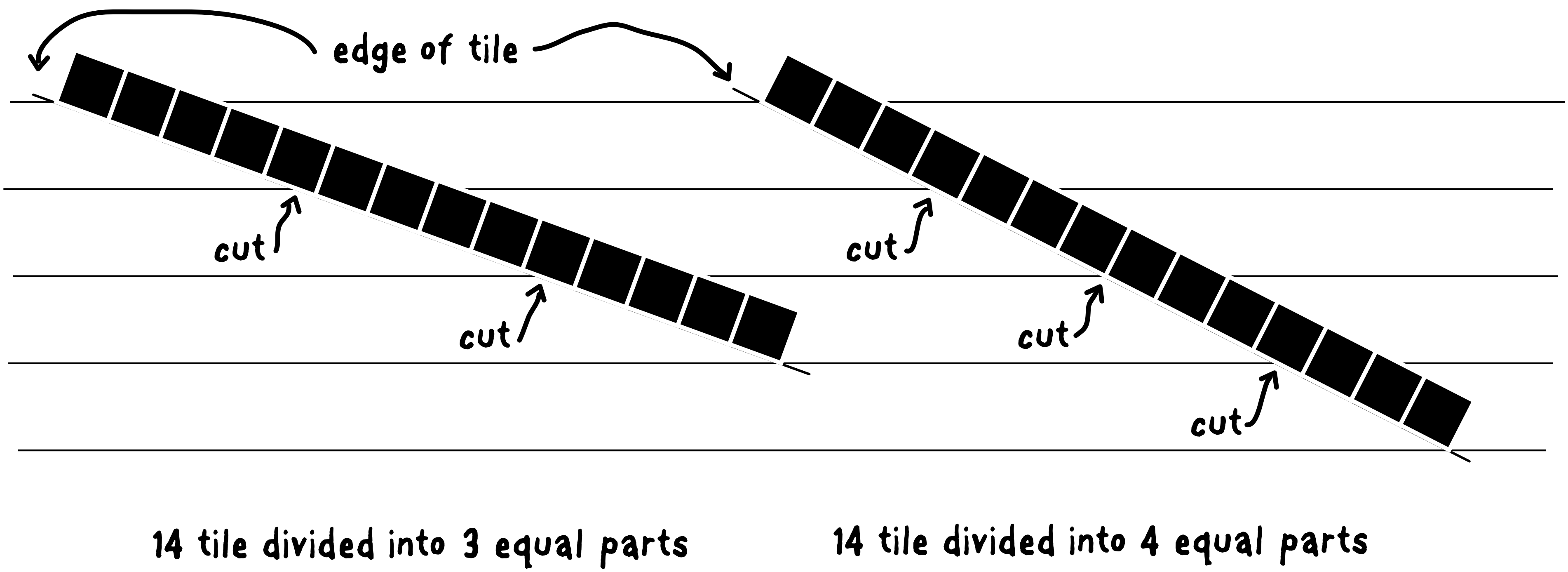


So we do what perhaps seems obvious: we cut the remaining 2 tile to fit in the array. One way to do this is to place them side by side and make 2 cuts to divide them into 3 equal parts, and place one part in each row. (Note that one of these parts consists of a portion from each of the two tile.) Now we have an array whose value is 14 with a left edge whose value is 3. The value of this bottom edge is the quotient $14 \div 3$.

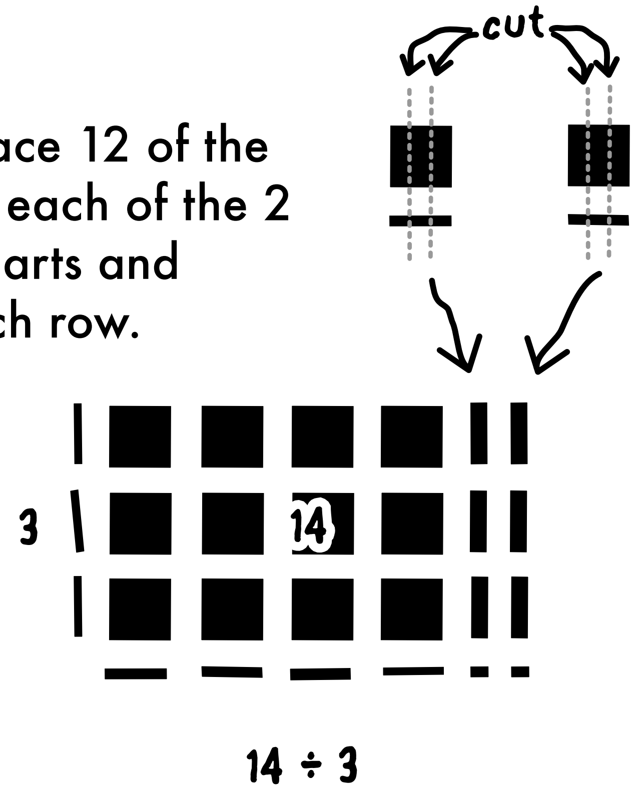
There are other ways we can divide the 14 tile into 3 rows. One way is to line up the 14 tile in a row and with 2 cuts divide the 14 tile into 3 equal parts. (***Click here*** to see how this can be done using equi-spaced parallel lines.) These 3 parts can then be arranged into a 3 high array.



