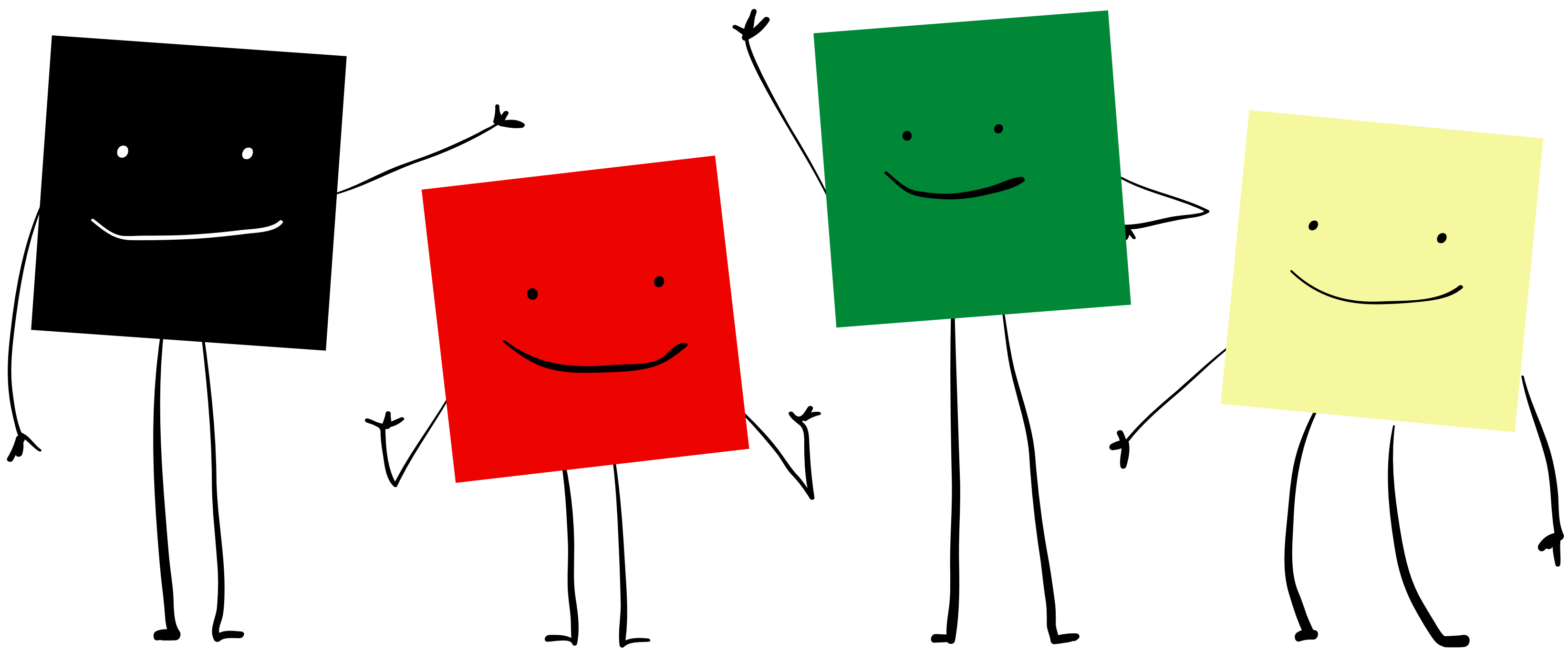


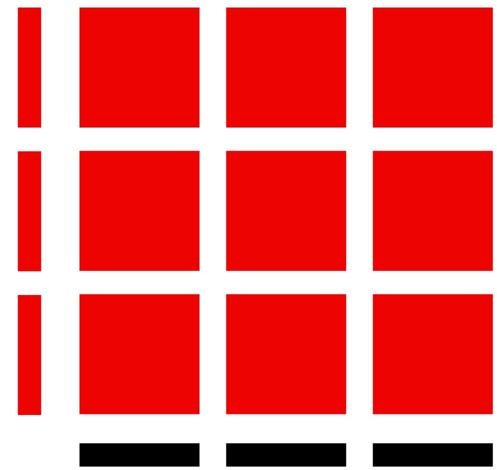
The Complex Numbers

The eighteenth scene in a series of articles on elementary mathematics.

written by Eugene Maier

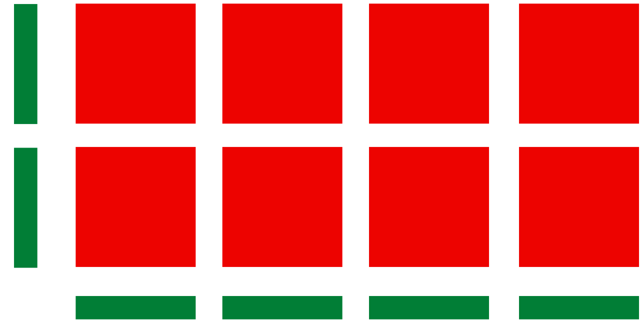
designed and illustrated by Tyson Smith

As described in Scene 16, if a square has adjacent edges which have the same value, then that value is the square root of the value of the square. The square shown here has value -9; adjacent edges have unequal values, namely 3 and -3. Hence, neither is a square root of -9. As a matter of fact, with the numbers and corresponding number pieces we have introduced so far, there is no common value for adjacent edges of a red square, that is, there is no square root for a negative number. In order to obtain square roots for negative numbers, we will have to create new number pieces and corresponding numbers.

