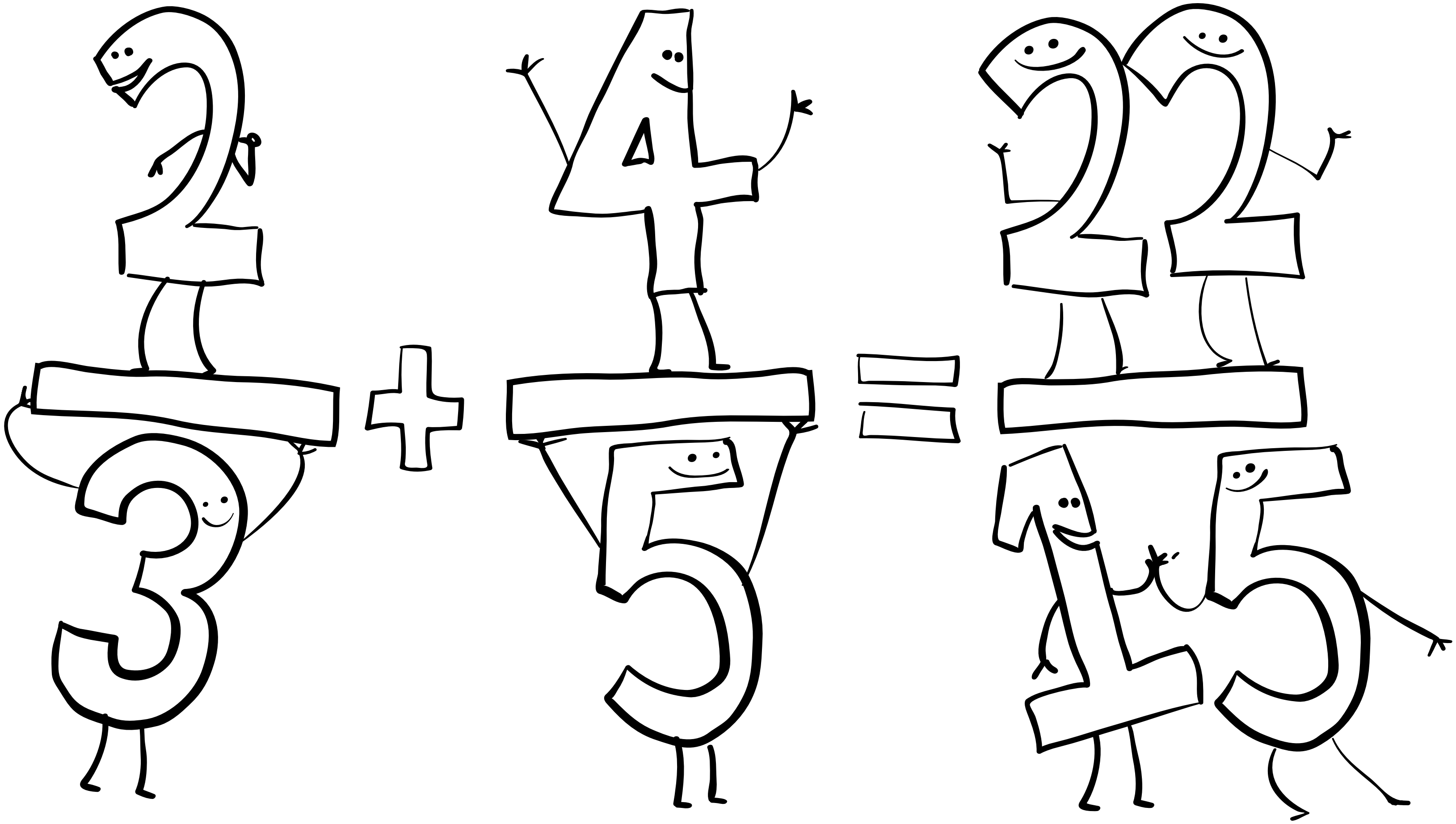
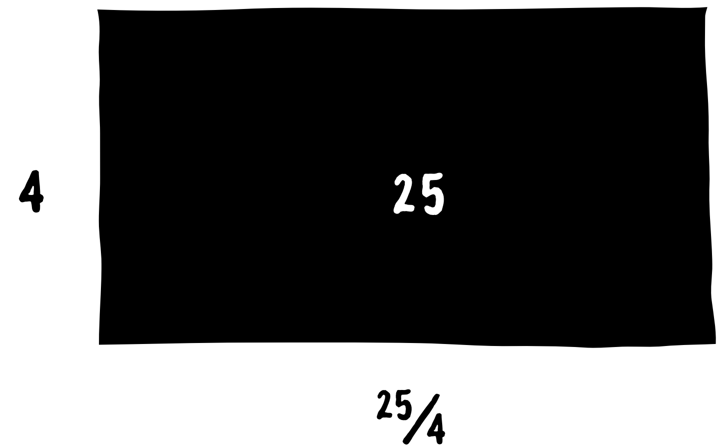