To determine where cuts should be made to divide a row of 14 tile into 3 equal parts, an edge of the row of tile can be placed on a sheet of equally spaced parallel lines so that the two ends of the edge lie on parallel lines that are 3 spaces apart. The cut points are the points where the edge intersects the 2 intervening parallel lines.

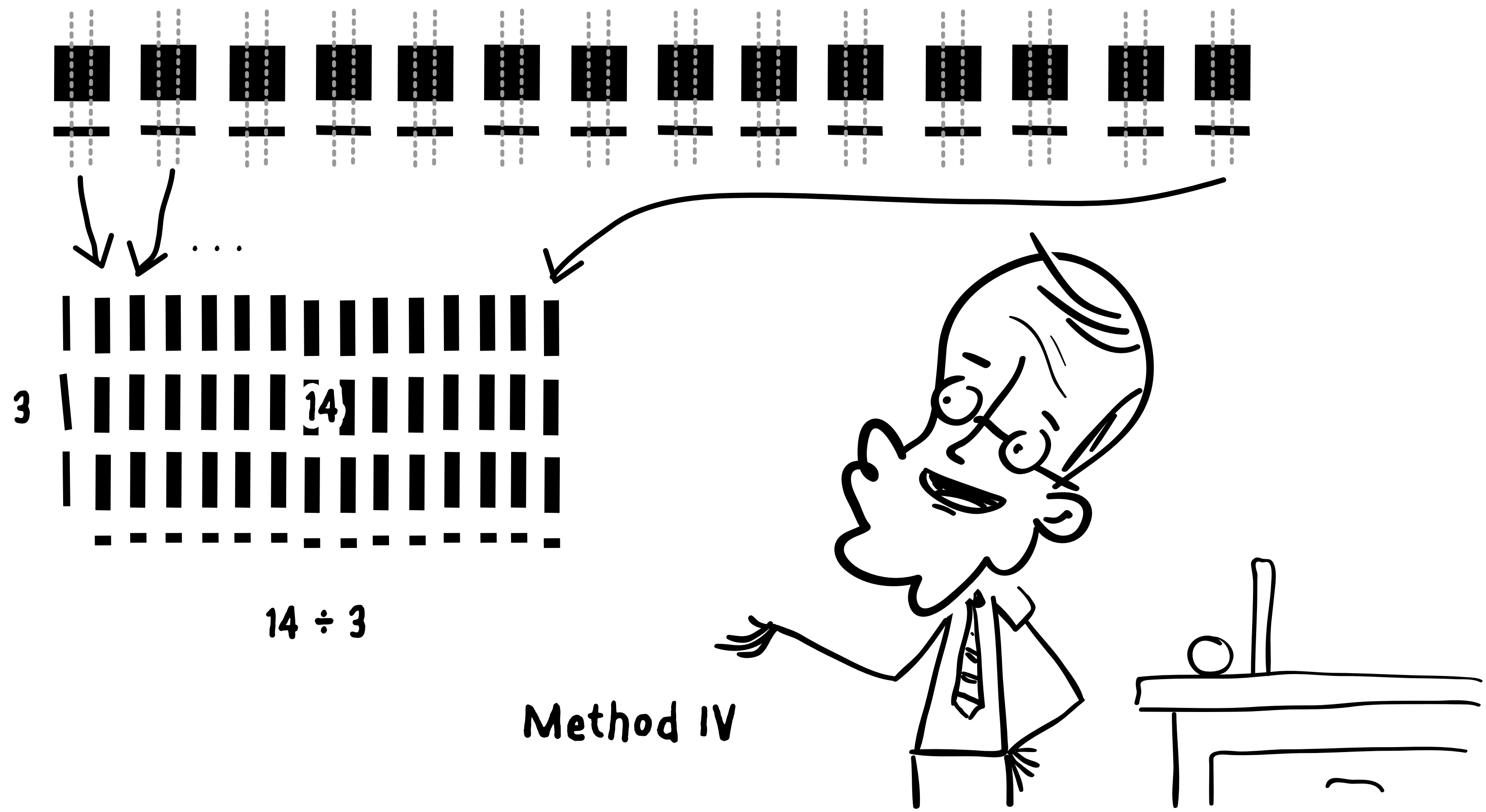


Or one could proceed to place 12 of the tile into 3 rows and then cut each of the 2 remaining tile into 3 equal parts and place 1 of these parts in each row.



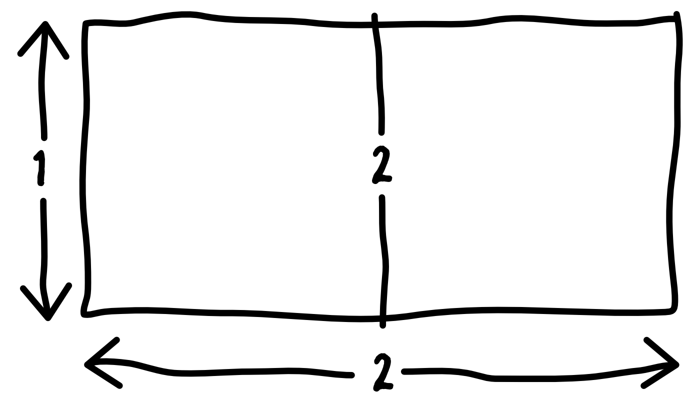
Method III

Notice that each of the first two methods of cutting required only 2 cuts while the third method requires 4 cuts. If one wanted to make lots of cuts, one could divide each tile into 3 equal parts and place 1 of these parts in each row.

