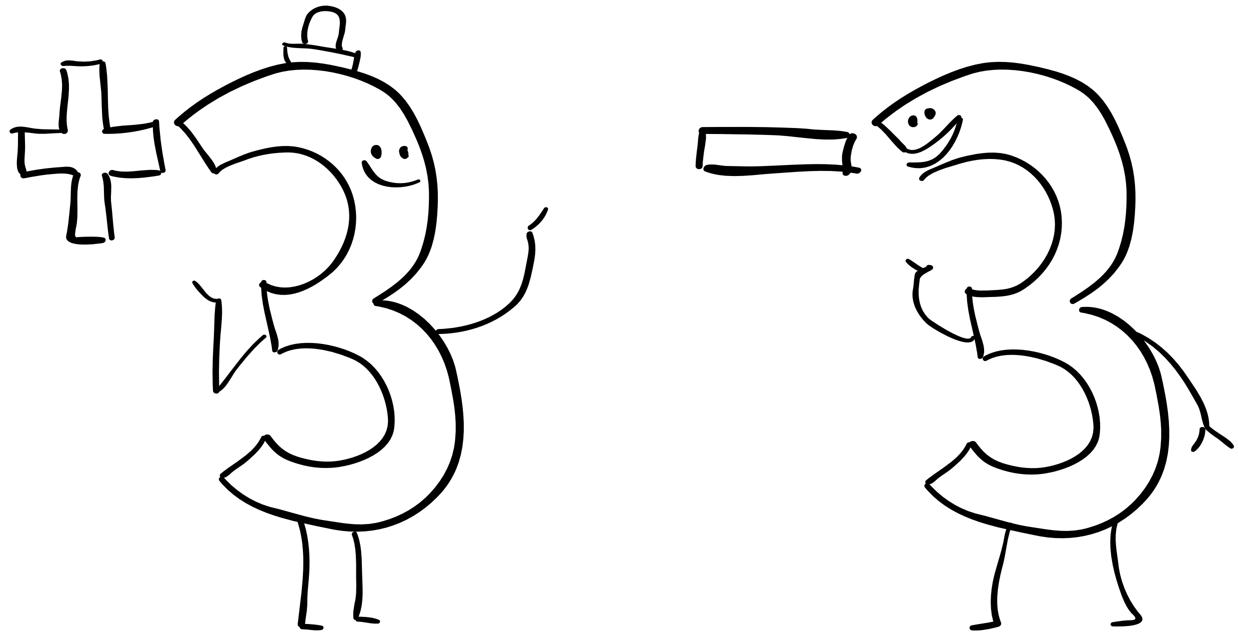
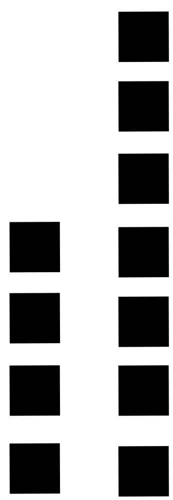


THE INTEGERS

The sixth scene in a series of articles
on elementary mathematics.

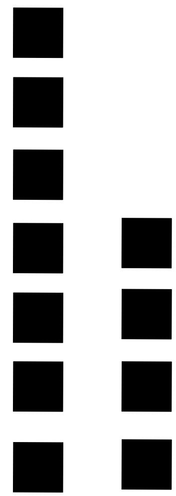
written by Eugene Maier
designed and illustrated by Tyson Smith

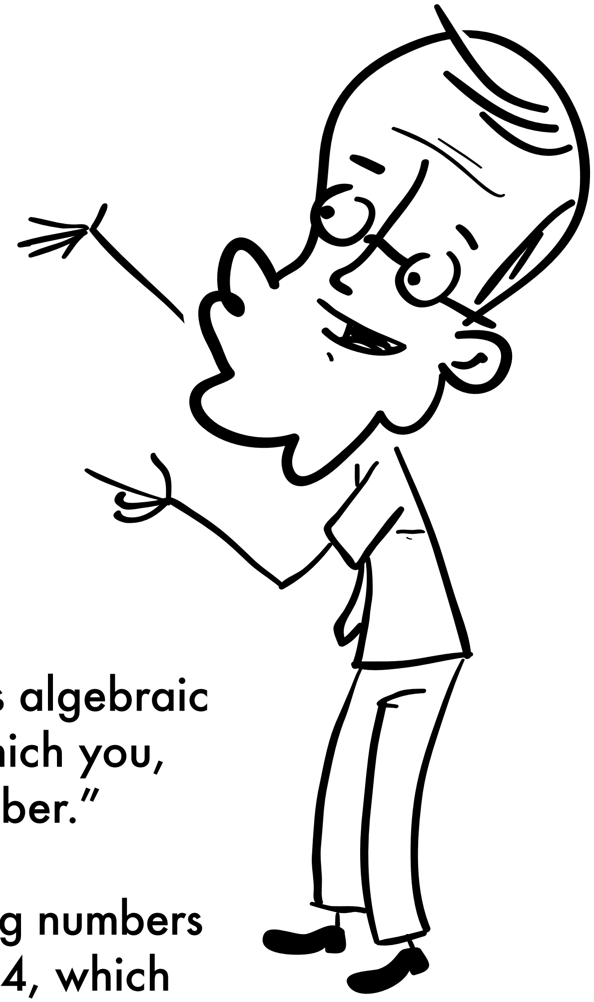
Consider the following two stacks of tile. If I asked you, “How many tile must be added to the first stack so it has the same number of tile as the second stack?” without much thought you would tell me, “Three.”



