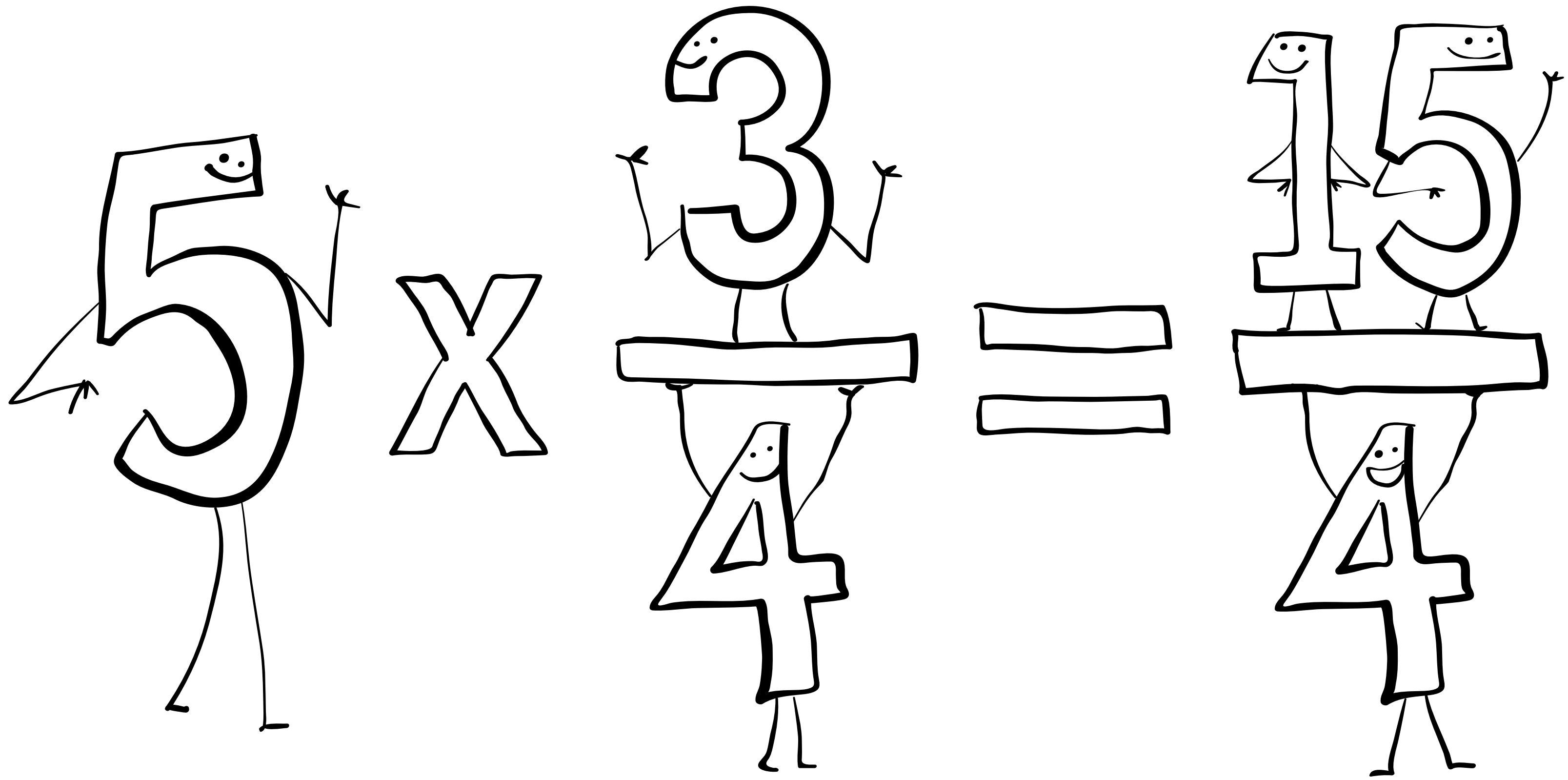
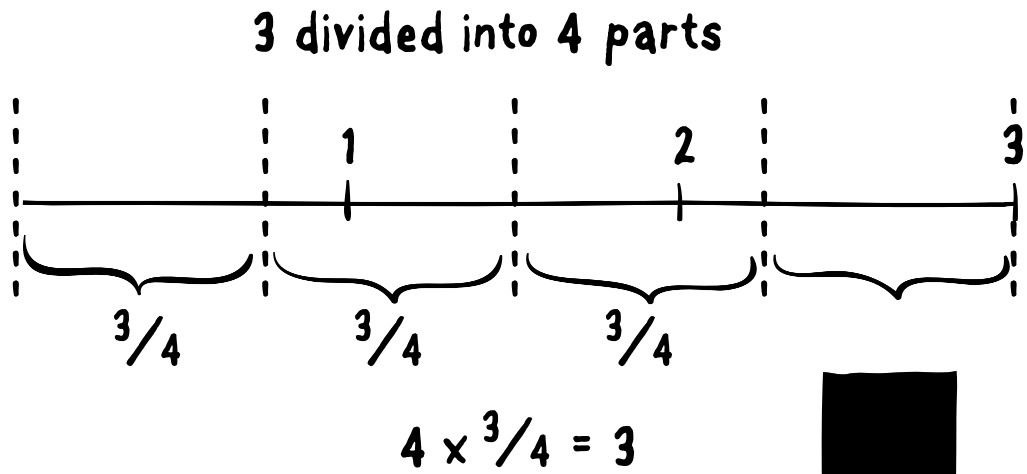