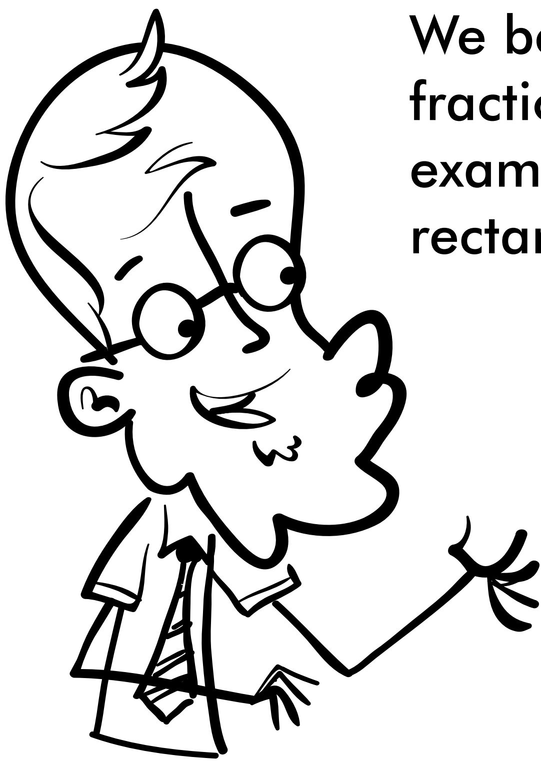
FRACTION ADDITION AND SUBTRACTION

The eleventh scene in a series of articles
on elementary mathematics.

written by Eugene Maier
designed and illustrated by Tyson Smith

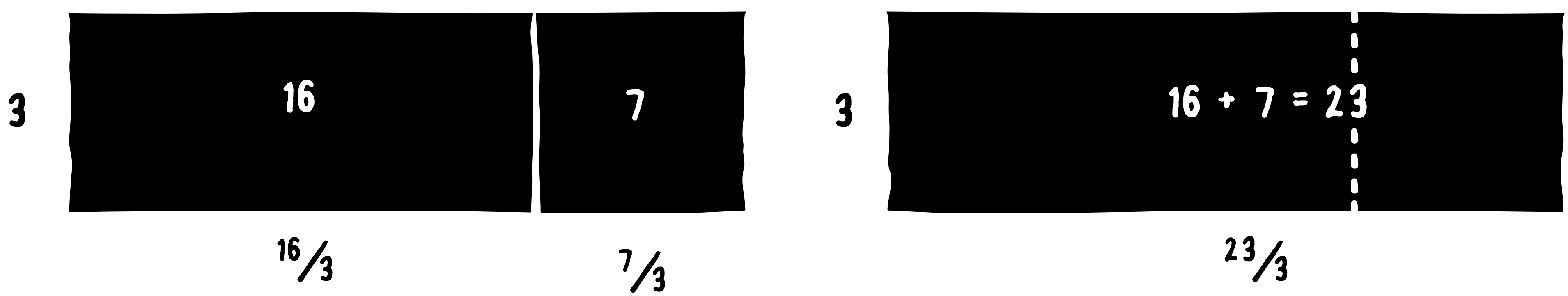
The area model is useful in describing fraction operations. In this model, described in Scene 9, a fraction is represented as the base of a rectangle whose area is the numerator of the fraction and whose height is the denominator of the fraction.





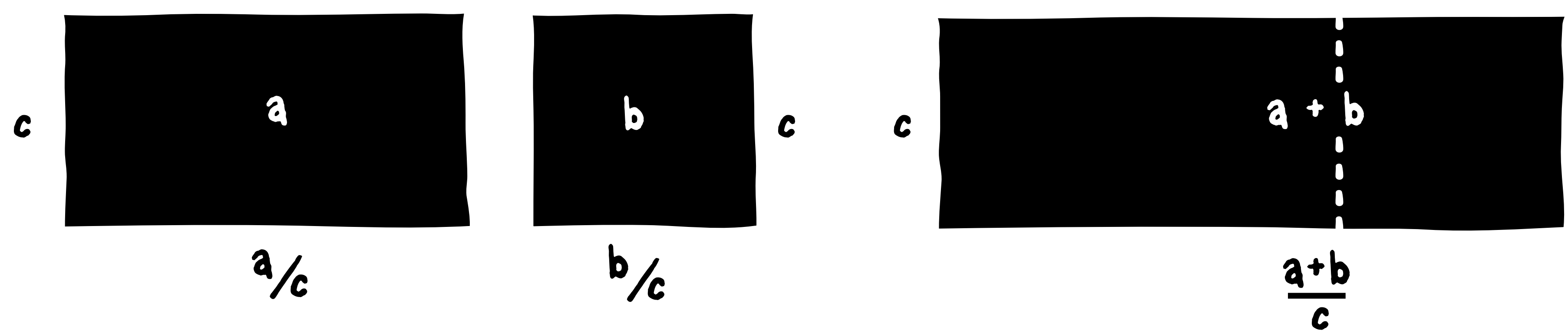
We begin our exploration of addition by considering fractions which have the same denominator. Consider, for example, $\frac{16}{3}$ and $\frac{7}{3}$, represented here as the bases of rectangles:

If I place these two rectangles side by side, as shown, I can form a third rectangle by removing the common side. The area of this new rectangle is 23 and its height is 3; so its base is $\frac{23}{3}$. But it's base is also the sum of the bases of the two original rectangles. Hence, $\frac{7}{3} + \frac{16}{3} = \frac{23}{3}$.



