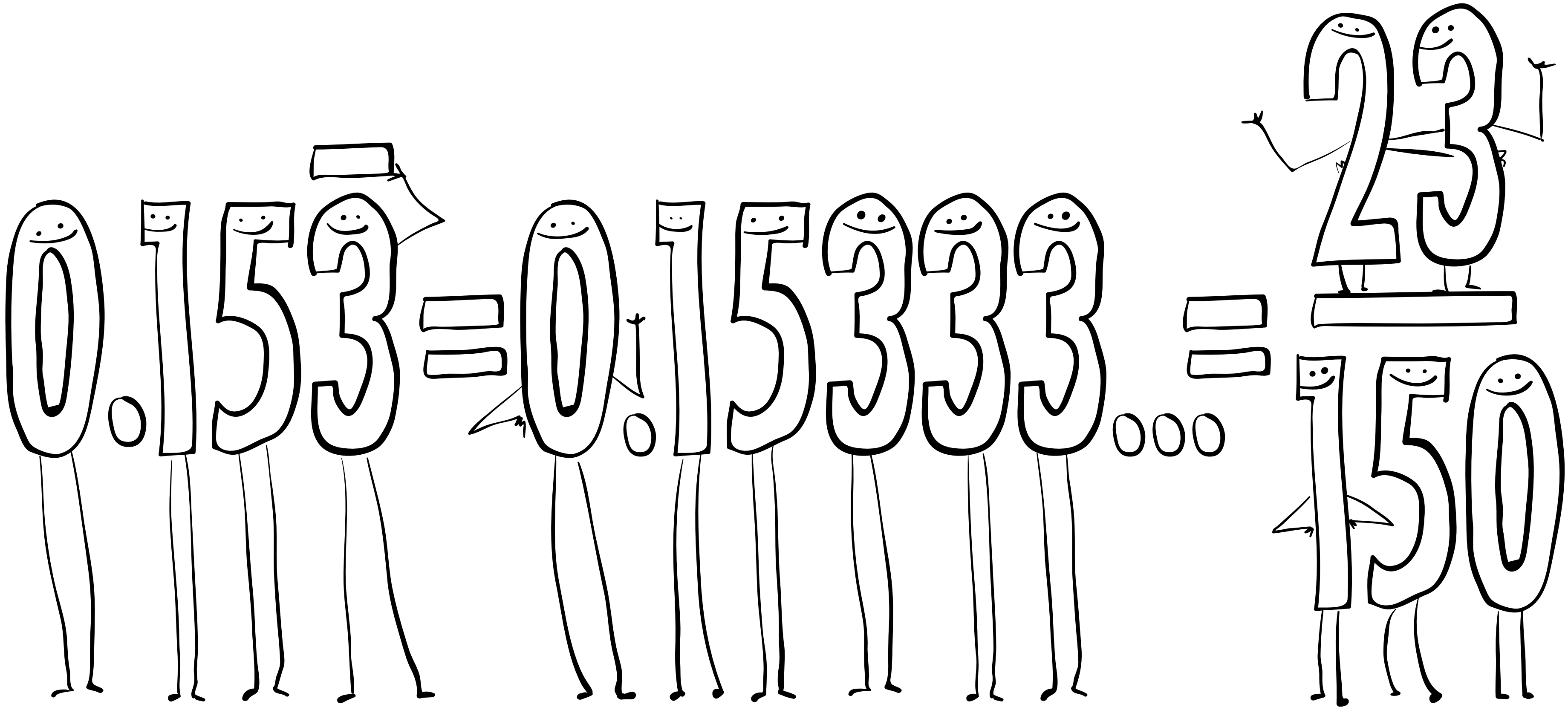


The Real Numbers

The seventeenth scene in a series of articles on elementary mathematics.

written by Eugene Maier
designed and illustrated by Tyson Smith

The set of ***real numbers*** is the set of all decimals. The set of ***rational numbers*** is the set of all decimals which terminate or repeat. The set of ***irrational numbers*** is the set of all real numbers which are not rational, that is the set of decimals which do not terminate nor repeat.

As we saw in Scene 16, A decimal which terminates or repeats can be written as a fraction and, conversely, every fraction can be written as a decimal which terminates or repeats, Thus another way of describing the rational numbers is the set of all fractions. Thus, a rational number is a number that can be expressed in the form a/b, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are integers with the stipulation that $\boldsymbol{b} \neq 0$.

$$\frac{5}{6},\quad -\frac{3}{4} = \frac{-3}{4},\quad 2\frac{3}{7} = \frac{17}{7}$$