FRACTION MULTIPLICATION AND DIVISION

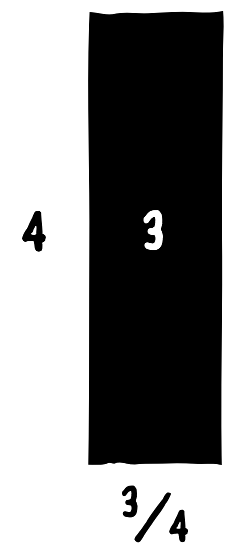
The twelth scene in a series of articles
on elementary mathematics.

written by Eugene Maier
designed and illustrated by Tyson Smith

We first observe the following: If a fraction is multiplied by its denominator, the result is its numerator. For example if $\frac{3}{4}$ is multiplied by 4, the result is 3. Recalling that $\frac{3}{4}$ is the quotient when 3 is divided by 4, all the previous sentence says is that if one divides 3 by 4 and then multiplies that result by 4, one is back to 3. (See the illustration.)

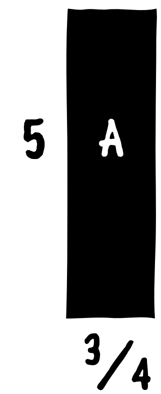


One can also deduce this statement from the area model. One can view $\frac{3}{4}$ as the base of a rectangle whose area is 3 and height is 4. But the area of a rectangle is the product of its dimensions, so $4 \times \frac{3}{4} = 3$.

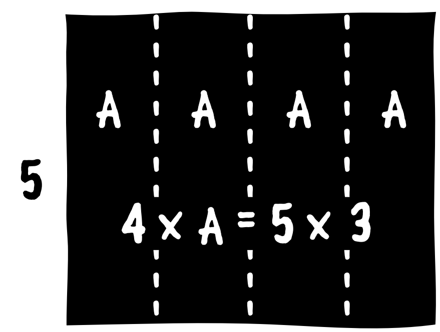


In general, if the fraction $\frac{a}{b}$ is multiplied by b, the result is a, that is $b \times \frac{a}{b} = \frac{a}{b} \times b = a$; which simply says that if a is divided by b and the result is then multiplied by b, one is back to the amount one started with, namely a.

Now suppose one wants to multiply a fraction by an integer other than it's denominator, say one wants to find the product $5 \times \frac{3}{4}$. One way is to use the area model. In this model $5 \times \frac{3}{4}$ is the area $\boldsymbol{A}$ of a rectangle whose dimensions are 5 and $\frac{3}{4}$. (The figure is not drawn to scale.)



To find the value of $\boldsymbol{A}$ we place 4 of these rectangles side-by-side and combine them into a single rectangle whose area is $4 \times \boldsymbol{A}$ and dimensions are 5 and $4 \times \frac{3}{4}$, which equals 3 (see the previous page). Since the area of a rectangle is the product of its dimensions, $4 \times \boldsymbol{A} = 5 \times 3 = 15$. Thus $\boldsymbol{A}$ is 15 divided by 4 or $\frac{15}{4}$. Hence $5 \times \frac{3}{4} = \frac{15}{4}$.