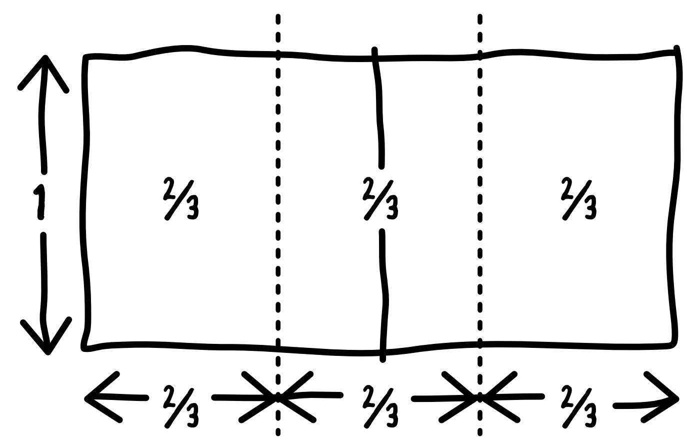


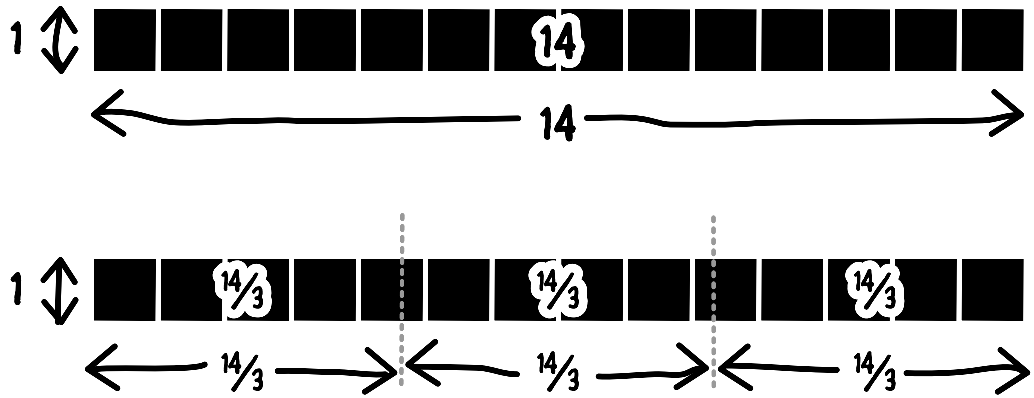
After a string of tile (or edge pieces) have been cut into equal parts, the amount in each part is designated by a ***fraction***. A fraction consists of two numbers, generally written one above the other separated by a bar (or one after another separated by a slanted line). The top number (or first number) is called the ***numerator***. It indicates the number of units—whatever they might be: tile, edge pieces, miles, pounds, etc.—that are being divided. The bottom number is called the ***denominator***. It indicates the number of equal parts into which the given units are being divided. In ***Method I*** above, 2 tile are divided into 3 equal parts. Hence, each part is $\frac{2}{3}$ tile (or, written in the "slanted line" notation, ⅔ tile.).

A 1 x 2 array:

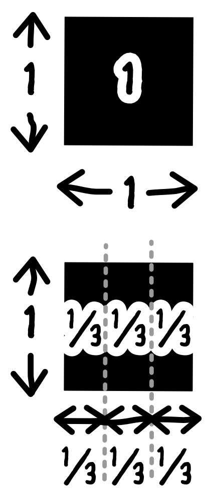


A 1 x 2 array cut into 3 equal parts:





In ***Method II***, 14 tile are divided into 3 equal parts. Hence each part is $\frac{14}{3}$ tile.



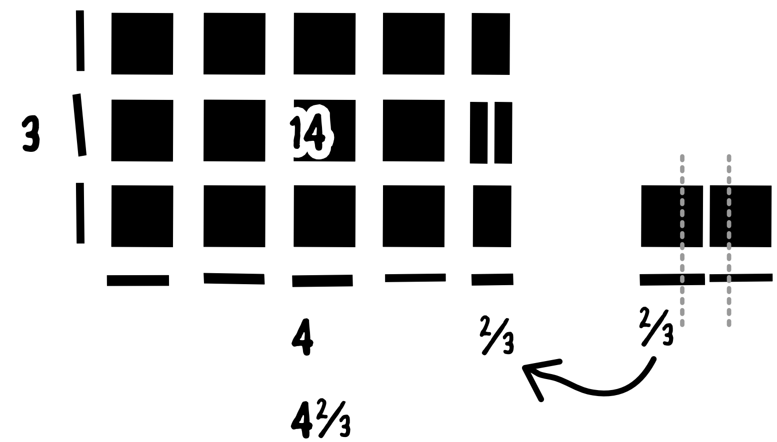
In ***Methods III*** and ***IV***, single tile are divided into 3 parts. Hence, each of these parts is $\frac{1}{3}$ tile.