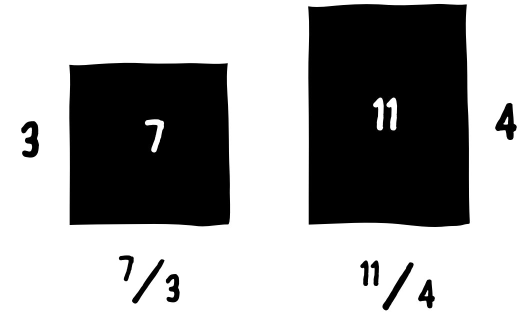
Note that (1) the numerator, 23, of the sum is the sum of the numerators of the two addends and (2) the denominator, 3, of the sum is the common denominator of the addends. This is portrayed by the rectangles: the area of the final rectangle is the sum of the areas of the two original rectangles and its height is the common height of the two originals.

The procedure described above works for any two fractions which have the same denominator. Suppose we represent two generic fractions $\frac{a}{c}$ and $\frac{b}{c}$ as bases of rectangles. Adjoining these rectangles and removing the common side results in a rectangle whose area is $\boldsymbol{a + b}$ and height is $\boldsymbol{c}$, so its base is $\frac{a+b}{c}$. But its base is also the sum of the bases of the original rectangles. Hence, $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$.



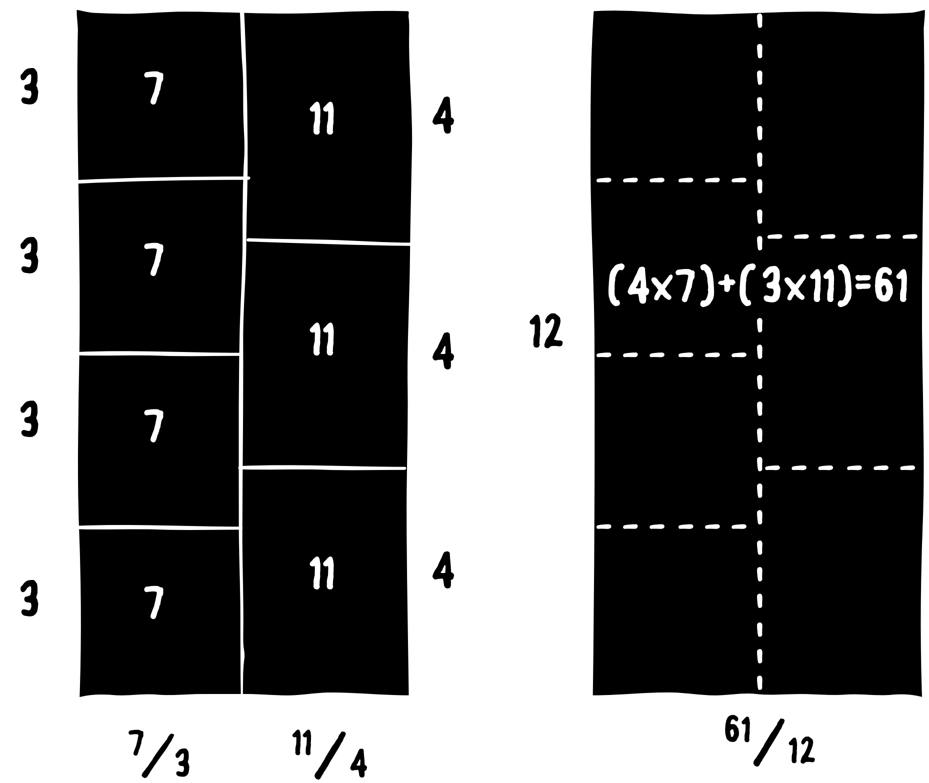
We have arrived at a general rule: To add fractions with like denominators, add the numerators and place over the common denominator.

We can adapt the above procedure to add fractions which have unlike denominators. Suppose, for example, we want to represent $\frac{7}{3} + \frac{11}{4}$ as a single fraction. We begin by representing $\frac{7}{3}$ and $\frac{11}{4}$ as the bases of rectangles. But now the rectangles have different heights, so when adjoined they do not form a rectangle.