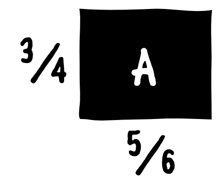


$4 \times 3/4 = 3$

Gene Says: Alternately, one can think of $\frac{3}{4}$ as 3 one-fourths, and thus 5 times this amount is 15 one-fourths which is the same as $\frac{15}{4}$.
In symbols: $5 \times \frac{3}{4} = 5 \times (3 \times \frac{1}{4}) = (5 \times 3) \times \frac{1}{4} = 15 \times \frac{1}{4} = \frac{15}{4}$

This way of thinking about the product of an integer and a fraction doesn't readily extend to determining the product of two fractions. As we shall see on the next page, the area model does.

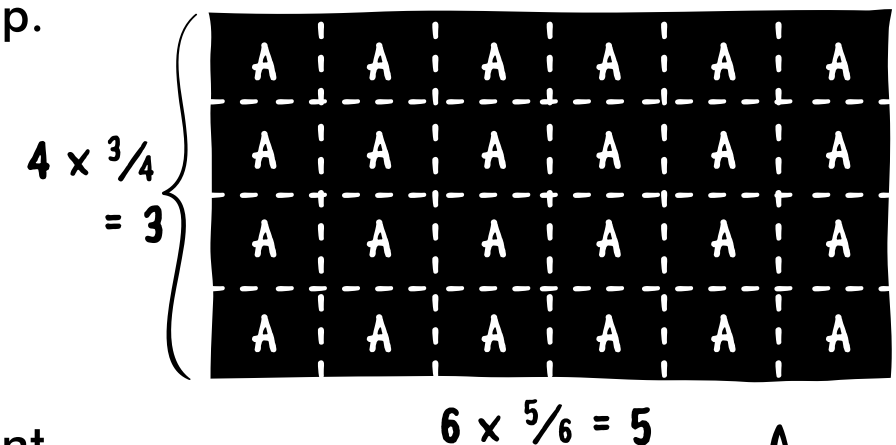
To illustrate the use of the area model to find the product of two fractions, we compute $\frac{3}{4} \times \frac{5}{6}$. To do this, we want to find the area ***A*** of a rectangle ***R*** whose dimensions are $\frac{3}{4}$ and $\frac{5}{6}$.



RECTANGLE R

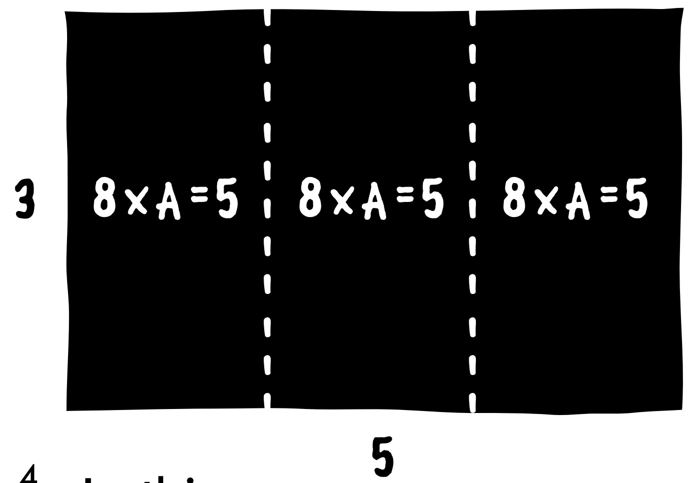
Starting with ***R***, we construct a new rectangle ***S*** that consists of 24 copies of ***R***, 6 across and 4 up. This new rectangle has a height which is $4 \times \frac{3}{4}$, or 3, and a base which is $6 \times \frac{5}{6}$, or 5. It's area is 24 times that of ***R***, or 24***A***. On the other hand, it's area is its width times its base, which is 3 x 5 or 15. Thus ***A*** is the amount obtained when 15 is divided into 24 parts, that is $A = \frac{15}{24}$.

