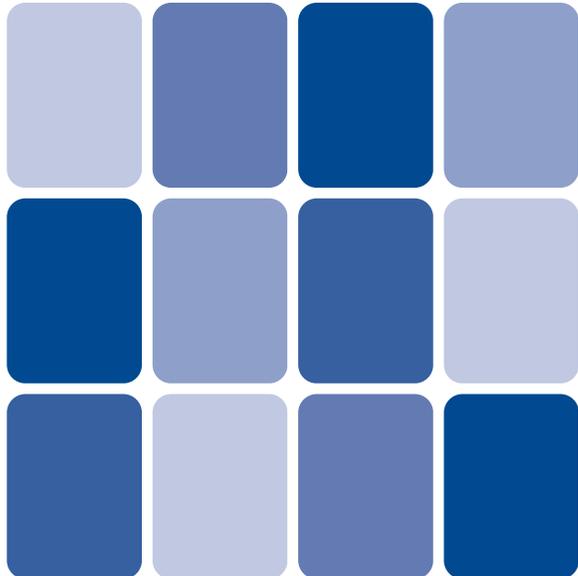
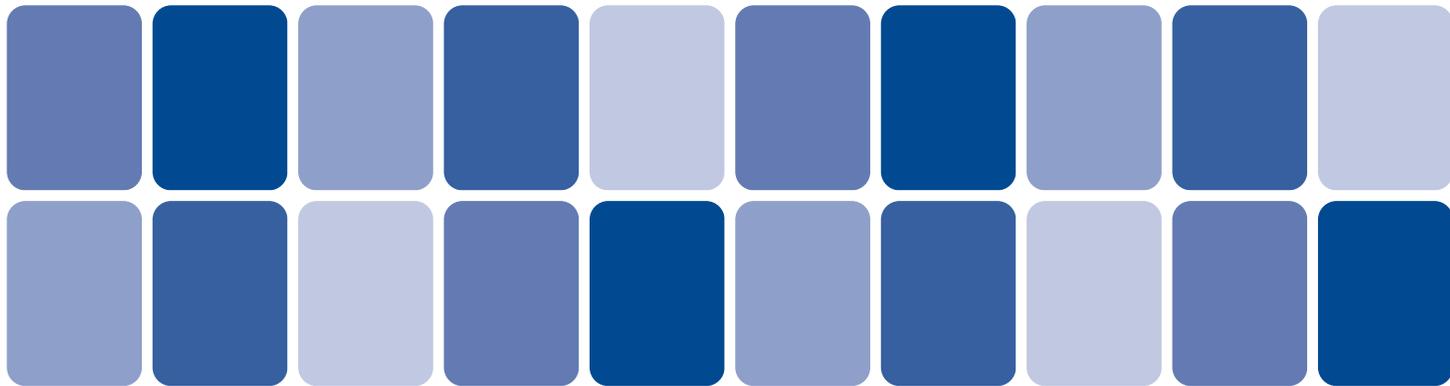


ALGEBRA THROUGH VISUAL PATTERNS

A Beginning Course in Algebra

Volume I



EUGENE MAIER
LARRY LINNEN



The MATH LEARNING CENTER



ALGEBRA THROUGH VISUAL PATTERNS

A Beginning Course in Algebra

Volume I



A Math Learning Center publication adapted and arranged by
EUGENE MAIER and **LARRY LINNEN**

ALGEBRA THROUGH VISUAL PATTERNS, VOLUME I

A Math Learning Center Resource

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ALGEBRA THROUGH VISUAL PATTERNS

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INTRODUCTION

Algebra Through Visual Patterns is a series of lessons that comprise a semester-long introductory algebra course, beginning with the development of algebraic patterns and extending through the solution of quadratic equations. In these lessons, students learn about and connect algebraic and geometric concepts and processes through the use of manipulatives, sketches, and diagrams and then link these visual developments to symbolic rules and procedures. The lessons can be used with students who are involved in learning first-year algebra wherever their instruction is taking place: in middle school, high school, community college, or an adult learning center.

Since the Algebra Through Visual Patterns lessons are designed to be accessible to students whatever their level of understanding, the lessons have been successfully used with students of varying background and ability, including Special Education students, students learning algebra for the first time, those who have struggled with the subject in previous courses, students who have been identified as talented and gifted, and students of various ages, from middle-schoolers to adult learners.

Algebra Through Visual Patterns offers a genuine alternative to the usual algebra course. It offers an approach to learning in which teachers and students collaborate to create a classroom in which learners

- explore algebraic concepts using manipulatives, models, and sketches,
- engage in meaningful discourse on their learning of mathematics,
- publicly present their understandings and solution to problems, both orally and in writing,
- build on their understandings to increase their learning.