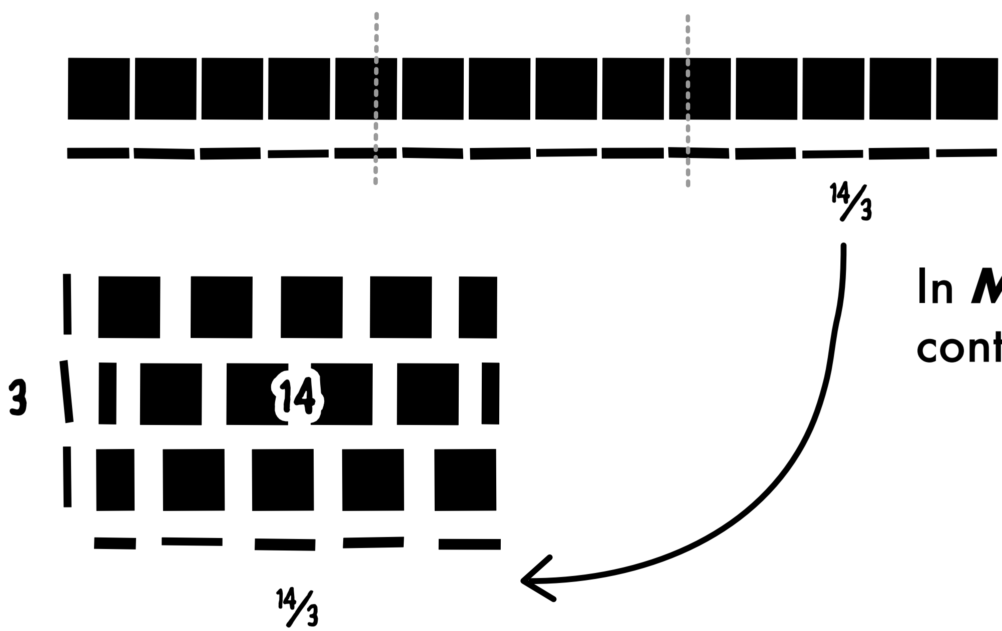
Gene says: The fraction $\frac{2}{3}$ is read "two-thirds" or, sometimes. "two over three". Typically, a fraction is read by naming its numerator as a cardinal number and the denominator as the plural of an ordinal number, except in the case of a unit fraction in which case the plural is not used. Thus, $\frac{27}{4}$ is read as "twenty-seven fourths" and $\frac{1}{4}$ as "one fourth".

The use of the ordinal form "fourth" in connection with a fraction is one of those anomalies of mathematical terminology that can make teaching mathematics a tricky business. We tell our students the cardinal form of a number is used to report "how many", as in "There are four quarters in a dollar," whereas the ordinal form of a number is used to report the position of something in a sequence, as in "April is the fourth month of the year." Then we cut a strip of paper into four equal parts and call each part "a fourth"—a usage that has nothing to do with position. The definite article, we say, will indicate which meaning of "fourth" is intended: "***the*** fourth" refers to a position which comes after "the third" position while "***a*** fourth" indicates the result of cutting something into 4 parts which, in terms of size, is less than "a third". Then students go to choir and the music teacher tells the class they sang "a fourth" when they should have been singing "a third," which is a smaller interval than "a fourth". Oh, well.

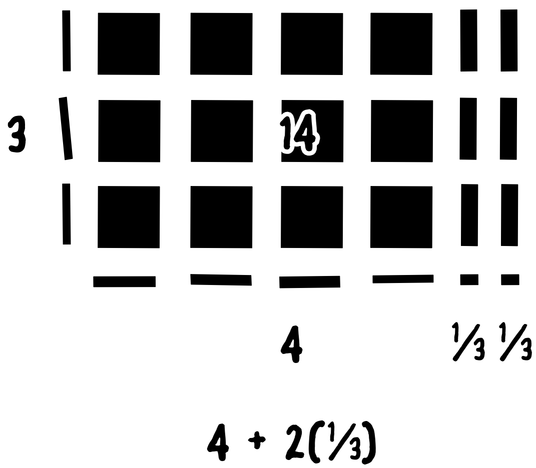
Fractions can be used to designate the value of the bottom edges in the 4 methods used to represent the quotient 14 ÷ 3.

In ***Method I***, the bottom edge consists of 4 and $\frac{2}{3}$ edge piece. Thus its value is 4 + $\frac{2}{3}$ which is customarily written $4\frac{2}{3}$, the "plus" sign being taken for granted.

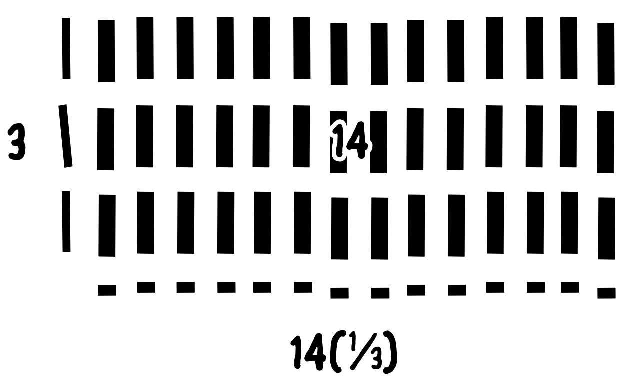




In ***Method II***, the bottom edge contains $\frac{14}{3}$ edge pieces; so its value is $\frac{14}{3}$.