RECTANGLE S

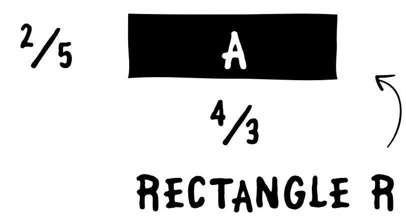


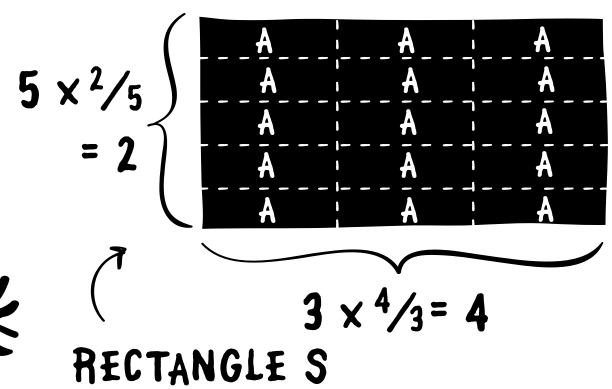
(We chose to make 24 copies of ***R***, 6 across and 4 up, because that produced a rectangle whose dimensions were integers and, thus, we could use our knowledge about multiplying integers to find the area of ***S***.)

We could divide the rectangle ***S***, shown on the previous page, into 3 rectangles. Each of these rectangles has area 5 and consists of 8 copies of ***R***. Hence 8 x ***A*** = 5 and, thus, ***A*** = $\frac{5}{8}$, which is a value of ***A*** that is in lowest terms.

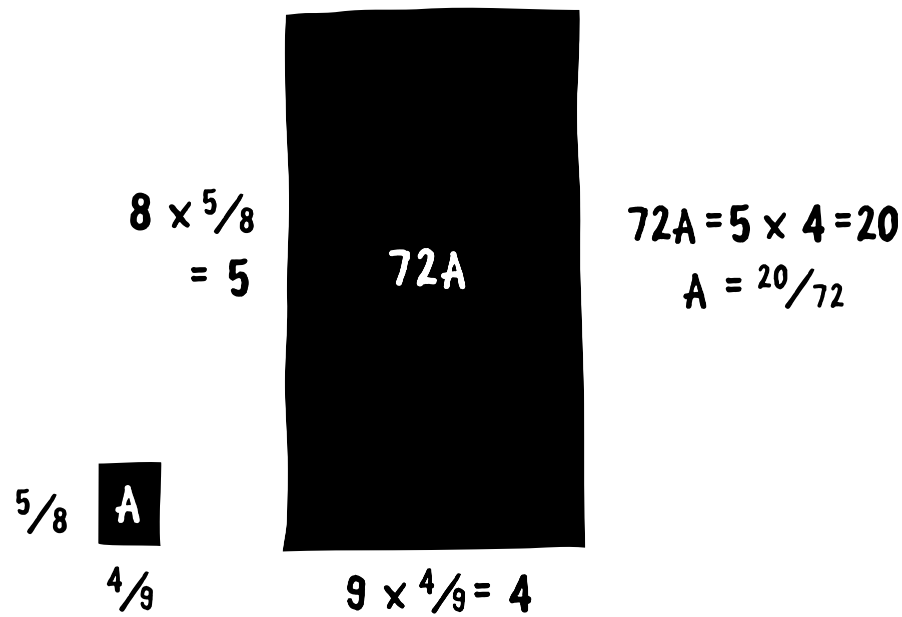


As another example, we compute $\frac{2}{5} \times \frac{4}{3}$, In this case, we want to find the area ***A*** of a rectangle ***R*** whose dimensions are $\frac{2}{5}$ and $\frac{4}{3}$. We create a new rectangle ***S*** composed of 15 copies of ***R***, 5 up and 3 across. The dimensions of ***S*** are 2 and 4, so its area is 8. But it's area is also 15 x ***A***. So ***A*** is obtained by dividing 8 into 15 parts, that is, ***A*** = $\frac{8}{15}$.