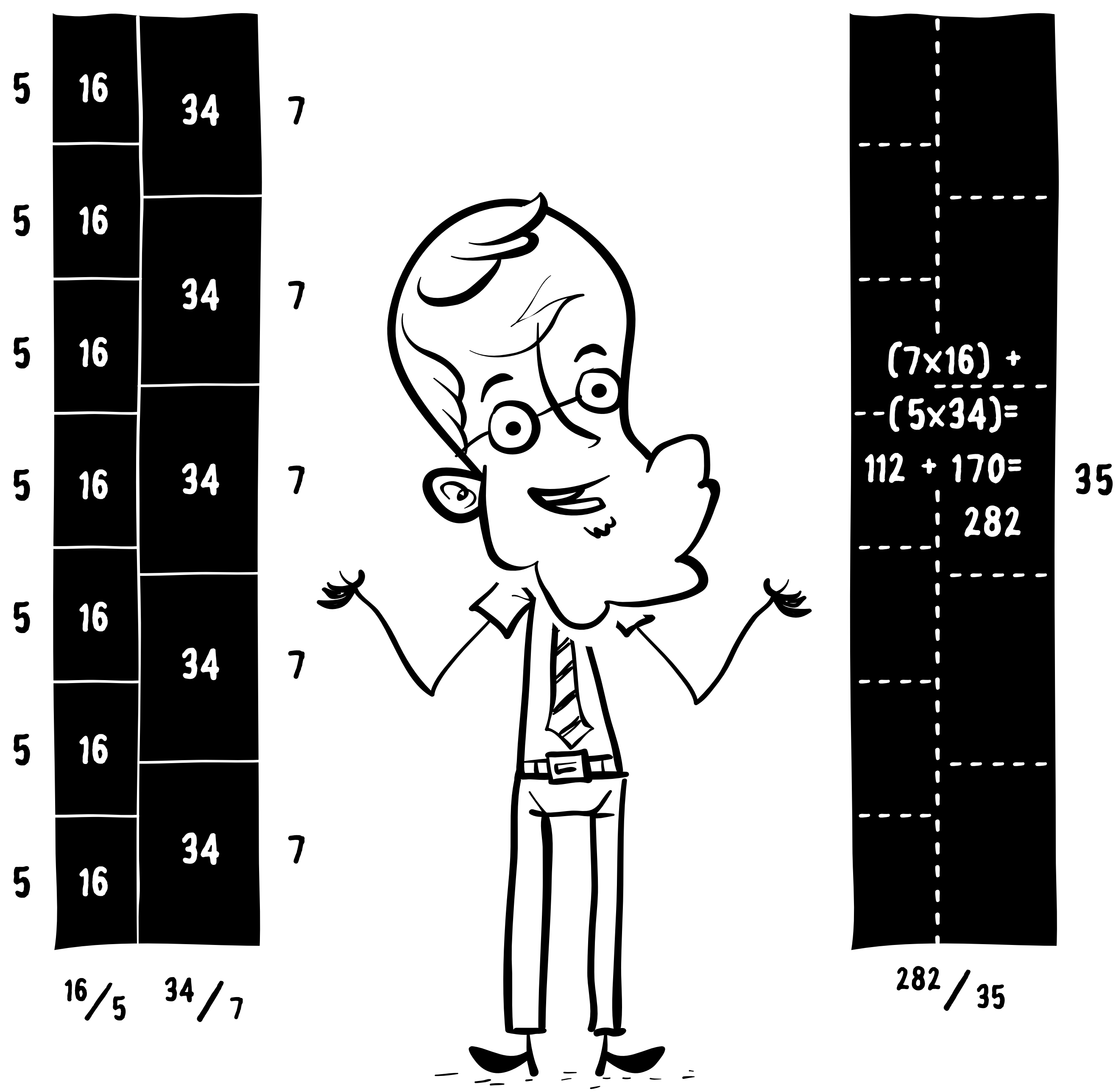


However, we can stack multiple copies of these rectangles atop one another until we obtain two stacks of rectangles that have a common height. In this case, stacking 4 copies of the first rectangle and 3 copies of the second rectangles accomplishes our purposes.

Removing all but the outer edges of this configuration of rectangles yields a rectangle whose height is 12 and area is (4 x 7) + (3 x 11), or 61. Hence, its base is $\frac{61}{12}$. But its base is also the sum $\frac{7}{3} + \frac{11}{4}$ of the two original rectangles. Thus, $\frac{7}{3} + \frac{11}{4} = \frac{61}{12}$.