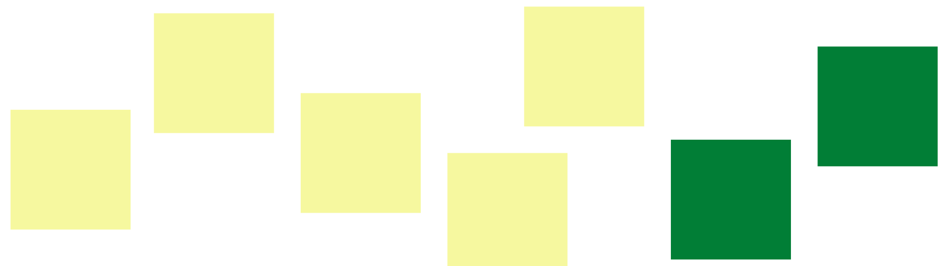


Gene Says: In the language of equations, since $3^2 = 9$ and $(-3)^2 = 9$, the equation $x^2 = 9$ has two solutions, namely 3 and -3. On the other hand, the equation $x^2 = -9$ has no solutions if the only numbers we have at our disposal are the real numbers. The numbers we create in this scene will provide solutions for this equation.

To accomplish our purpose, we introduce new number pieces which are green on one side and yellow on the other. The purpose of these pieces is to provide square roots for negative numbers. In particular, if both edges of an array are green, the pieces in the array will be red.

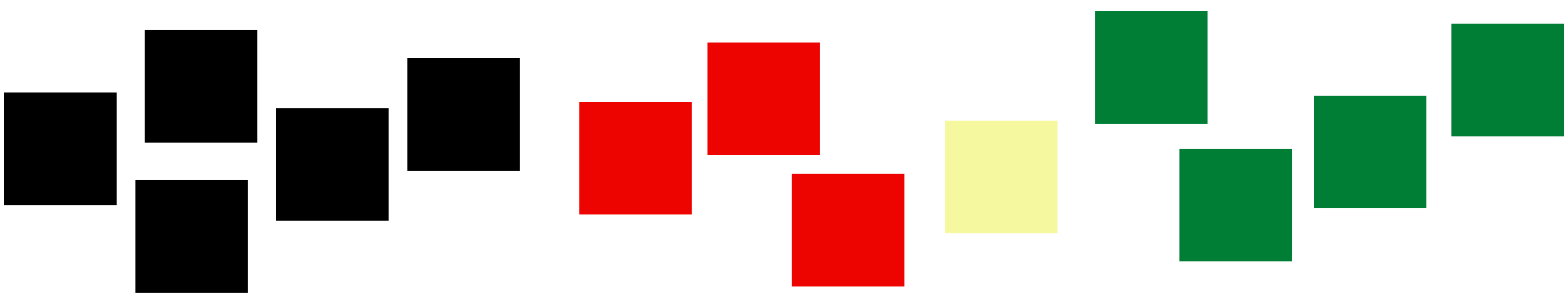


Also, green and yellow are opposites. Thus, for example, the net value of a collection of 5 yellow pieces and 2 green pieces is 3 yellow pieces.



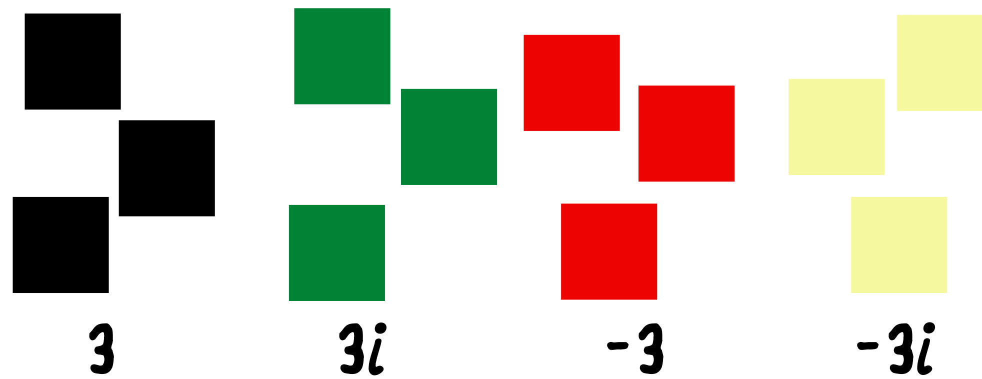
net value: 3 yellow

Collections can contain black, red, green and yellow pieces. The net value of the following collection is 2 black plus 3 green

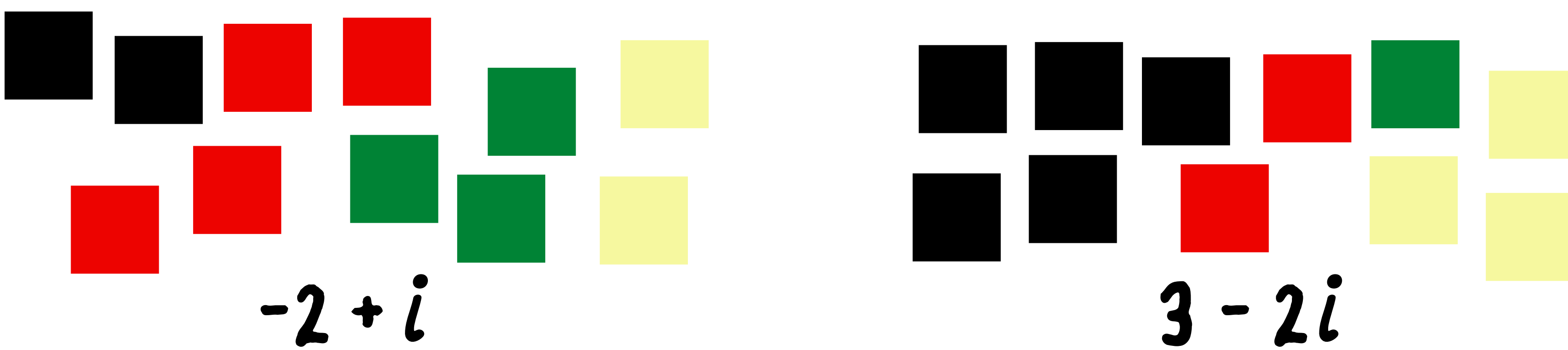


net value: 2 black plus 3 green

In order to distinguish between net values in the black/red system and those in the green/yellow system, the letter ***i*** will be used to indicate net values in the green/yellow system. Thus, a collection of 3 black tile has net value 3 while a collection of 3 green tile has net value 3***i***. A collection of 3 red tile has net value –3 while a collection of 3 yellow tile has value –3***i***.