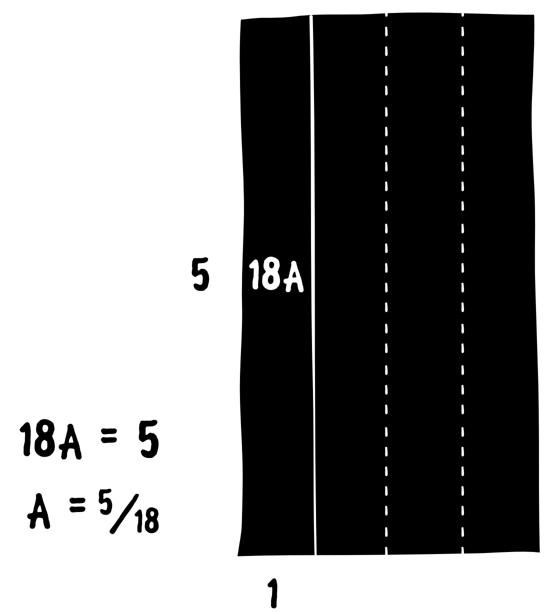




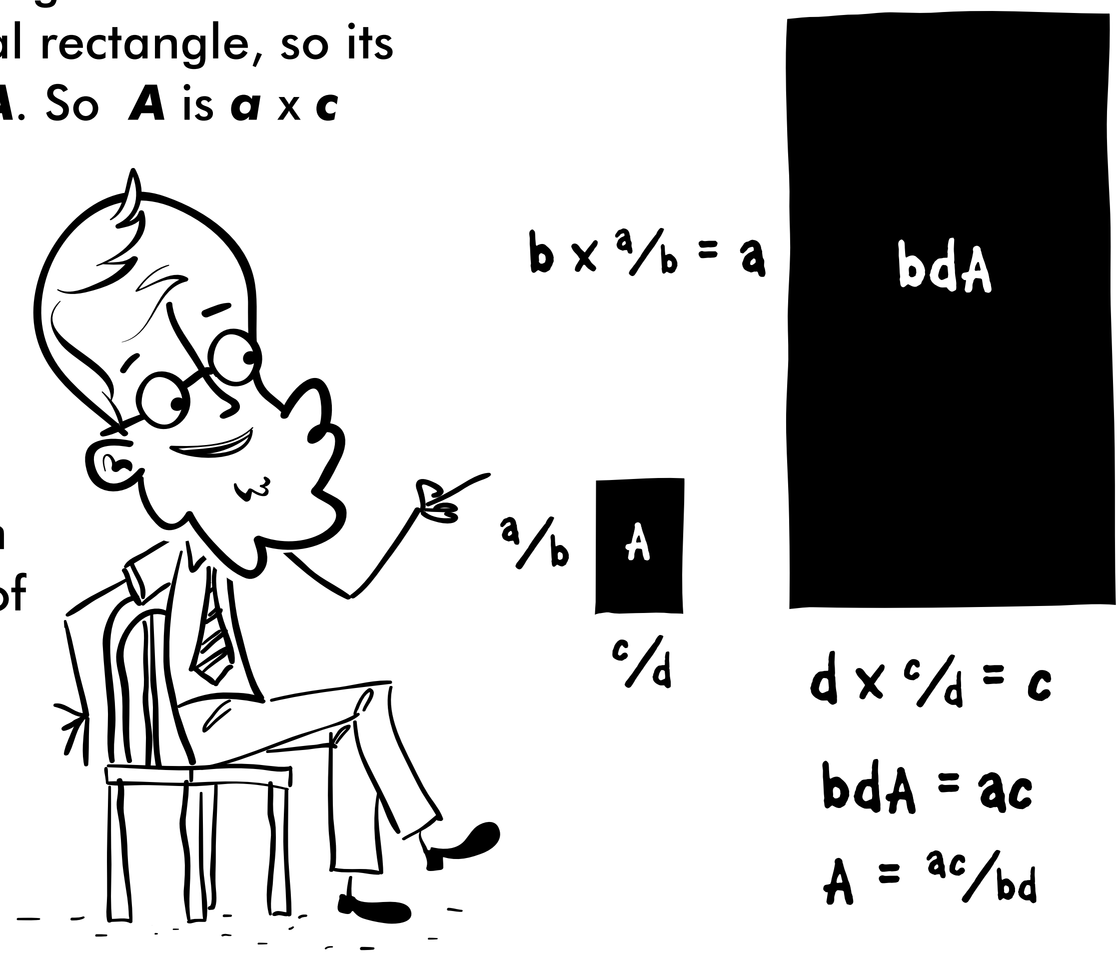
Sketches need not be drawn to scale. Shown are rough sketches that aid our thinking in computing $\frac{5}{8} \times \frac{4}{9}$. We start with a $\frac{5}{8}$ by $\frac{4}{9}$ rectangle of area $\boldsymbol{A}$. We increase its height by a factor of 8 and its width by a factor of 9 to obtain a 5 by 4 rectangle containing 72 copies of our original rectangle. Hence $72 \times \boldsymbol{A} = 20$, and $\boldsymbol{A} = \frac{20}{72}$.