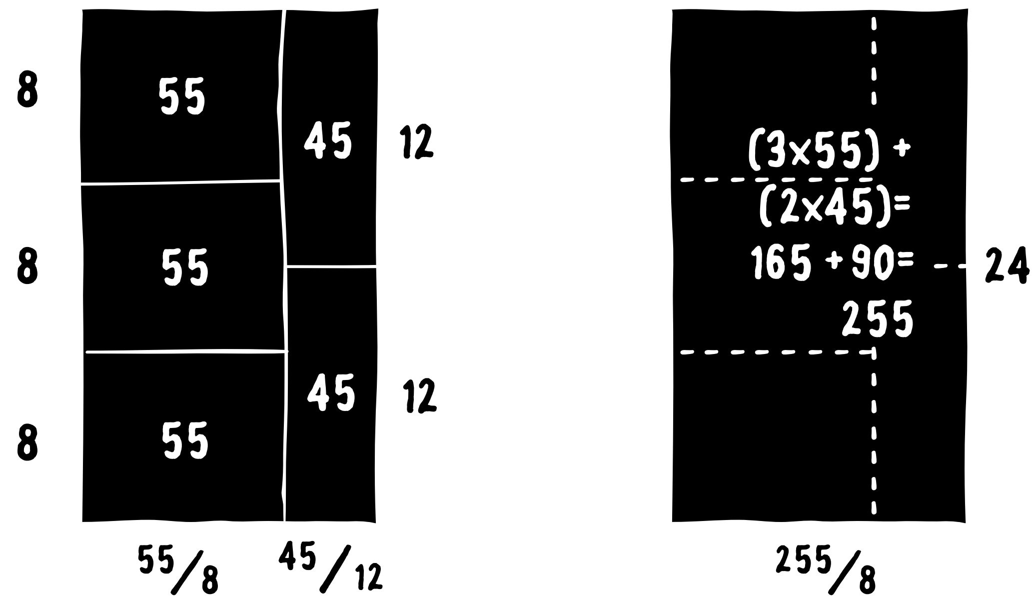


Given two rectangles with integral heights, one can always stack copies of these rectangles so that the two stacks have a common height. For example, if the height of one rectangle is 5 and the height of a second is 7, stacking 7 copies of the first and 5 copies of the second, will create stacks that have a common height of 35, as illustrated in the following sketches showing that $\frac{16}{5}+\frac{34}{7}=\frac{282}{35}$.



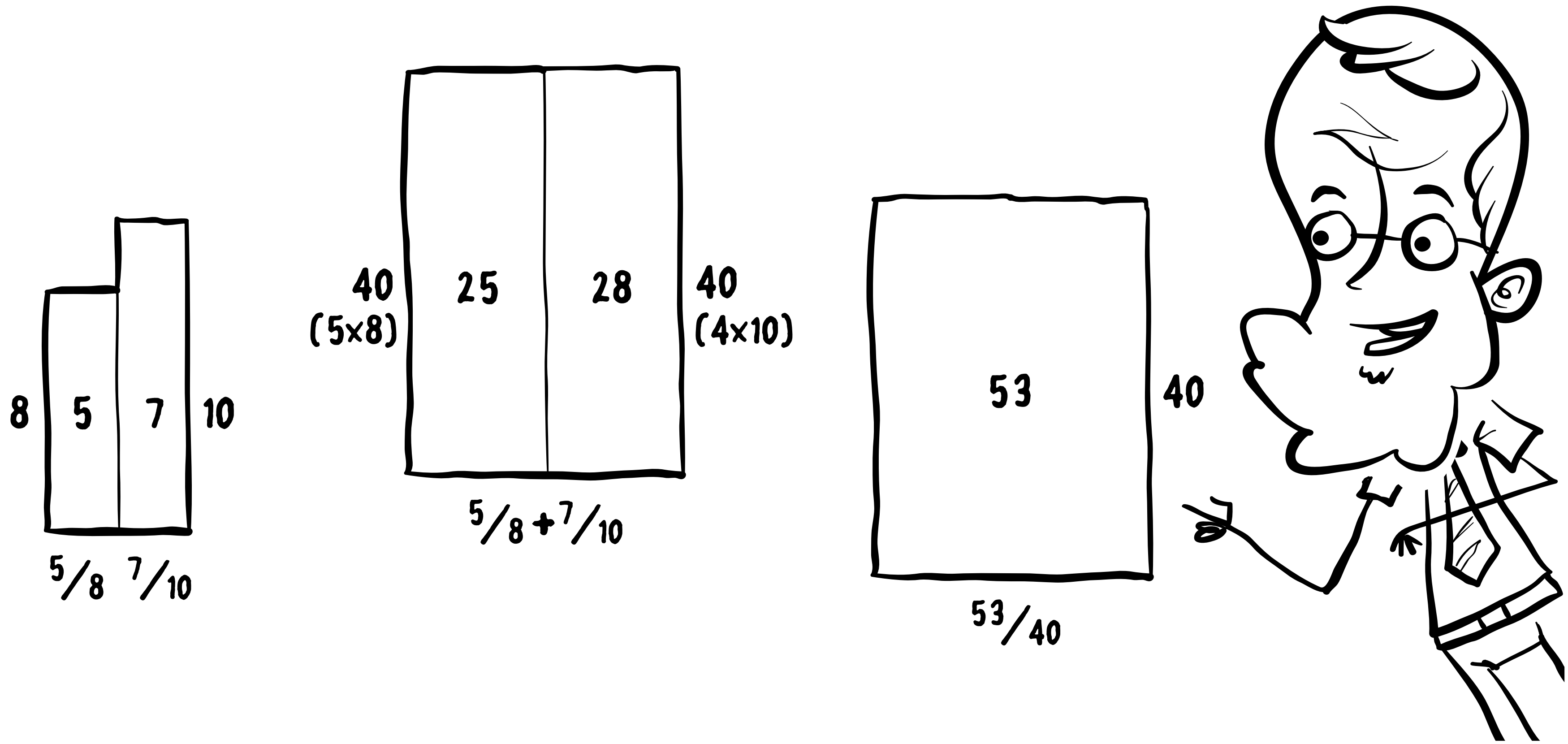
As in the previous example, one is always able to stack rectangles associated with two fractions to a common height which is the product of the denominators of the fractions. However, sometimes a smaller common height is possible as is the case with the rectangles associated with $\frac{55}{8}$ and $\frac{45}{12}$.

Whereas the product of these denominators is 96, one is able to stack 3 copies of the rectangle associated with $\frac{55}{8}$ and 2 copies of the rectangle associated with $\frac{45}{12}$ to reach a common height of 24. Doing this provides a rectangle of area 255 and height 24. Hence, $\frac{55}{8}+\frac{45}{12}=\frac{255}{24}$.