The lessons are designed in such a way as to render them useful as a stand-alone curriculum, as replacement lessons for, or as a supplement to, an existing curriculum. For example, you might decide to begin with a manipulative approach to factoring quadratic expressions that would lead to symbolic approaches for the same concept. This approach is built into Visual Algebra and thus could be used instead of simply a symbolic approach to factoring quadratics. The likelihood of learning for all students would be enhanced and the end result would be that students would understand factoring as well as increasing their competency to factor quadratics.

Each lesson includes a Start-Up, a Focus, and a Follow-Up. The Focus is the main lesson, while the Start-Up sets the stage for the Focus or connects it to a previous lesson, and the Follow-Up is a homework and/or assessment activity.

Together, Volumes 1 and 2 of Algebra Through Visual Patterns constitute a stand-alone semester course in algebra or a yearlong course when used in conjunction with other text materials. In the latter instance, lessons from Algebra through Visual Patterns can be used to provide an alternative to the purely symbolic developments of traditional algebra texts.

THE BIG IDEA

Tile patterns provide a meaningful context in which to generate equivalent algebraic expressions and develop understanding of the concept of a variable. Such patterns are a useful context for developing techniques for solving equations and for introducing the concept of graphing.

START-UP

Overview

A tile pattern provides the context for generating equivalent expressions, formulating equations, and creating bar graphs.

Materials

- Red and black counting pieces, 60 per student.
- Start-Up Master 1.1, 2 copies per student and 1 transparency.
- Start-Up Master 1.2, 1 transparency (optional).
- Start-Up Master 1.3, 1 copy per student and 1 transparency.
- Start-Up Master 1.4, 1 copy per student and 1 transparency.
- $\frac{1}{4}$ " grid paper (see *Blackline Masters*), 1 sheet per student and 1 transparency.
- Black counting pieces for the overhead.

FOCUS

Overview

Tile patterns are used to generate equivalent expressions, formulate equations, solve equations, and introduce coordinate graphs.

Materials

- Red and black counting pieces, 25 per student.
- $\frac{1}{4}$ " grid paper (see Appendix), 2 sheets per student and 1 transparency.
- Focus Master 1.1-1.2, 1 copy of each per student and 1 transparency.
- Focus Master 1.3, 1 copy per student.
- Black counting pieces for the overhead.

FOLLOW-UP

Overview

Students find patterns in arrangements of tile, write equivalent algebraic expressions for the number of tile in the n th arrangement, and create coordinate graphs to show the number of tile in certain arrangements.

Materials

- Follow-Up 1, 1 copy per student.
- Square tile or counting pieces (optional) for student use at home.
- $\frac{1}{4}$ " grid paper (see Appendix).

START-UP

Overview

A tile pattern provides the context for generating equivalent expressions, formulating equations, and creating bar graphs.

Materials

- Red and black counting pieces, 60 per student.
- Start-Up Master 1.1, 2 copies per student and 1 transparency.
- Start-Up Master 1.2, 1 transparency (optional).
- Start-Up Master 1.3-1.4, 1 copy of each per student and 1 transparency.
- $\frac{1}{4}$ " grid paper (see Appendix), 1 sheet per student and 1 transparency.
- Black counting pieces for the overhead.

ACTIONS

1 Distribute counting pieces to each student or group of students. Display the following sequence of 3 tile arrangements on the overhead. Have the students form this sequence of arrangements. Then have them form the next arrangement in the sequence. Ask the students to leave their sequence of arrangements intact so it can be referred to later.



2 Ask the students to consider the sequence of arrangements in which the 4th arrangement is the one illustrated in Comment 1. Ask them to imagine the 20th arrangement and to determine the number of tile required to build it. Ask for a volunteer to describe their method of determining this number. Illustrate their method on the overhead, using a transparency of Start-Up Master 1.1.

COMMENTS

1 In this and subsequent activities, the counting pieces are often referred to as *tile*. In later activities the color of the counting pieces is relevant; however, color is not relevant in this activity.

Many students will form the 4th arrangement as shown to the right. If someone forms another arrangement, acknowledge it without judgment, indicating there are a number of ways in which a sequence can be extended.

In Action 7 the students will be asked to convert their sequence of arrangements into a bar graph.