We could divide our large rectangle into fourths to obtain a rectangle of area 5 containing 18 copies of the original rectangle. Hence $18 \times \boldsymbol{A} = 5$ and $\boldsymbol{A} = \frac{5}{18}$, which is a value of $\boldsymbol{A}$ in lowest terms.

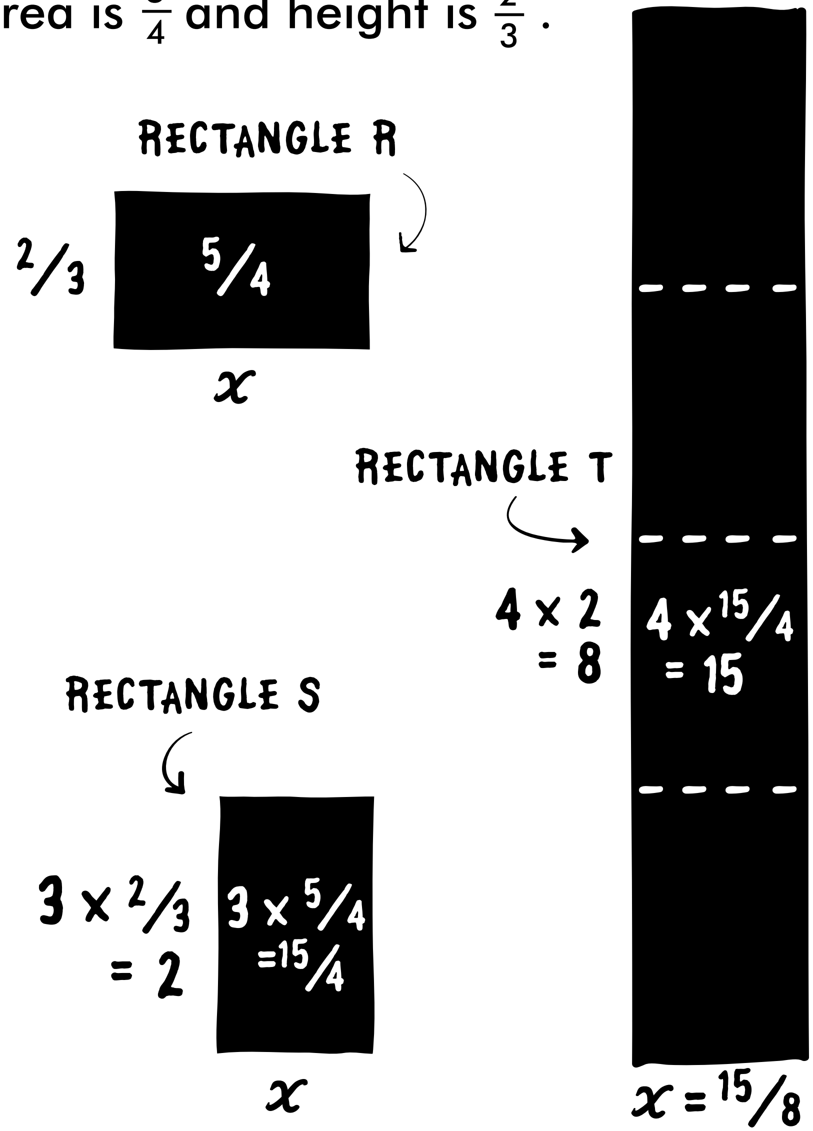


FYI. We can use the area method to obtain a formula for multiplying fractions. Suppose we wish to compute $\frac{a}{b} \times \frac{c}{d}$. This product is the area A of a rectangle whose dimensions are $\frac{a}{b}$ and $\frac{c}{d}$. If we make b copies up and d copies across of this rectangle, we obtain a new rectangle whose dimensions are a and c, and hence has area $a \times c$. On the other hand, this new rectangle contains $b \times d$ copies of our original rectangle, so its area is $b \times d$ times A. So A is $a \times c$ divided into $b \times d$ parts, that is, $A = \frac{a \times c}{b \times d}$. But, A is also the product of the dimensions of the original rectangle, hence $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$. In words: the product of two fractions is the product of their numerators divided by the product of their denominators.