In general, if a larger rectangle is to be constructed from two smaller rectangles in the above manner, the height of the larger rectangle must be a common multiple of the two original rectangles. Thus, the height of the smallest rectangle that can be constructed will be the least common multiple of the heights of the two original rectangles.

Gene says: In practice it is not necessary to draw sketches to scale. Rough sketches that capture the essence of the process will do, as shown in the following computation showing that $\frac{5}{8}+\frac{7}{10}=\frac{53}{40}$.

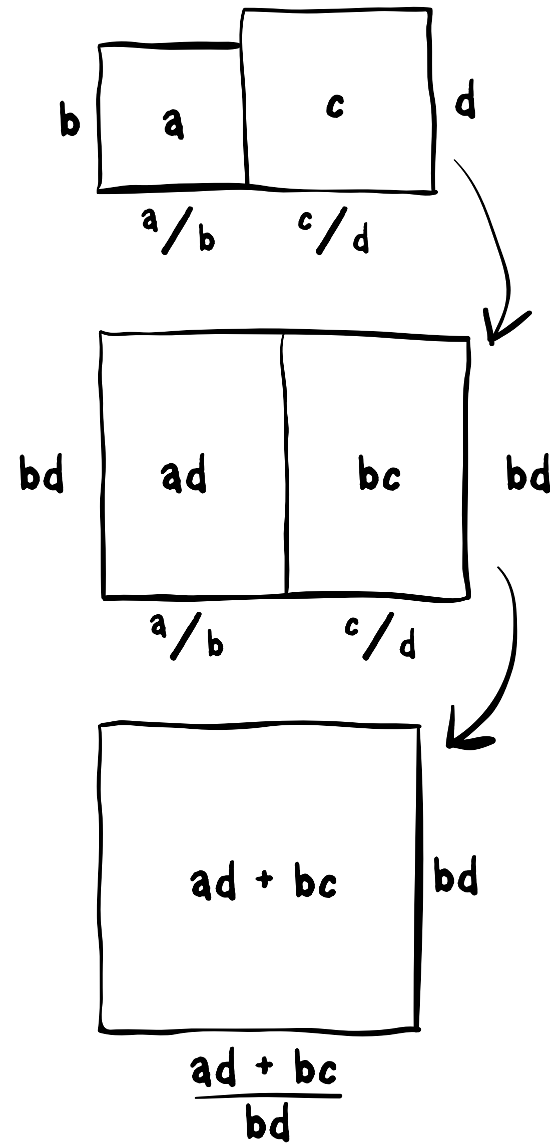


FYI: The above procedure can be used to find a general formula for the sum of two fractions $\frac{a}{b}$ and $\frac{c}{d}$.

The first of these fractions can be represented as the base of a rectangle of area ***a*** and height ***b***, the second as the base of a rectangle of area ***c*** and height ***d***.

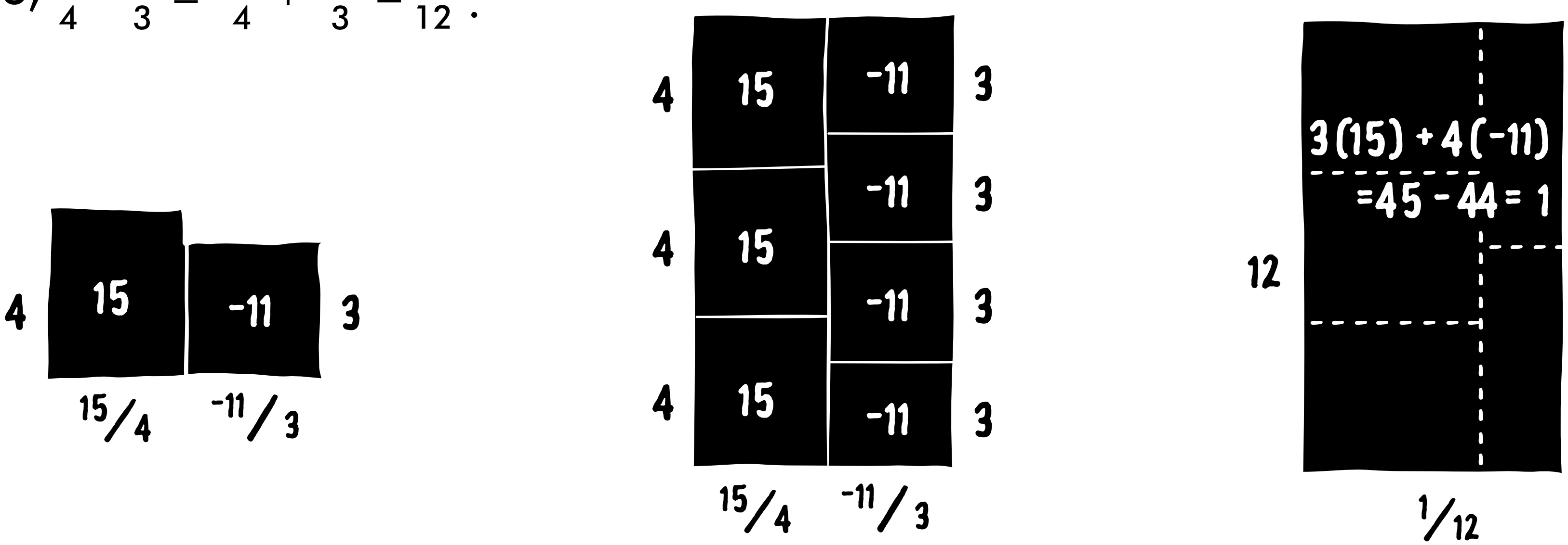
Stacking ***d*** copies of the first rectangle produces a rectangle of area ***ad*** and height ***bd***; stacking ***b*** copies of the second rectangle produces a rectangle of area ***bc*** and height ***bd***.