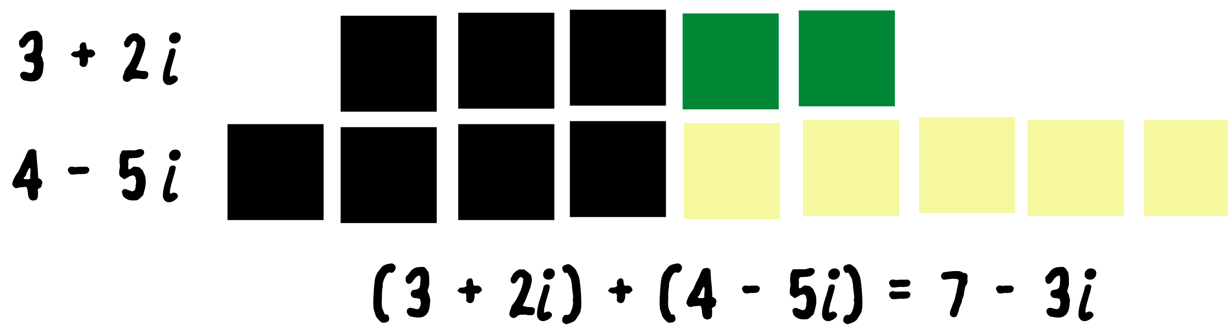


Here are some other collections and their net values:



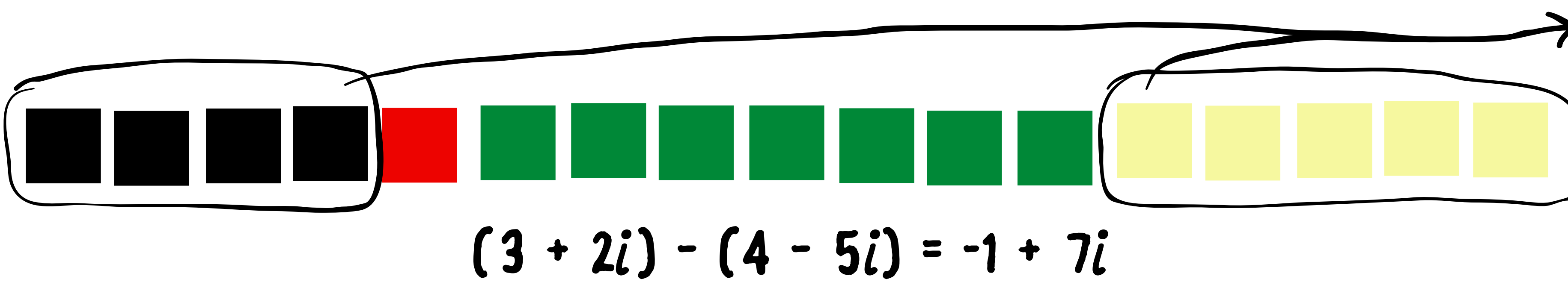
FYI: A number of the form ***a*** + ***bi***, where a and b are real numbers, is a ***complex number***. An ***imaginary number*** is a complex number for which ***b*** ≠ 0. A ***pure imaginary number*** is a complex number for which ***a*** = 0.

The sum of two complex numbers can be found by combining collections, e. g., to find $(3 + 2i) + (4 - 5i)$, a collection whose value is $3 + 2i$ is combined with a collection whose value is $4 - 5i$. The resulting collection has a net value of $7 - 3i$. Thus $(3 + 2i) + (4 - 5i) = 7 - 3i$.

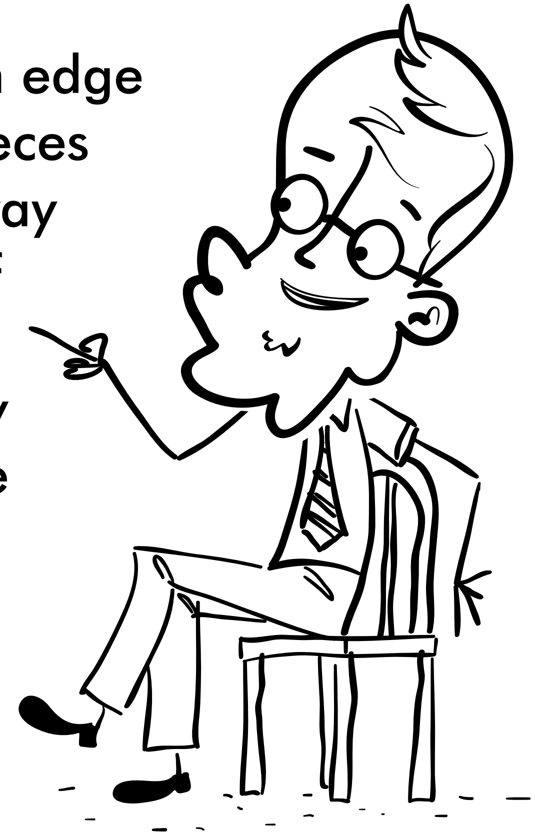


The difference $(3 + 2i) - (4 - 5i)$ can be found by adding the opposite of $4 - 5i$ to $3 + 2i$: $(3 + 2i) - (4 - 5i) = (3 + 2i) + (-4 + 5i) = -1 + 7i$.