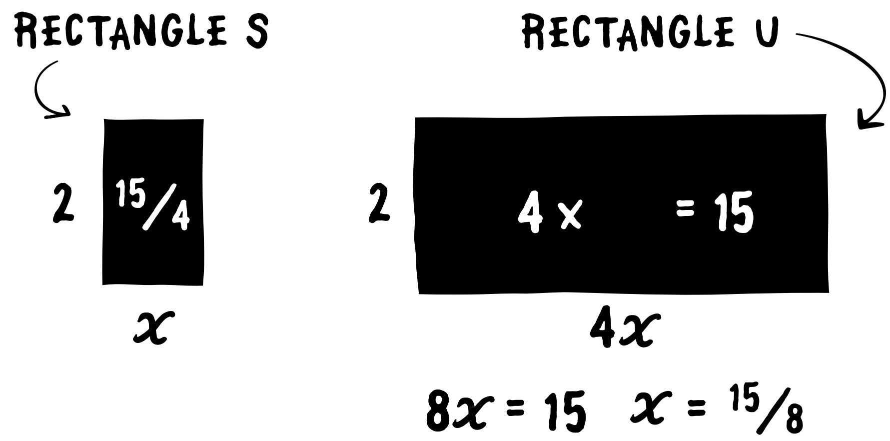


The division of fractions can also be accomplished using the area model. As an example, suppose we wish to evaluate the quotient $\frac{5}{4} \div \frac{2}{3}$. In the area model, this amounts to finding the length ***x*** of the base of a rectangle ***R*** whose area is $\frac{5}{4}$ and height is $\frac{2}{3}$.

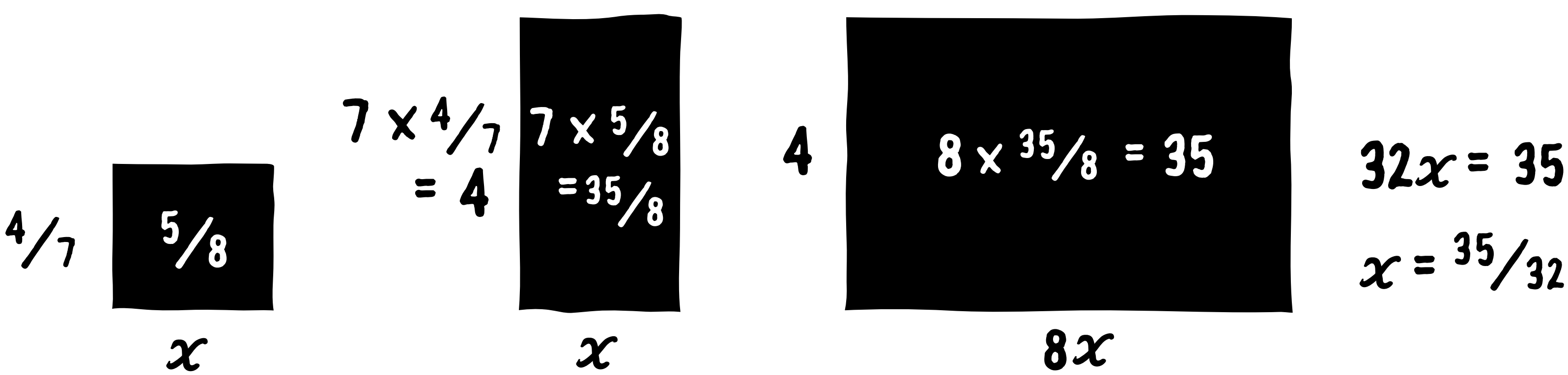
There are a couple of ways we can proceed. One way is to stack 3 copies of ***R*** one atop the other to obtain a new rectangle ***S*** whose base is still ***x*** but height is 3 x $\frac{2}{3}$, or 2, and area is 3 x $\frac{5}{4}$ which, using our knowledge of fraction multiplication, is $\frac{15}{4}$. (The figure is not drawn to scale.) We then stack 4 copies of ***S*** one atop another to obtain rectangle ***T*** whose base is still ***x***, but whose height is 8 and area is 4 x $\frac{15}{4}$, or 15. Since the base of a rectangle is its area divided by its height, ***x*** is $\frac{15}{8}$. Thus, $\frac{5}{4} \div \frac{2}{3} = \frac{15}{8}$.