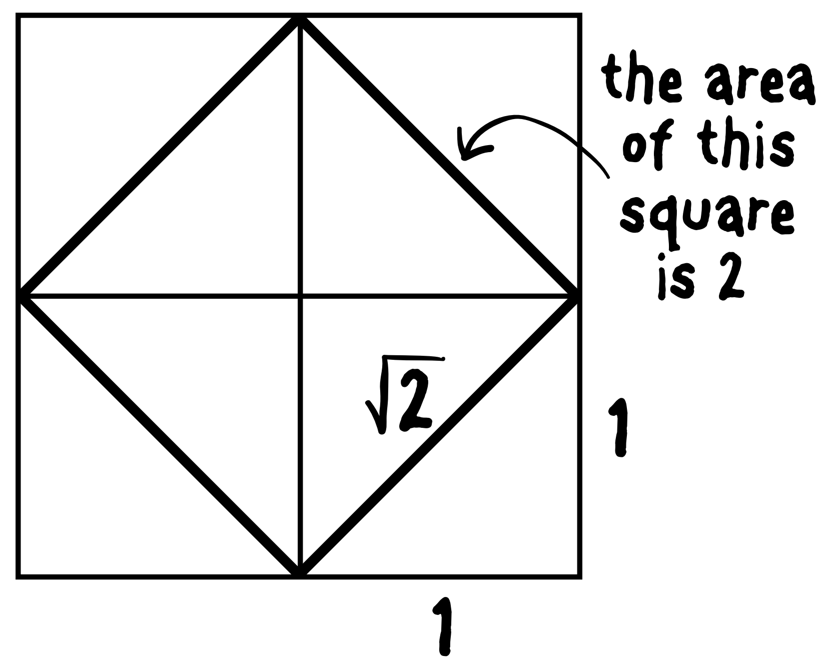
$$0.15\overline{3} = 0.15333\ldots = \frac{23}{150}$$

$$2.\overline{36} = 2.363636\ldots = \frac{26}{11}$$

some rational numbers

Gene says: The words "real" and "irrational" when referring to numbers are not to be confused with common usage of these words. When applied to a number, the adjective "real" is used to distinguish that number from a so-called "imaginary" number, which we will encounter in the next scene. However, an imaginary number is as authentic as a real number; they are both valid and useful mathematical constructs. Also, an irrational number is as logical concept as a rational number. Used in reference to a number, "rational" means that it can be expresses as the ratio of two integers.

Many of the numbers encountered in elementary mathematics are irrational. For example, π, the ratio of the circumference of a circle to its diameter is irrational—the rational number $^{22}/_{7}$ is often used as an approximation to π; it differs from the actual value of π by less than two-thousandths.

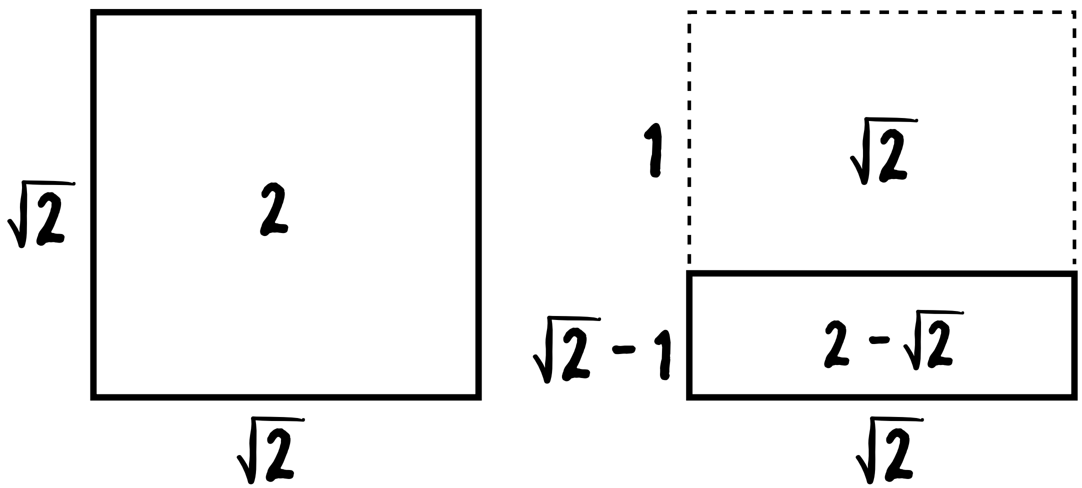


The number $\sqrt{2}$, the length of the diagonal of a unit square, is also irrational. In fact, the square roots of all counting numbers, other than perfect squares, are irrational. For a method of establishing this, turn the page.

We begin by showing that √2 is irrational. To do this, we will assume it is rational and show that this assumption leads to a contradiction, that is, a situation where both a statement and its negation are established. Since this is an impossibility, we conclude that our assumption was incorrect and hence √2 must not be rational, in other words, it must be irrational.

So suppose √2 is rational. Then it can be written as a fraction whose numerator and denominator are positive integers. In particular, there will be a fraction ***a*/*b*** which equals √2 and whose denominator ***b*** will be the smallest among all fractions whose value is √2. (For example, among all the fractions which have the same value as 15/20, the one which has the smallest denominator is 3/4.)