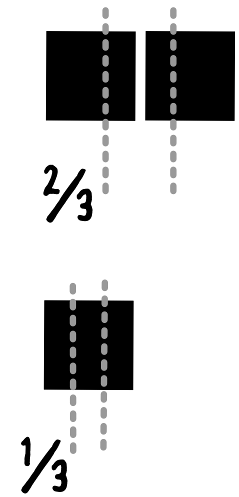


In ***Method III***, the bottom edge contains 4 edge pieces plus 2 partial edge pieces, each of value $\frac{1}{3}$. Thus the value of the edge is 4 + $\frac{1}{3}$ + $\frac{1}{3}$. Using the repeated addition model of multiplication, we can write $\frac{1}{3}$ + $\frac{1}{3}$ as 2($\frac{1}{3}$) and, accordingly, the value of the edge as 4 + 2($\frac{1}{3}$).



In ***Method IV***, the bottom edge consists of 14 partial edge pieces, each of value $\frac{1}{3}$, so its value is 14($\frac{1}{3}$).

Since, in all four methods, the edges have the same value, it follows that $14 \div 3 = 4\frac{2}{3} = \frac{14}{3} = 4 + 2(\frac{1}{3}) = 14(\frac{1}{3})$. Since $4\frac{2}{3} = 4 + 2(\frac{1}{3})$, it follows that $\frac{2}{3} = 2(\frac{1}{3})$, that is, if 2 is divided into 3 equal parts, each part is twice the portion one gets when 1 is divided into 3 parts. (See the figure.) Also, we have $\frac{14}{3} = 14(\frac{1}{3})$, that is, if 14 divided into 3 parts, each part is the same as 14 times the portion one gets when 1 is divide into 3 parts, or, to put it another way, fourteen-thirds (i.e., fourteen divided by 3) is the same as fourteen one-thirds (i.e., 14 copies of 1 divided by 3)

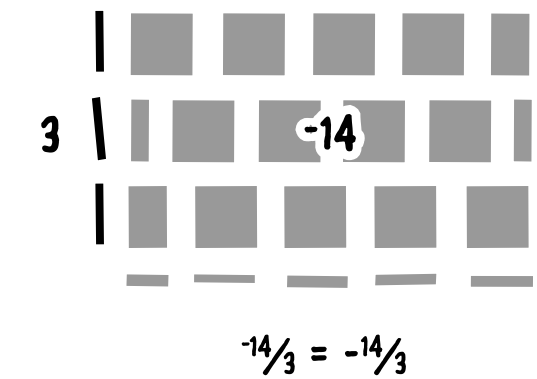


FYI: A fraction such as $\frac{1}{3}$, whose numerator is 1, is called a ***unit*** fraction—it's the result of dividing a single unit into equal parts. A number such as $4\frac{2}{3}$ which is the sum of a an integer and a fraction is called a ***mixed*** number. A fraction such as $\frac{2}{3}$ in which the numerator is less than the denominator is called a ***proper*** fraction and a fraction such as $\frac{14}{3}$ in which the numerator is larger than the denominator is called an ***improper*** fraction.

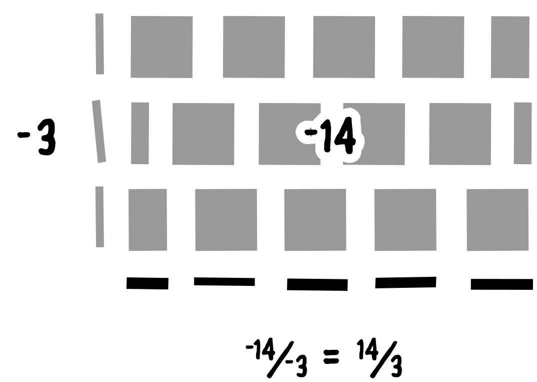
Gene says: I find the adjectives "proper" and "improper" as used in referring to fractions as misleading. An improper fraction is every bit as proper as a proper fraction. There is much to-do made in some classes about changing improper fractions into mixed numbers in which the fractional part is proper. I've never understood the urgency to do so. For some purposes it's simpler to leave a fraction in improper form. For example, if one wants the decimal equivalent of a fraction such as $\frac{175}{32}$ it's extra bother to turn this improper fraction into a proper fraction before dividing the numerator by the denominator.

So far all of our examples of fractions have involved only positive integers. However, a fraction might involve any integer—positive, negative, or zero. The denominator of a fraction may be either positive or negative, but it can't be 0 since, as pointed out in Scene 8, division by 0 is not allowed. The numerator of a fraction may be any integer. If it is 0, the value of the fraction is also 0 since, as also pointed out in Scene 8, 0 divided by any non-zero integer is 0.

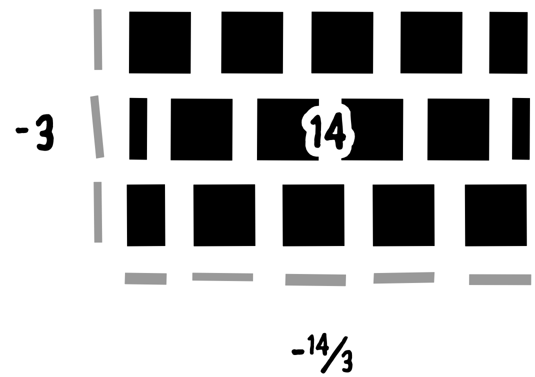
A fraction involving negative integers can also be modeled by arrays. For example, in an array model, the quotient $\frac{-14}{3}$ is the value of the bottom edge of an array which contains a collection of red tile whose value is -14 and whose left edge is 3 black edge pieces. Hence the bottom edge is red and is the opposite of a black edge whose value is $\frac{14}{3}$. Thus, $\frac{-14}{3} = -\frac{14}{3}$. In words, this says that the quotient obtained when -14 is divided by 3 is the opposite of the quotient obtained when 14 is divided by 3.