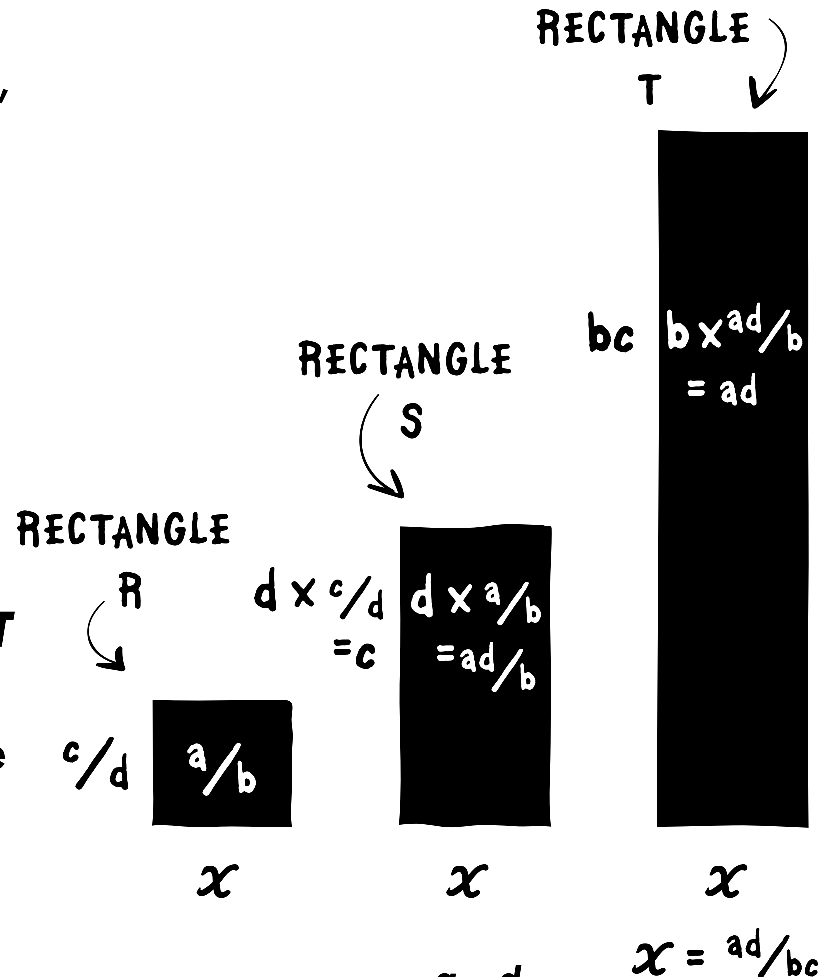
Alternatively, instead of stacking 4 copies of ***S*** one atop another, one could stack 4 copies of S side by side to get a rectangle ***U*** which has a height of 2, an area of $4 \times \frac{15}{4}$, or 15, and a base of $4 \times \boldsymbol{x}$. Since the area of ***U*** is the product of its base and height, we have $8 \times \boldsymbol{x} = 15$. Thus, as before, $\boldsymbol{x} = \frac{15}{8}$.



Gene Says: As is the case with multiplication, one can draw rough sketches that aren't to scale to aid one's thinking. The following rough sketches help us determine that $\frac{5}{8} \div \frac{4}{7} = \frac{35}{32}$.



FYI: As with fraction multiplication, one can develop a formula for fraction division. To determine $\frac{a}{b} \div \frac{c}{d}$, we want to find the base $\boldsymbol{x}$ of a rectangle $\boldsymbol{R}$ whose area is $\frac{a}{b}$ and height is $\frac{c}{d}$. Stacking $\boldsymbol{d}$ copies of $\boldsymbol{R}$ atop one another produces a rectangle $\boldsymbol{S}$ whose base is still $\boldsymbol{x}$ but has a height of $\boldsymbol{d} \times \frac{c}{d}$, which is $\boldsymbol{c}$, and an area of $\boldsymbol{d} \times \frac{a}{b}$, which is $\frac{ad}{b}$. Then stacking $\boldsymbol{b}$ copies of $\boldsymbol{S}$ atop one another, produces a rectangle $\boldsymbol{T}$ whose base is $\boldsymbol{x}$, height is $\boldsymbol{bc}$ and area $\boldsymbol{b} \times \frac{ad}{b}$,which equals $\boldsymbol{ad}$. Hence $\boldsymbol{x}$ is $\frac{ad}{bc}$. Thus, $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.



Note that the resulting fraction is equal to the product $\frac{a}{b} \times \frac{d}{c}$. Thus, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, which is the familiar rule: to divide two fractions, invert the divisor and multiply.

END of SCENE 12:
FRACTION MULTIPLICATION AND DIVISION

For comments and questions
please email Gene Maier at...
genem@mathlearningcenter.org