Thinking of these stacks of tile as representatives of counting numbers, the question might have been phrased, “What counting number must be added to 4 in order to get 7?” or, casting things in an algebraic mode, “For what counting number n is $4 + n = 7$?”

Now suppose we reverse the stacks and again ask the question "How many tile must be added to the first stack so it has the same number of tile as the second stack?" I don't know what your response might be—perhaps "What do you mean?" or "That doesn't make sense?" or "That's impossible."

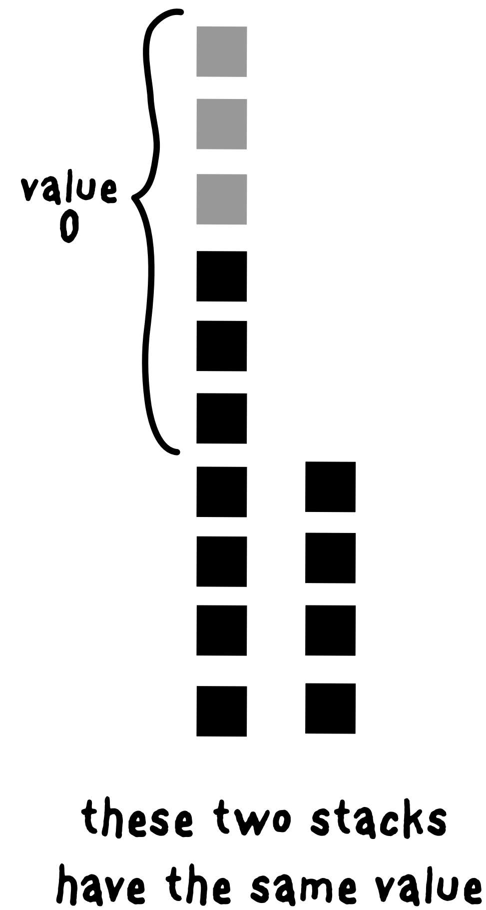




Thinking again of the stacks as representatives of counting numbers, the question becomes "What counting number must be added to 7 in order to get 4?" or, in its algebraic form, "For what counting number n is $7 + n = 4$?" to which you, quite properly, respond "There is no such counting number."

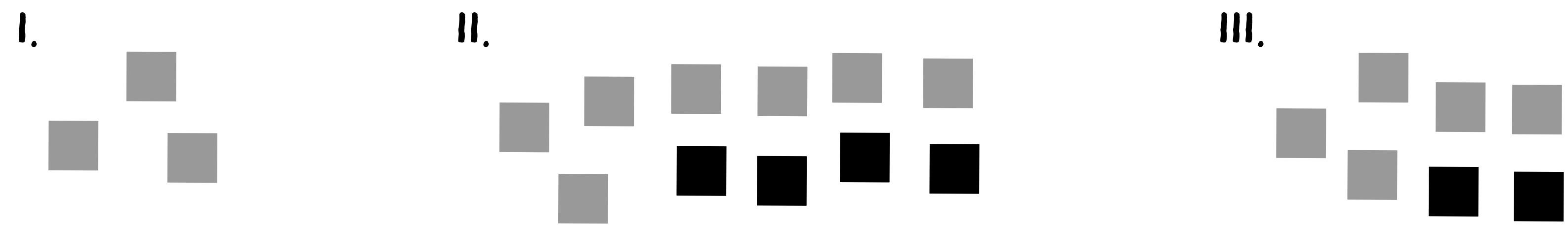
Thus, from a mathematical perspective, the set of counting numbers 1, 2, 3, ..., is lacking: There are equations, like $7 + n = 4$, which have no solution among the counting numbers. To remedy this situation, we extend our set of numbers to include solutions for these equations.

We will continue to use collections of tile to model numbers. However, we will switch from focussing on the number of tile in a collection to the value of the collection. Rather than asking what must be added to a stack of 7 tile in order to get a stack of 4 tile, we ask “What must be added to a stack of tile whose value is 7 black tile in order to get a stack whose value is 4 black tile?” To be able to answer this, and similar questions, we introduce red tile, with the understanding that a red tile cancels the value of a black tile. Thus adding 3 red tile to 7 black tile cancels the value of 3 of these black tile, leaving a collection of tile whose value is 4 black tile. Thus, the answer to our above question is 3 red tile must be added to the stack of 7 black tile to get a stack where the value is 4 black tile.



There are several things to note about collections containing both black and red tile:

Many different collections of tile have the same value. For example, the following collections all have value 3 red. Notice that adding the same number of black and red tile to a collection, or removing the same number of black and red tile from a collection, does not change its value.