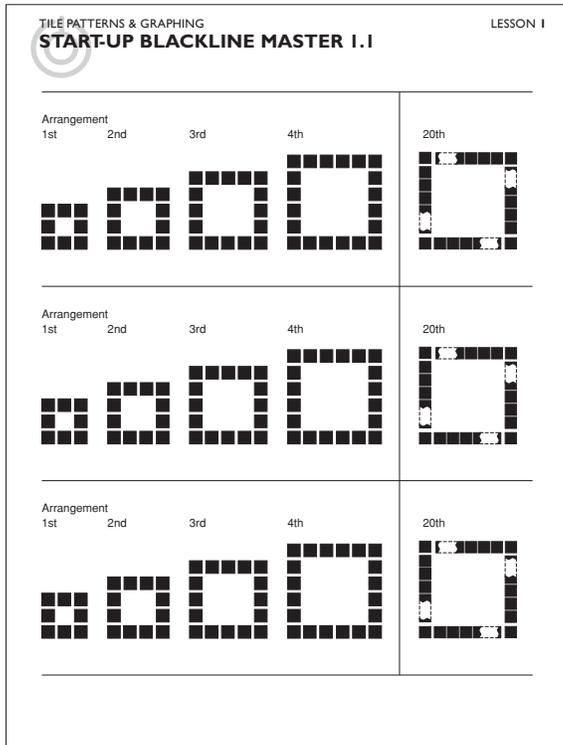


2 There are various ways to arrive at the conclusion that 84 tile are required to build the 20th arrangement. Here is one possible explanation: "There are just as many tile between the corners of an arrangement as the number of the arrangement, for example, in the 3rd arrangement there are 3 tile between the corners, in the 4th arrangement there are 4 tile between corners, and so forth. So in the 20th arrangement there will be 20 tile between corners on each side. Since there are 4 sides and 4 corners, there will be 4 times 20 plus 4 tile." This way of viewing tile arrangements can be illustrated as shown below.

continued next page

START-UP

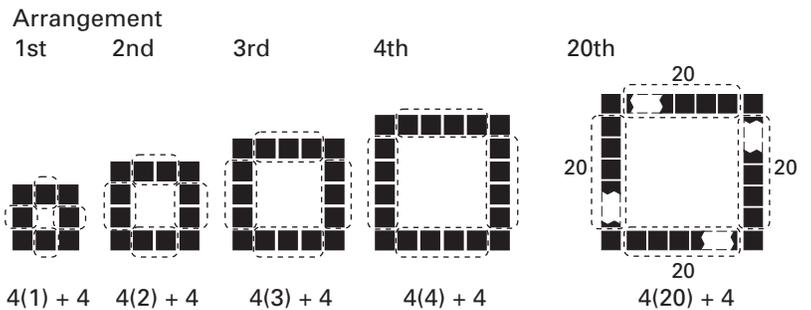
ACTIONS



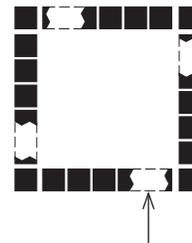
3 Distribute 2 copies of Start-Up Master I.1 to each student. Have the students record on a copy of Start-Up Master I.1 the method of viewing the arrangements described in Comment 1. Then ask for volunteers to describe other ways of determining the number of tile in the 20th arrangement. Illustrate these methods on the overhead, using a transparency of Start-Up Master I.1. Have the students make a record of these methods on their copies of Start-Up Master I.1. Continue until 5 or 6 different methods have been recorded.

COMMENTS

2 continued



Some students may write formulas to represent the number of tile in any arrangement. If so, you might ask them to relate their formula to the 20th arrangement. This will be helpful for other students who need time to work with specific cases before generalizing in Action 4.



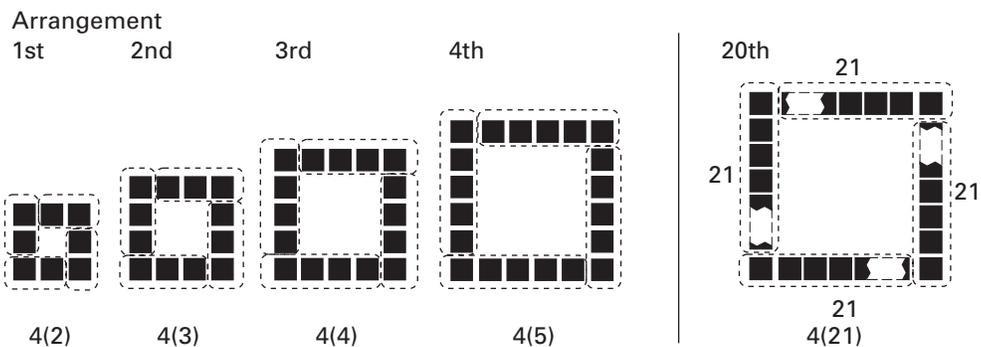
Note that the sections of white space in the strips of tile forming the 20th arrangement on Master I.1 are intended to suggest there are missing tile in the arrangement. That is, by mentally elongating the strip, one can imagine it contains 20 tile.