If we change the left edge of the above array from black to red, then the bottom edge is black. Thus, $\frac{-14}{-3} = \frac{14}{3}$, that is, the quotient obtained when -14 is divided by -3 equals the quotient obtained when 14 is divided by 3.



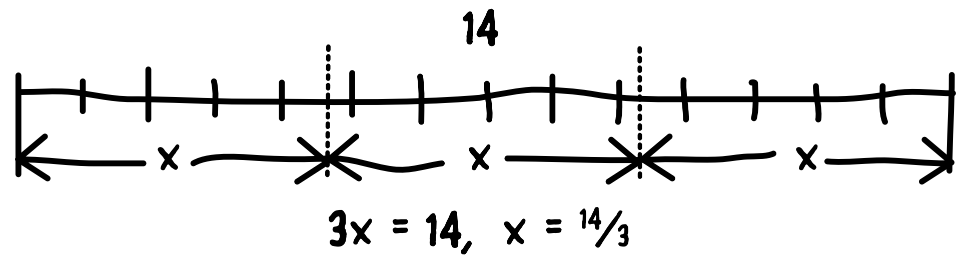
If we leave the array black and change the left edge to red, then the bottom edge is also red. Hence, $\frac{14}{-3} = -\frac{14}{3}$, that is, the quotient obtained when 14 is divided by −3 is the opposite of the quotient obtained when 14 is divided by 3.



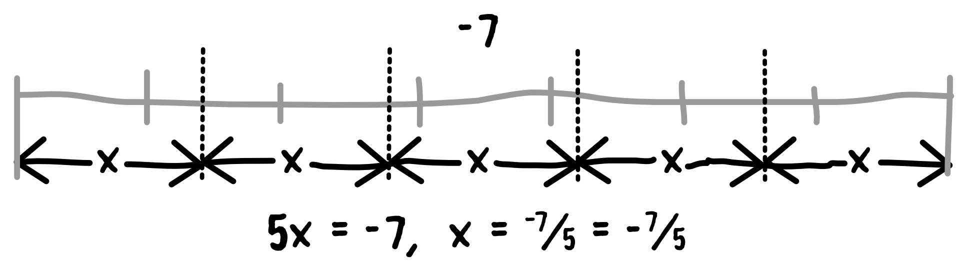
To summarize $\frac{-14}{3} = -\frac{14}{3}$ and $\frac{14}{-3} = -\frac{14}{3}$ and $\frac{-14}{-3} = \frac{14}{3}$.

In general if ***m*** and ***n*** are positive integers, then $\frac{-m}{n} = \frac{m}{-n} = -\frac{m}{n}$ and $\frac{-m}{-n} = \frac{m}{n}$. Thus, any fraction which has a negative numerator and/or denominator can always be expressed as a fraction, or the opposite of a fraction, in which both the numerator and denominator are positive.

Expanding our collection of numbers to include fractions enables us to solve equations like that referred to at the beginning of this scene. Whereas there is no integer $\boldsymbol{x}$ such that $3\boldsymbol{x} = 14$, there is a fraction $\boldsymbol{x}$ such that $3\boldsymbol{x} = 14$: $\boldsymbol{x}$ is the fraction obtained when 14 is divided into 3 equal parts, that is $\boldsymbol{x} = \frac{14}{3}$. In general, if $\boldsymbol{bx} = \boldsymbol{a}$, where $\boldsymbol{a}$ is an integer and $\boldsymbol{b}$ is a positive integer then $\boldsymbol{x} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$, that is, $\boldsymbol{x}$ is the fraction obtained when $\boldsymbol{a}$ is divided into $\boldsymbol{b}$ equal parts.



If $\boldsymbol{b}$ is a negative integer, consider the equation obtained by taking opposites of both sides. For example, if $-5\boldsymbol{x} = 7$, then, taking opposites, $5\boldsymbol{x} = -7$. So $\boldsymbol{x}$ is the fraction obtained when -7 is divided into 5 equal parts, that is, $\boldsymbol{x} = \frac{-7}{5} = -\frac{7}{5}$.



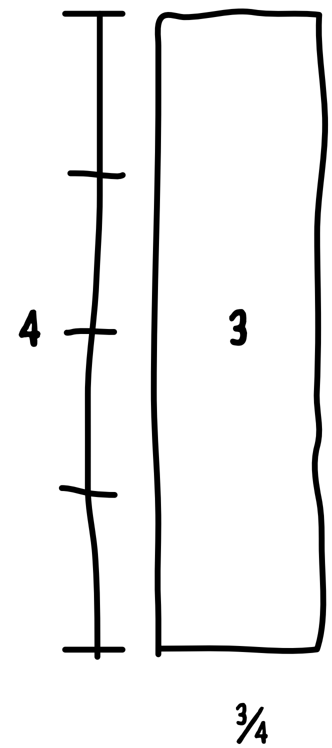
With fractions available, the equation $\boldsymbol{bx} = \boldsymbol{a}$ has a solution for all integers $\boldsymbol{a}$ and $\boldsymbol{b}$ as long as $\boldsymbol{b} \neq 0$.