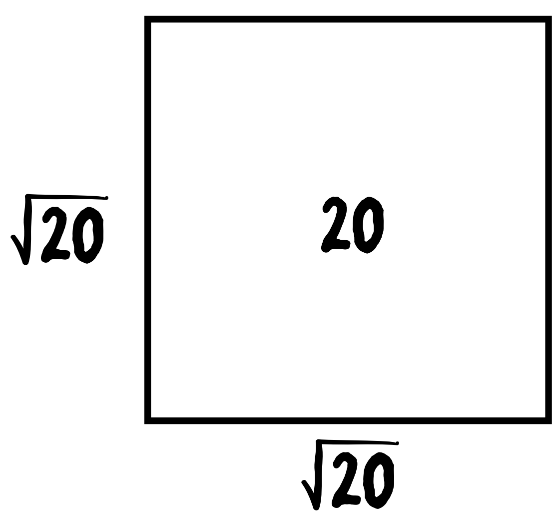
Now $\sqrt{2}$ is the length of the side of a square of area 2. Cut off a $1 \times \sqrt{2}$ rectangle from the top of this square. In so doing we have reduced the side of the square by 1 and its area by $\sqrt{2}$. Thus the remaining rectangle has height $\sqrt{2} - 1$ and area $2 - \sqrt{2}$ while its base remains $\sqrt{2}$. Since the base of a rectangle equals its area divided by its height, we have $\sqrt{2} = \frac{2 - \sqrt{2}}{\sqrt{2} - 1}$. Since $\sqrt{2} = \frac{a}{b}$, we have $\sqrt{2} = \frac{2 - \sqrt{2}}{\sqrt{2} - 1} = \frac{2 - \frac{a}{b}}{\frac{a}{b} - 1} = \frac{b}{b} \times \frac{2 - \frac{a}{b}}{\frac{a}{b} - 1} = \frac{2b - a}{a - b}$, which is also a rational number since $2b - a$ and $a - b$ are integers.

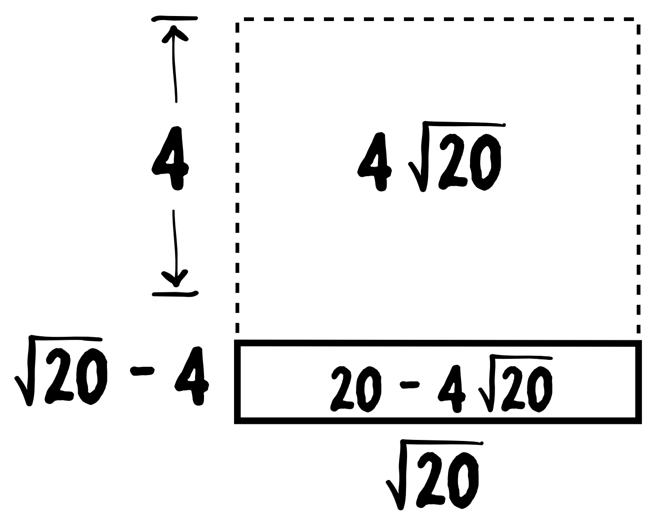


But $1 < \sqrt{2} < 2$, thus $1 < \frac{a}{b} < 2$. Hence $b < a < 2b$ and $0 < a - b < b$. Thus $a - b$ is smaller than b and the denominator of a fraction which equals $\sqrt{2}$. Hence b is not the smallest denominator of a fraction which equals $\sqrt{2}$, which is a contradiction. Thus the assumption that $\sqrt{2}$ is rational is false, so $\sqrt{2}$ is irrational.

The above argument can be adapted to show that the square root of any positive integer, other than perfect squares, is irrational. If ***n*** is not a perfect square, instead of cutting off a rectangle of height 1 from a square whose area is ***n***, cut off a rectangle whose height is the greatest integer less than $\sqrt{n}$.

Suppose, for example that n = 20. Now $4^2 = 16$ and $5^2 = 25$, so $\sqrt{20}$ is between 4 and 5 and the greatest integer less than $\sqrt{20}$ is 4. If we subtract a rectangle of height 4, from a square whose area is 20, what remains is a rectangle of area $20 - 4\sqrt{20}$, height $\sqrt{20} - 4$ and base $\sqrt{20}$. Therefore, $\sqrt{20} = \frac{20 - 4\sqrt{20}}{\sqrt{20} - 4}$. Now, suppose $\sqrt{20}$ is rational and equals a/b where b is the smallest possible denominator of all fractions which equal $\sqrt{20}$. Then $\sqrt{20} = \frac{20 - 4\frac{a}{b}}{\frac{a}{b} - 4} = \frac{b}{b} \times \frac{20b - 4a}{a - 4b}$. Since a and b are integers, so are $20b - 4a$ and $a - 4b$, But $4 < a/b < 5$, So $4b < a < 5b$ and, thus, $0 < a - 4b < b$ which contradicts that b is the smallest possible denominator.





END of SCENE 17:
THE REAL NUMBERS

For comments and questions please email Gene Maier

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