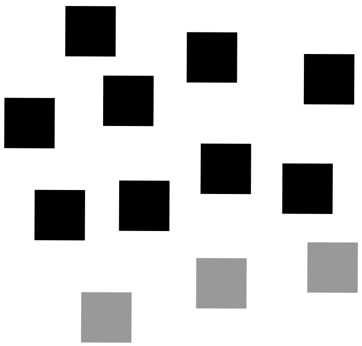


all these collections have value 3 red

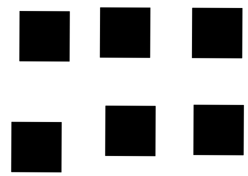
A collection which contains the same number of black and red tile has value 0.

all these collections have value 0

If a collection whose value is not 0 contains both black and red tile, pairs of black and red tile can be removed from the collection without changing its value until only one color of tile remains. This will be the smallest collection having that particular value and is called the ***minimal*** collection for that value. In general, the number of tile in a collection will differ numerically from its value. However, for minimal collections, the number of tile and the value are numerically equal.



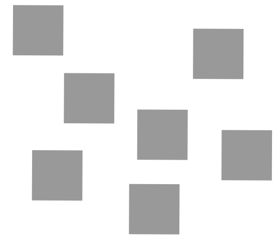
This collection is not minimal. It contains 12 tile. It's value is 6 black.



This collection is minimal. It contains 6 tile. It's value is 6 black.

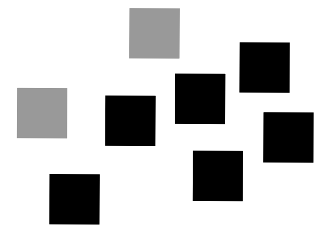
Gene says: Coins provide another example where the number of objects in a collection and its value differ: a collection of 3 dimes, 2 nickels and 4 pennies contains 9 coins but has a value of 44 cents. Also, different collections of coins can have the same value: a collection of 1 quarter and 19 pennies and a collection of 1 dime, 5 nickels and 9 pennies are other collections whose value is 44 cents.

The value of collections are denoted by ***signed*** counting numbers. A black value is denoted by a counting number prefixed by a plus sign in superscript position, e.g. the value "3 black" is denoted "$^{+}3$", read "plus three"; a red value is denoted by a counting number prefixed by a minus sign in superscript position, e. g. the value "3 red" is denoted "$^{-}3$", read "minus three." Also, a black value is said to be ***positive*** and a red value is said to be ***negative***. The set of all values, positive, negative and zero is called the ***integers***.



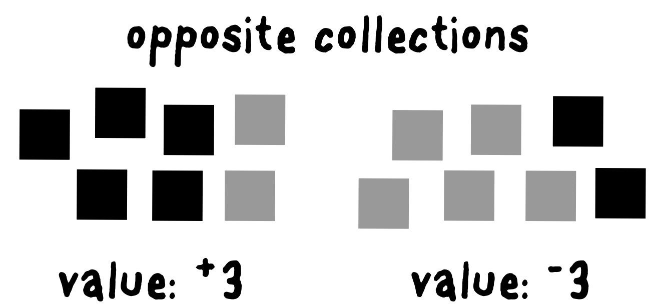
a minimal collection with a value of $^{-}7$

a non-minimal collection with a value of $^{+}4$



FYI: The use of black to represent positive and red to represent negative stems from the accounting practice that used black ink to indicate a gain and red ink to indicate a loss. This is counter to the practice of indicating a positive pole of a battery by red and a negative pole by black.

If the pieces in a collection of tile are turned over, the result is the ***opposite*** collection. The values of opposite collections are said to be opposites of one another. Thus $^{+}3$ and $^{-}3$ are opposites of one another, that is the opposite of $^{+}3$ is $^{-}3$ and the opposite of $^{-}3$ is $^{+}3$. For the time being we will use the letter o and parentheses to indicate the opposite of an integer, e.g. the notation $o(^{+}3)$ stands for “the opposite of $^{+}3$. Thus: $o(^{+}3) = {}^{-}3$ and $o(^{-}3) = {}^{+}3$. Note that the opposite of the opposite of a collection is the collection itself—turn over a collection twice and its back to its original state—so, for example, $o(o(^{+}3)) = {}^{+}3$. Also note that the opposite of a collection whose value is 0 also has value 0, that is, $o(0) = 0$.



Gene says: Later we will adopt the standard practice of using a minus sign to indicate the opposite of a number, e.g., rather than writing $o(^{+}3)$ to indicate the opposite of $^{+}3$ we will write $-(^{+}3)$. Also later we will write the signs of signed numbers in standard, rather than superscripted, position, e.g. we will write -4 instead of $^{-}4$. For the time being we will continue to use the ‘o’ notation and superscripted signs to help distinguish between the various meanings attached to + and – signs. In normal usage, the plus sign is used in two distinct ways: as a symbol for addition and also as the sign of a positive signed number. The minus sign is used in three distinct ways: as a symbol for subtraction, as the sign of a negative signed number, and to indicate the opposite of a number.

END of SCENE 6: THE INTEGERS

For comments and questions please email
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coming up next...

SCENE 7: ADDING & SUBTRACTING INTEGERS