FYI: Viewing a fraction as the quotient of two numbers is referred to as the ***division model*** of fractions.

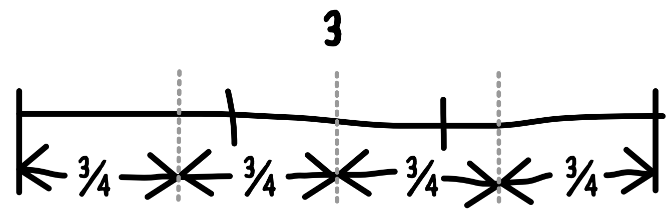
One can portray a fraction using the array model—also known as the area model—introduced in Scene 5. For example, $\frac{3}{4}$ is the value of the bottom edge of an array of value 3 whose left edge has value 4. Or, in terms of area, $\frac{3}{4}$ is the base of a rectangle whose area is 3 and height is 4. As we shall see in later scenes, the array model is especially helpful in developing the arithmetic of fractions.

One can also picture $\frac{3}{4}$ as the length of the parts obtained when a segment of length 3 is divided into 4 equal parts. We call this the ***linear model*** of $\frac{3}{4}$.

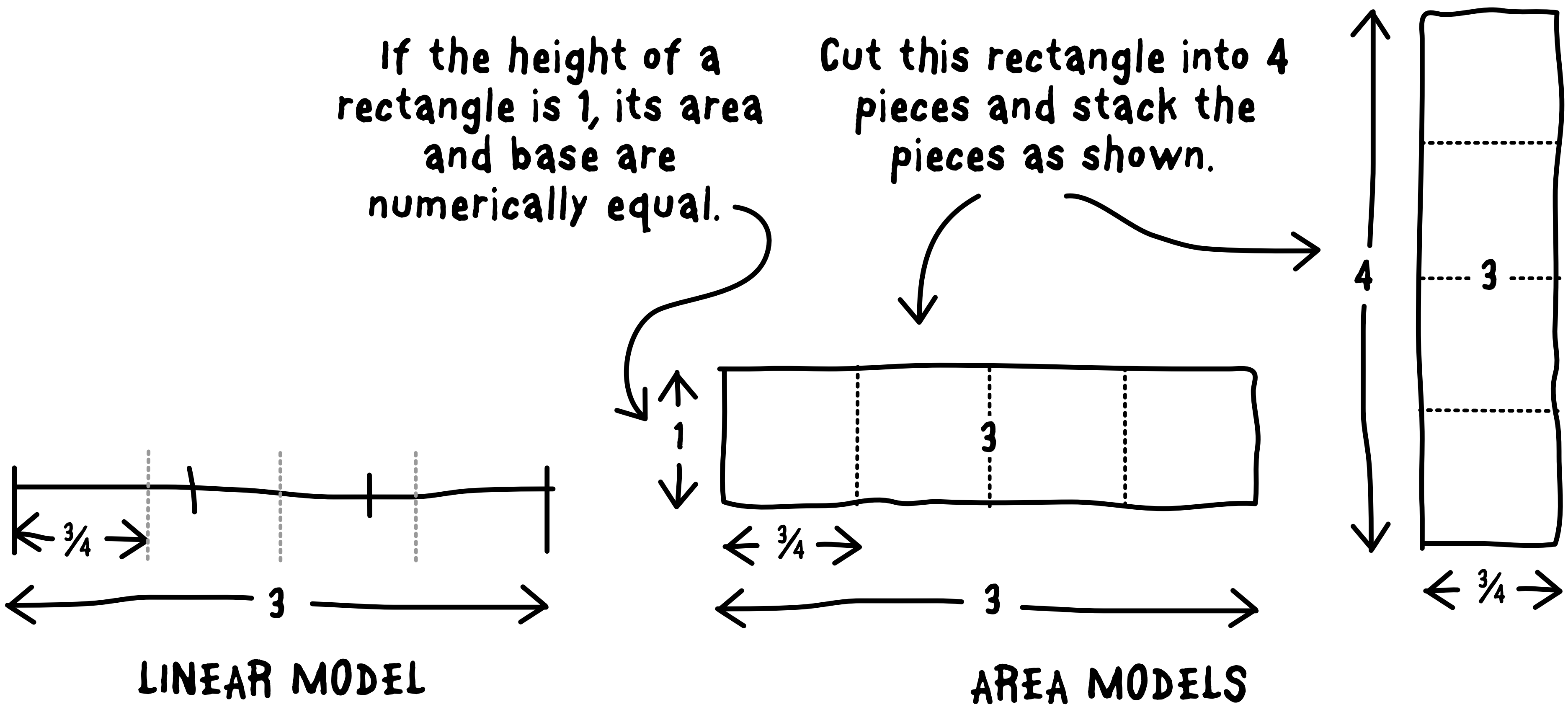
AREA MODEL OF 3/4:



LINEAR MODEL OF 3/4:

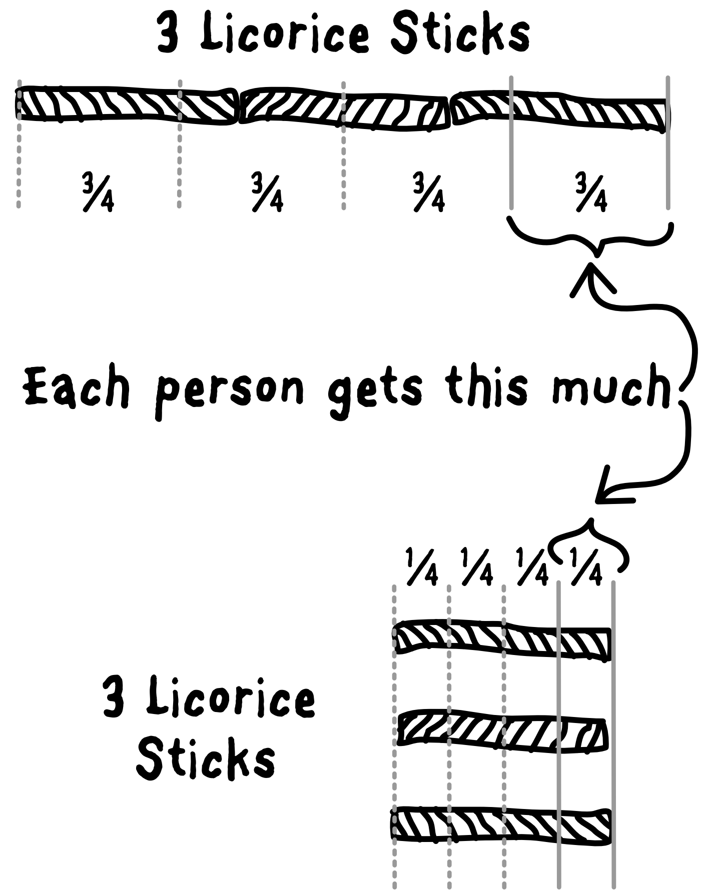


The linear model of $\frac{3}{4}$ can be converted to the area model of $\frac{3}{4}$ by constructing a rectangle and then dissecting and rearranging it as shown.

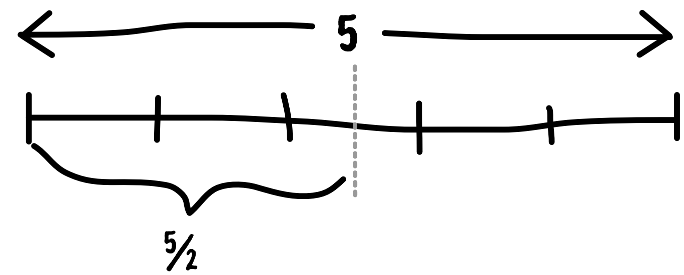


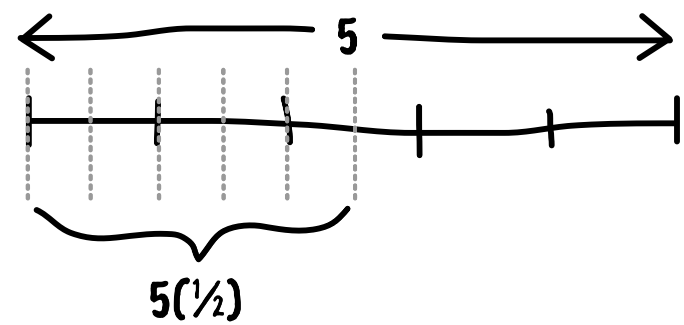
Often in school, another model is used to introduce fractions: the so-called ***parts-to-whole*** model. In this model, the denominator tells how many equal parts into which a "whole", i.e., 1 unit, is to be divided and the numerator tells you how many of these parts are to be taken. Thus, $\frac{3}{4}$ is what one obtains by dividing 1 into 4 parts and taking 3 of them, which, symbolically might better be written as $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$ or $3(\frac{1}{4})$. Rather than as the quotient $\frac{3}{4}$.

The division model and the parts to whole model lead to the same result. Suppose, for example, I want to divide 3 licorice sticks among 4 people. Using the division model, I could lay the sticks end to end and make 3 cuts to divide the sticks into 4 equal parts. so each person got a total of $\frac{3}{4}$ of a stick. Or, using the parts to whole model, I could divide each stick into 4 parts and give each person 1 part of every stick, so they received $3(\frac{1}{4})$ sticks. By either method, each person gets their fair share—the amount each person gets with the first method is the same as with the second: 3 divided by 4 is the same as 3 times the result of dividing 1 by 4.



Gene says: In some elementary curricula only the parts to whole interpretation of fractions is introduced. This is unfortunate since in algebra the expression $\frac{x}{y}$ generally stands for x divided by y—it rarely, if ever, is taken to mean $\frac{1}{y}$ taken x times. Confusion often results since the teacher and students are operating with different interpretations of a fraction. I remember from my days of teaching Math for Elementary Teachers that students were nonplussed when I located $\frac{5}{2}$ on a number line by dividing the interval from 0 to 5 in half. To locate $\frac{5}{2}$ they would either mark off $\frac{1}{2}$ five times or first convert it to $2\frac{1}{2}$—the simple matter of dividing an interval in half was obfuscated by their parts-to-whole view of fractions and/or their reluctance to deal with an improper fraction. It was difficult for many of them to alter their views. Students would be served best if their early instruction introduced them to both the division and parts-to-whole model of fractions and all forms of fractions, proper and improper,were treated equitably.





END of SCENE 9: FRACTIONS

For comments and questions please email
Gene Maier at genem@mathlearningcenter.org

coming up next...

SCENE 10: EQUIVALENT FRACTIONS