Alternatively, this difference can be found by forming a collection with net value $3 + 2i$ from which a collection with value $4 - 5i$ can be removed:



Products can be found by forming arrays with edge pieces. In doing so, recall that if two edge pieces are green, the corresponding piece in the array will be red. Also, in building the arithmetic of the complex numbers, 1 will maintain its role as a multiplicative identity, that is 1 times any number will be that number. In terms of edge pieces, this means that if a piece in an array has one black edge piece. the color of the piece will be the same as the color of the other edge piece:



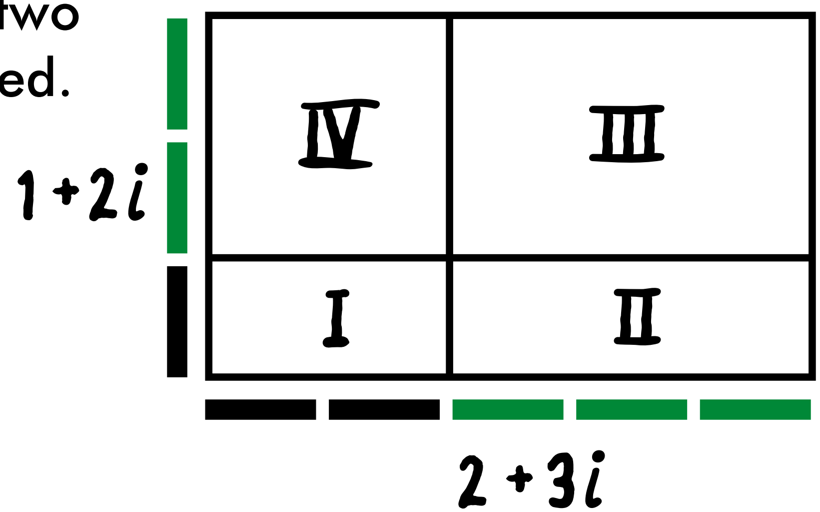
$1 \times 1 = 1$

$1 \times (-1) = -1$

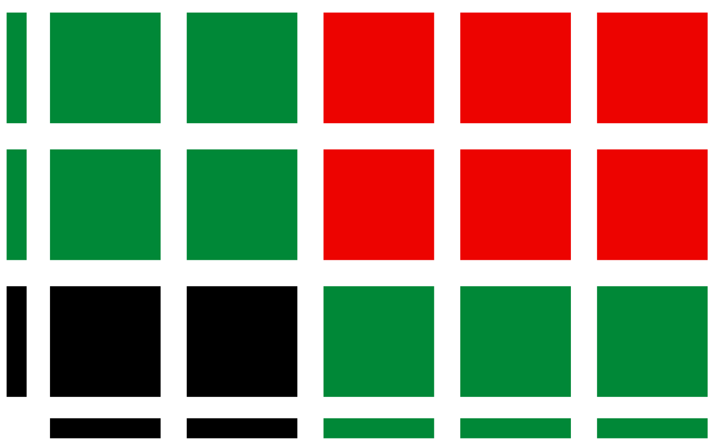
$1 \times i = i$

$1 \times (-i) = -i$

To illustrate the array model, we find the product (1 + 2***i***)(2 + 3***i***). An array whose edges have values 1 + 2***i*** and 2 + 3***i*** will have four sections as shown. Tile in section I have two black edges and hence will be black. Tile in section III have two green edges and hence will be red. Tile in sections II and IV have one black edge and one green edge, and hence will be green.



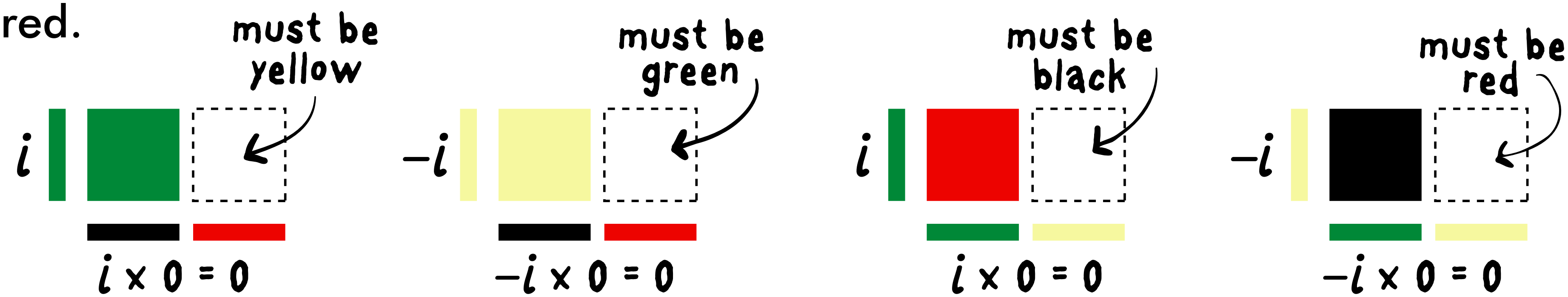
The completed array has value –4 + 7***i***.
Thus (1 + 2***i***)(2 + 3***i***) = –4 + 7***i***.



Before proceeding, we need to determine what happens for other possible edge colors. We know that if a piece has a black edge, it will be the color of the other edge. We also know that a piece with 2 red edges is black and a piece with 2 green edges is red. Thus far, then, we know the information indicated in the chart.

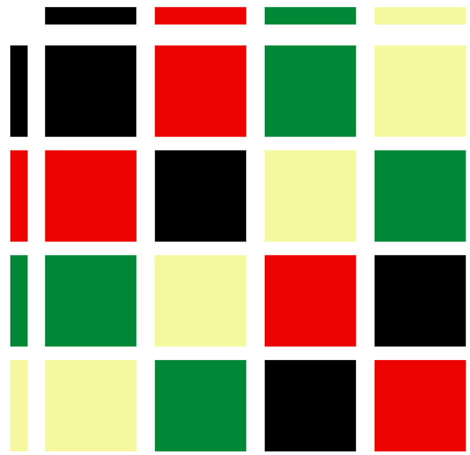
edge 2 \ edge 1	B	R	G	Y
B	B	R	G	Y
R	R	B		
G	G		R	
Y	Y			

We can find the remaining possibilities by considering the arrays shown below. In each case, one edge of the array has value 0 in which case the array must have value 0—if we agree that multiples of 0 should be 0 in the complex numbers as they are in the real numbers. The first array shows us that a piece that has one green edge and one red edge must be yellow. The next array shows us that a piece with one red edge and one yellow edge must be green. The third array shows us that a piece with one green edge and one yellow edge must be black and, using that information, the last array shows us that a piece with two yellow edges must be red.