These two rectangles combine to form a rectangle of area ***ad*** + ***bc*** and height ***bd***. The base of this rectangle is $\frac{ad + bc}{bd}$. But its base is also the sum of the bases of the two original rectangles.
Hence, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

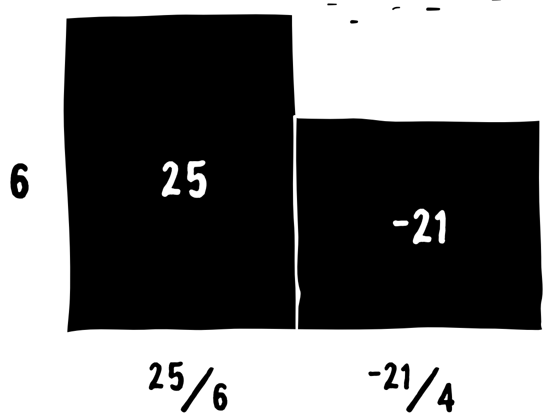


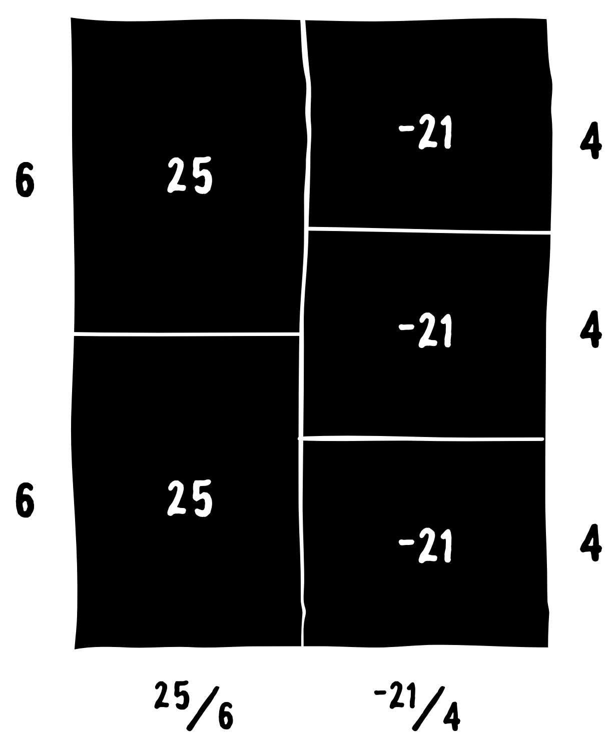
The procedure developed for the addition of fractions can be adapted to subtraction, since subtracting a second number from a first number is equivalent to adding the opposite of the second number to the first (Confer the concluding paragraph of ***Scene 7, Adding and Subtracting Integers***). Thus, for example, $\frac{15}{4}-\frac{11}{3}=\frac{15}{4}+\left(-\frac{11}{3}\right)$. As pointed out in ***Scene 9, Fractions***, $-\frac{11}{3}=\frac{-11}{4}$, so $\frac{15}{4}-\frac{11}{3}=\frac{15}{4}+\frac{-11}{3}$. Thinking in terms of value rather than area, the first fraction in this sum can be represented as the base of a rectangle whose value is 15 and height is 4 while the second fraction can be represented as the base of a rectangle whose value is -10 and height is 3. Now 3 copies of the first rectangle and 4 of the second can be combined to form a rectangle whose value is 1 and height is 12. Thus the value of its base is $\frac{1}{12}$. But this base is also the sum of the bases of the two original rectangles.

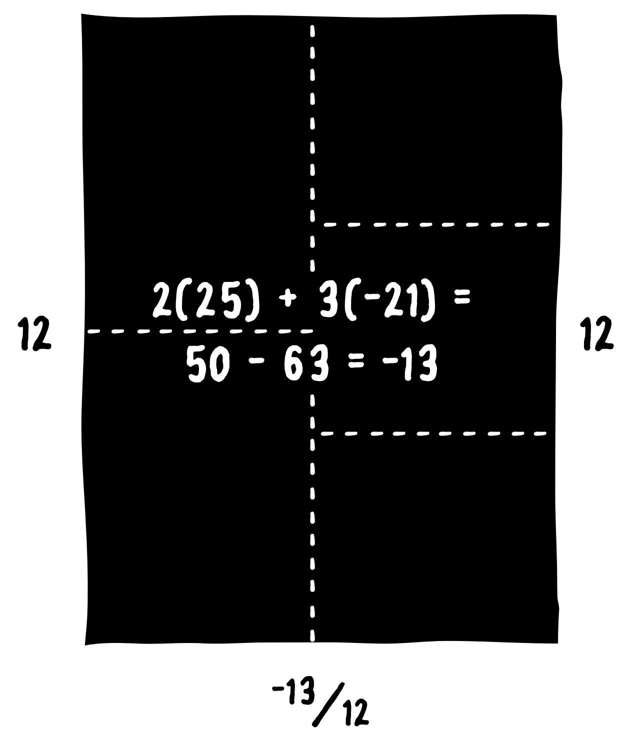
Hence, $\frac{15}{4}-\frac{11}{3}=\frac{15}{4}+\frac{-11}{3}=\frac{1}{12}$.