3 Normally, in a group of 25 to 30 students, there will be several different methods proposed for determining the number of tile in the 20th arrangement. In order to obtain a variety of ways of viewing the arrangements beyond those suggested by students, you can ask the students to devise additional ways, or you can devise other ways. Shown on the two following pages are 5 ways of counting the number of tile in the 20th arrangement.

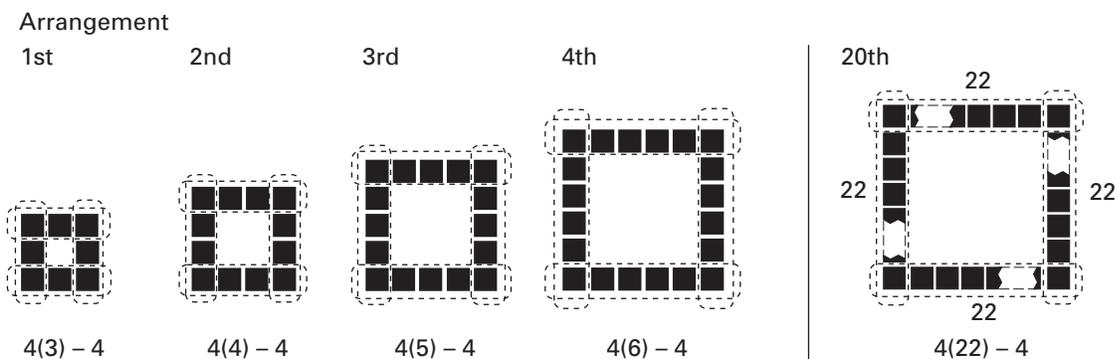
START-UP

ACTIONS

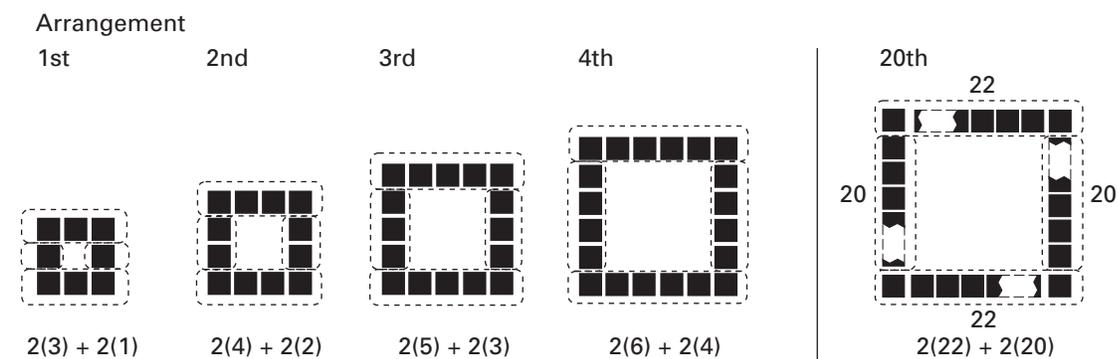
COMMENTS



There are 21 tile on each side, starting at one corner and ending before the next corner.



There are 22 tile on each side, counting each corner twice.



There are 22 tile on the top and the bottom and 20 on each side between the top and the bottom.

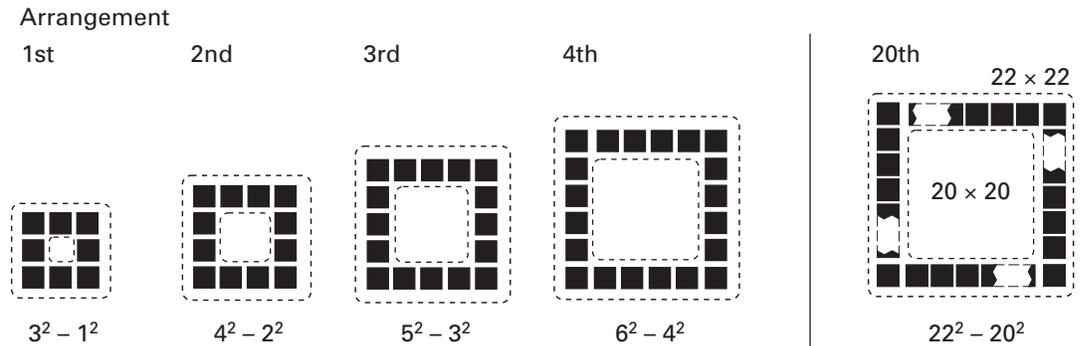
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START-UP