The completed table is on the left below, The middle table provides this information using colors instead of letters. The table on the right is obtained by replacing colors by their values.

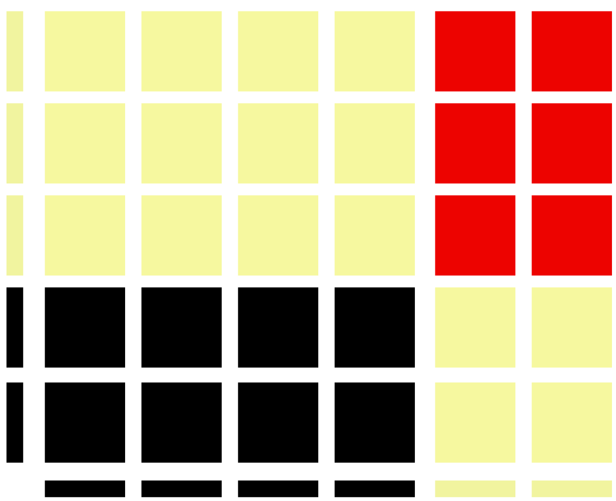
edge 2 \ edge 1	B	R	G	Y
B	B	R	G	Y
R	R	B	Y	G
G	G	Y	R	B
Y	Y	G	B	R



×	1	−1	i	$-i$
1	1	−1	i	$-i$
−1	−1	1	$-i$	i
i	i	$-i$	−1	1
$-i$	$-i$	i	1	−1

Here are some examples of other products:

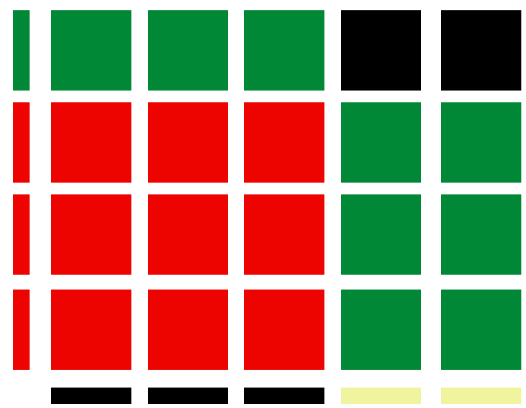
$2 - 3i$



$4 - 2i$

$(2 - 3i) \times (4 - 2i) = 2 - 16i$

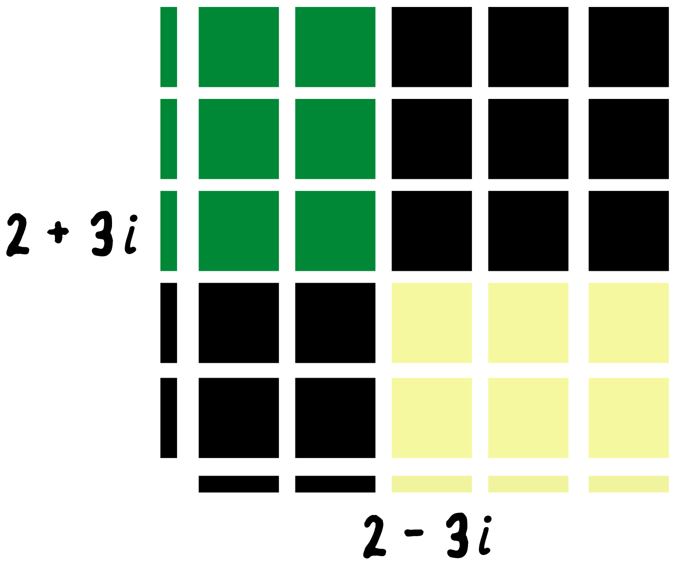
$-3 + i$



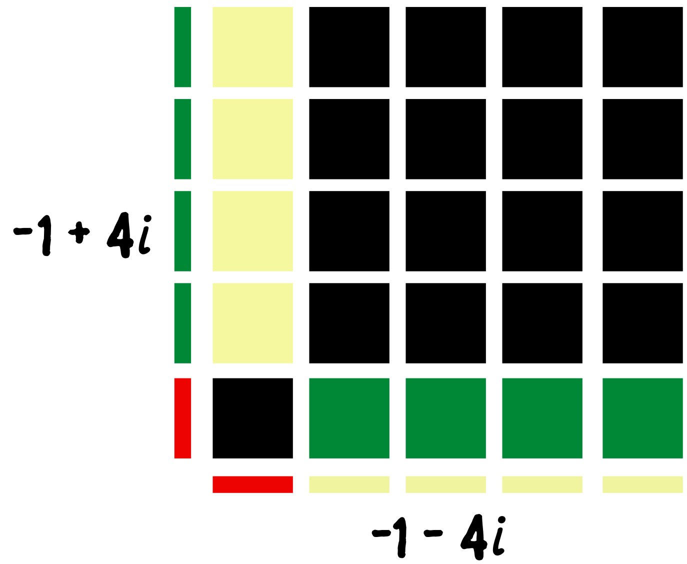
$3 - 2i$

$(-3 + i)(3 - 2i) = -7 + 9i$

The ***conjugate*** of a complex number is the number obtained by changing the imaginary part of a number to its opposite. Thus the conjugate of $2 + 3\boldsymbol{i}$ is $2 - 3\boldsymbol{i}$; the conjugate of $4 - 7\boldsymbol{i}$ is $4 + 7\boldsymbol{i}$. Notice that the product of a complex number and its conjugate is a positive real number, as shown in the following examples.