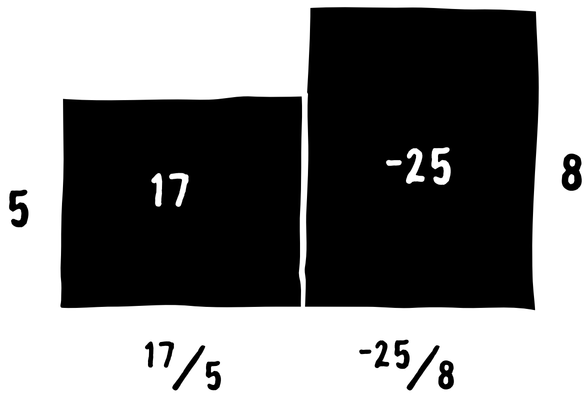


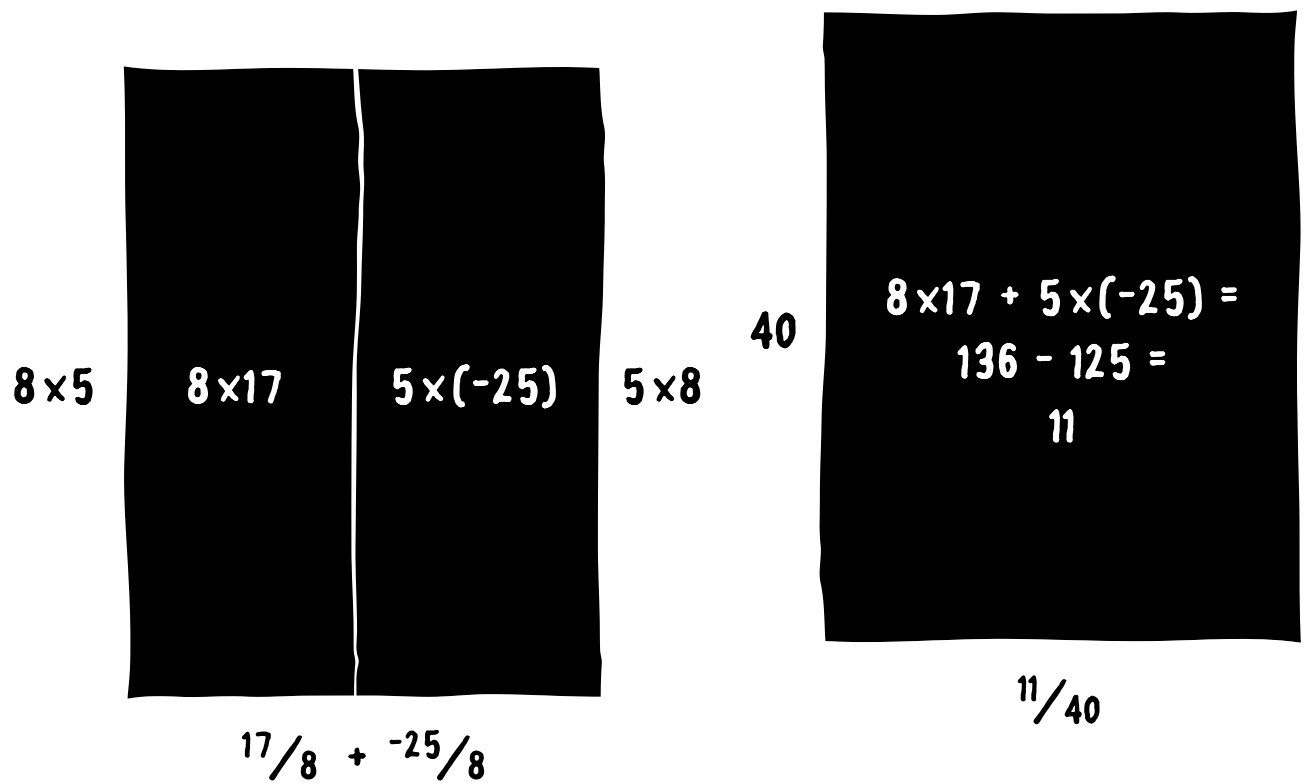
As another example of subtraction, the following sketches show that $\frac{25}{6} - \frac{21}{4} = -\frac{13}{12}$.











As a final example, the above rough sketches, neither drawn to scale nor color-coded, show that $\frac{17}{5} - \frac{25}{8} = \frac{11}{40}$.

END of SCENE 11:
FRACTION ADDITION AND SUBTRACTION

For comments and questions
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