$$(2 + 3i) \times (2 - 3i) = 13$$

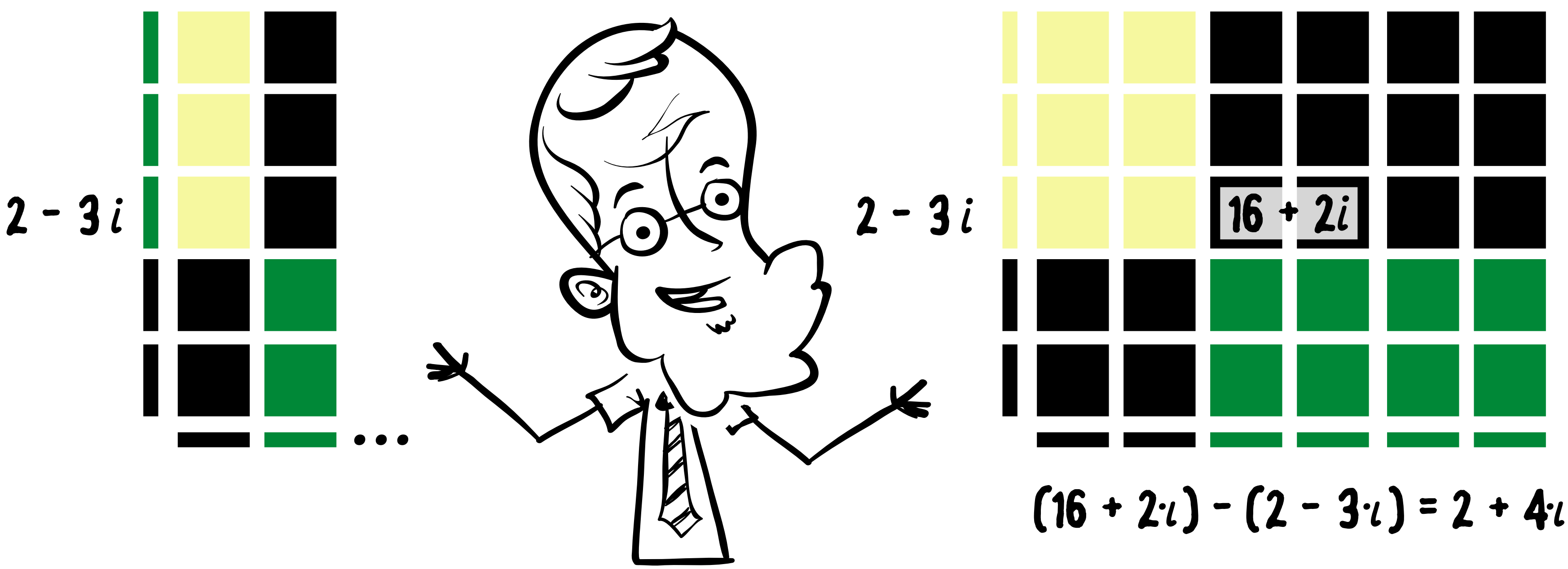


$$(-1 + 4i) \times (-1 - 4i) = 17$$

FYI: One may also find products arithmetically, using results from the multiplication table developed earlier, e.g.,

$$(3 + 4\boldsymbol{i})(2 - 5\boldsymbol{i}) = 3(2 - 5\boldsymbol{i}) + 4\boldsymbol{i}(2 - 5\boldsymbol{i}) = 6 - 15\boldsymbol{i} + 8\boldsymbol{i} - 20\boldsymbol{i}^2 = 6 - 7\boldsymbol{i} - 20(-1) = 26 - 7\boldsymbol{i}$$

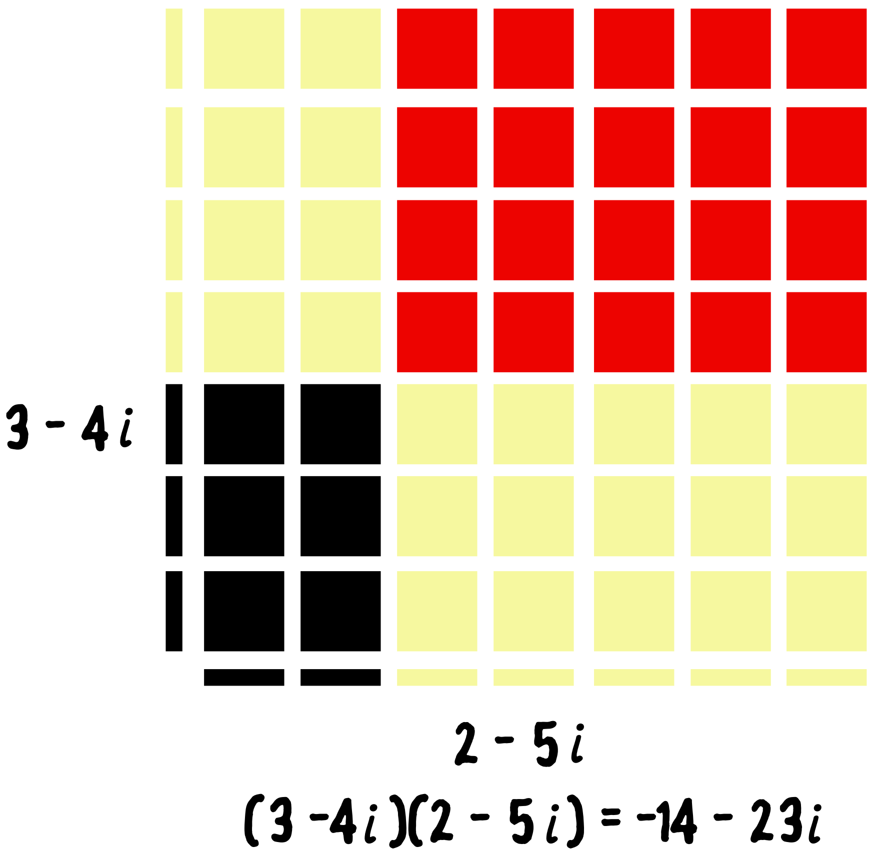
Division may also be performed by forming arrays. To find the quotient $(16 + 2\boldsymbol{i}) \div (2 - 3\boldsymbol{i})$, one wants to form an array whose value is $16 + 2\boldsymbol{i}$ and has an edge whose value is $2 - 3\boldsymbol{i}$. The quotient will be the value of the other edge. We begin by laying out an edge of 2 black and 3 yellow pieces and then determine the other edge to get an array that has 16 black pieces and 2 more green than yellow. We will get black pieces if the bottom edge contains black and green edge pieces. Every black piece in the bottom edge will lead to 2 black and 3 yellow pieces in the array while a green edge piece will lead to 2 green and 3 black pieces. Picking 2 black edge pieces and 4 green edge pieces gives the desired array. Hence, $(16 + 2\boldsymbol{i}) \div (2 - 3\boldsymbol{i}) = 2 + 4\boldsymbol{i}$.

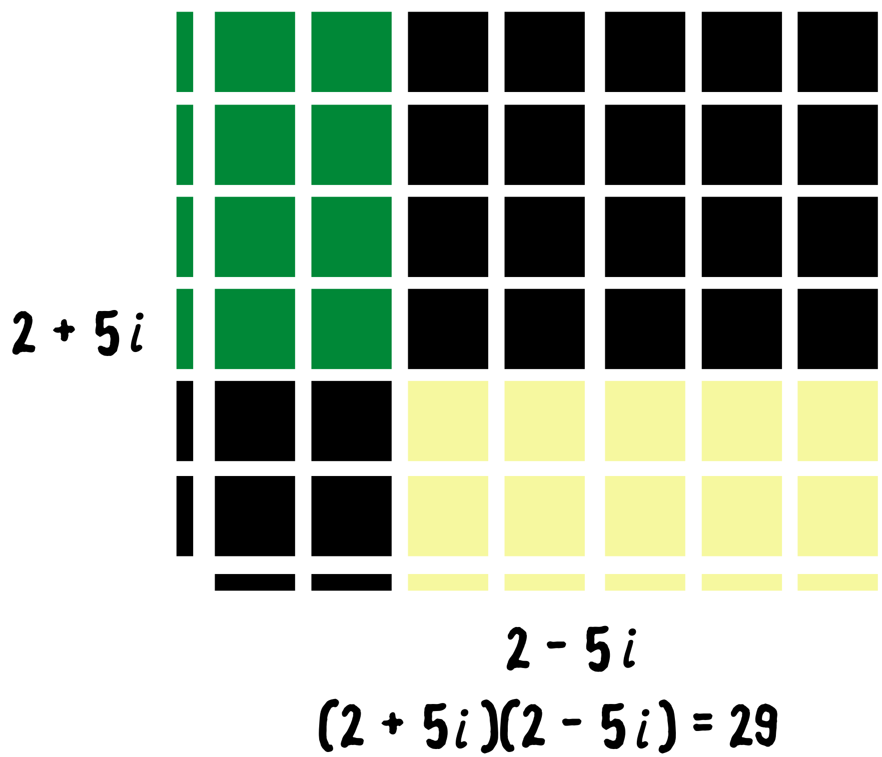


A more systematic method for division entails using conjugates to convert a division into multiplications.

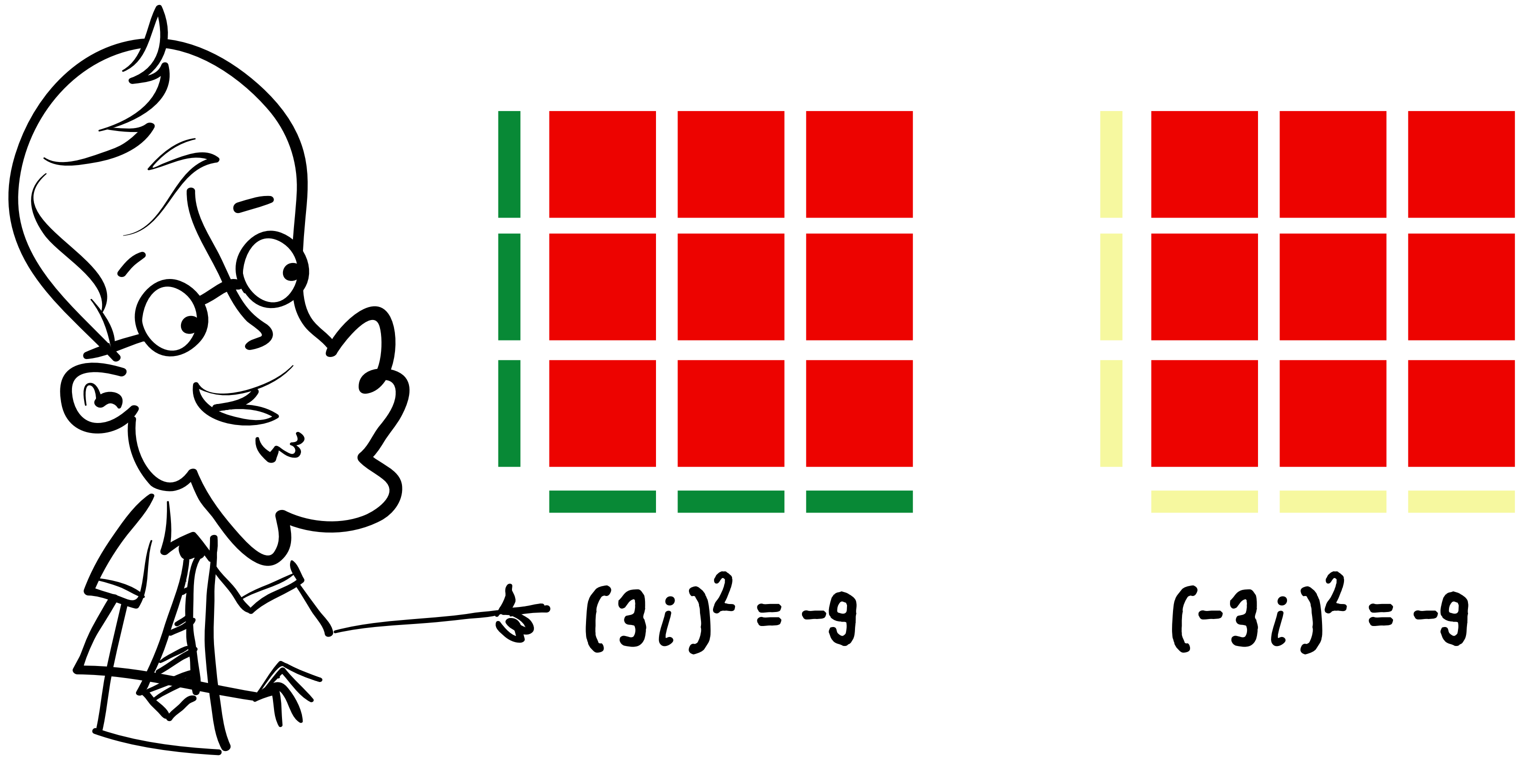
To find a quotient by this method, one writes the quotient as a fraction and multiplies the numerator and denominator by the conjugate of the denominator. Since this is essentially multiplying by 1, it doesn't change the value of the quotient, but it does change the denominator into a real number, as shown in the following example.

$$(3-4\boldsymbol{i}) \div (2+5\boldsymbol{i}) = \frac{3-4\boldsymbol{i}}{2+5\boldsymbol{i}} \times \frac{2-5\boldsymbol{i}}{2-5\boldsymbol{i}} = \frac{(3-4\boldsymbol{i})\times(2-5\boldsymbol{i})}{(2+5\boldsymbol{i})\times(2-5\boldsymbol{i})} = \frac{-14-23\boldsymbol{i}}{29} = -\frac{14}{29} - \frac{23}{29}\boldsymbol{i}$$



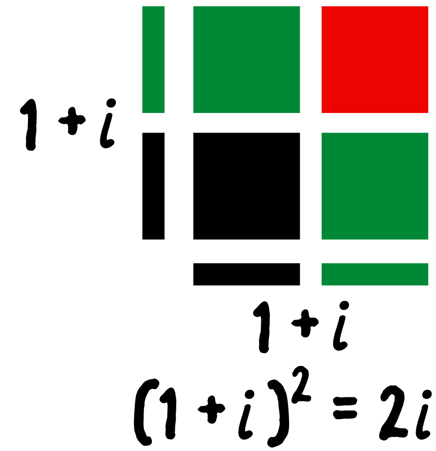


Note that we now have solutions for the equation, $x^2 = -9$, mentioned at the beginning of the scene. It has two solutions, namely $3\boldsymbol{i}$ and $-3\boldsymbol{i}$.

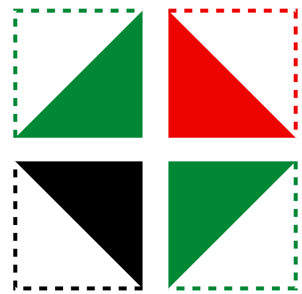


One might ask if one needs to introduce new pieces to obtain square roots for imaginary numbers. Is it possible, for example, to construct an array whose value is $\boldsymbol{i}$ and has adjacent edges of equal value? The answer is "Yes", as shown on the following page, if one is allowed pieces of non-integral dimensions.

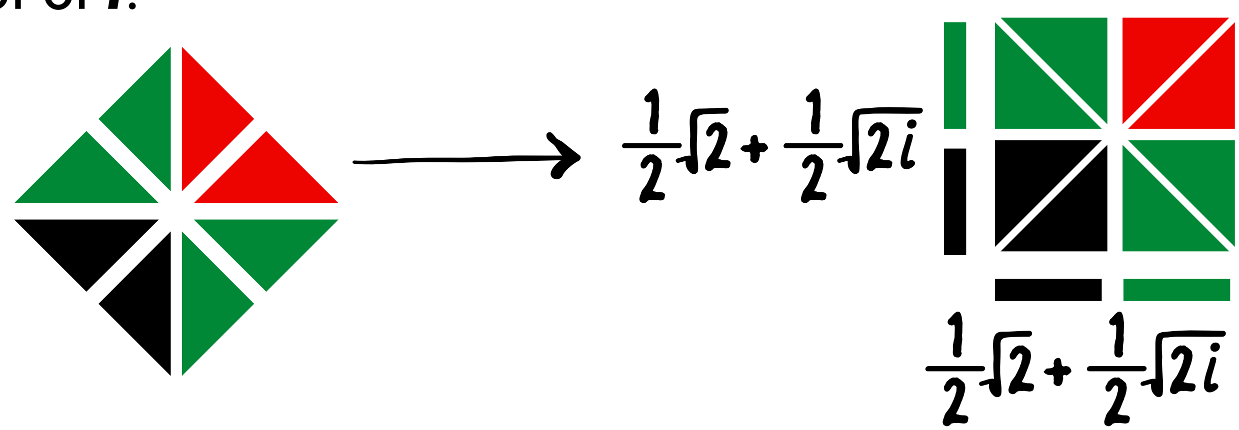
The array to the right has value 2$\boldsymbol{i}$ and adjacent edges each of whose value is 1 + $\boldsymbol{i}$. Thus 1 + $\boldsymbol{i}$ is a square root of 2$\boldsymbol{i}$, that is 1 + $\boldsymbol{i}$ is a solution of the equation $\boldsymbol{X}^2$ = 2$\boldsymbol{i}$.



If we cut the pieces in half, as shown, one obtains a square whose value is $\boldsymbol{i}$.



If these pieces are cut in half again they can be formed into a square with adjacent edges of equal value. The length of each edge piece is half the length of the diagonal of a unit square, or $\frac{1}{2}\sqrt{2}$. Thus $\frac{1}{2}\sqrt{2} + (\frac{1}{2}\sqrt{2})\boldsymbol{i}$ is a square root of $\boldsymbol{i}$.



END of SCENE 18:
THE COMPLEX NUMBERS

For comments and questions please email Gene Maier
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