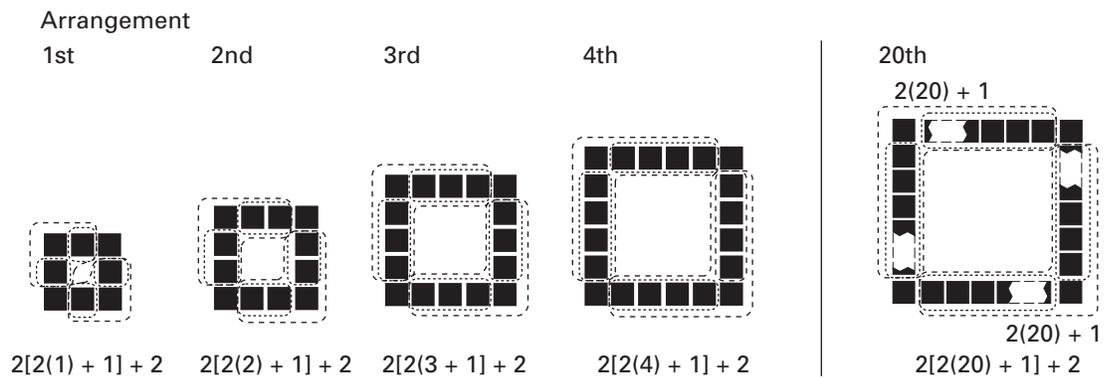
ACTIONS

COMMENTS

3 continued



There is a 22×22 square with a 20×20 square removed from inside it.



There are 2 "L-shapes," each containing 2 times 20 plus 1 tile, and 2 corners.

4 Tell the students that one of the things discovered about the arrangements is that the number of tile in the bottom row of an arrangement contains two more tile than the number of the arrangement and, thus, the following statements are true:

Arrangement 1 contains $1 + 2$ tile in the bottom row.

Arrangement 2 contains $2 + 2$ tile in the bottom row.

Arrangement 3 contains $3 + 2$ tile in the bottom row.

4 The statements can be written one at a time on the overhead as you make them or you can prepare an overhead transparency from Start-Up Master 1.2 and reveal the statements one at a time as you make them.

A *variable* is a letter used to designate an unspecified or unknown number. Variables allow for great economy in mathematical discourse. In this instance, the use of a variable enables one to replace an infinite collection of statements with a single statement.

Initially, some students may not easily grasp the concept of variable. However, as variables become a part of classroom discussion, these students generally come to understand and use them appropriately.


START-UP
ACTIONS

Arrangement 4 contains $4 + 2$ tile in the bottom row.

Arrangement 5 contains $5 + 2$ tile in the bottom row.

Arrangement 6 contains $6 + 2$ tile in the bottom row.

Comment that one could continue making such statements indefinitely.

Point out that the statements all have the same form, namely:

Arrangement $_$ contains $_ + 2$ tile in the bottom row, where the blank is filled by one of the counting numbers, 1, 2, 3, 4,

Tell the students that in mathematical discourse, instead of using a blank, it is customary to use a letter, for example:

Arrangement n contains $n + 2$ tile in the bottom row, where n can be replaced by any one of the counting numbers 1, 2, 3, 4,

Introduce the term *variable*.

5 Distribute a copy of Start-Up Master 1.3 to each student (see following page). For one of the methods of viewing arrangements discussed above, illustrate how the n th arrangement would be viewed for that method. Then write an expression for the number of tile in the n th arrangement. Ask the students to do this for the other methods discussed. For each method, ask for a volunteer to show their sketch and corresponding formula. Discuss.

COMMENTS

5 Shown on the following page are illustrations for the 6 methods described earlier, with corresponding formulas. During the discussion, you can point out notation conventions which may be unfamiliar to the students, such as the use of juxtaposition to indicate multiplication, e.g., $4n$, and the use of grouping symbols such as parentheses to avoid ambiguities, e.g., writing $4(n + 1)$ to indicate that $n + 1$ is to be multiplied by 4 in contrast to writing $4n + 1$ which indicates that n is to be multiplied by 4 and then 1 is to be added to that product.

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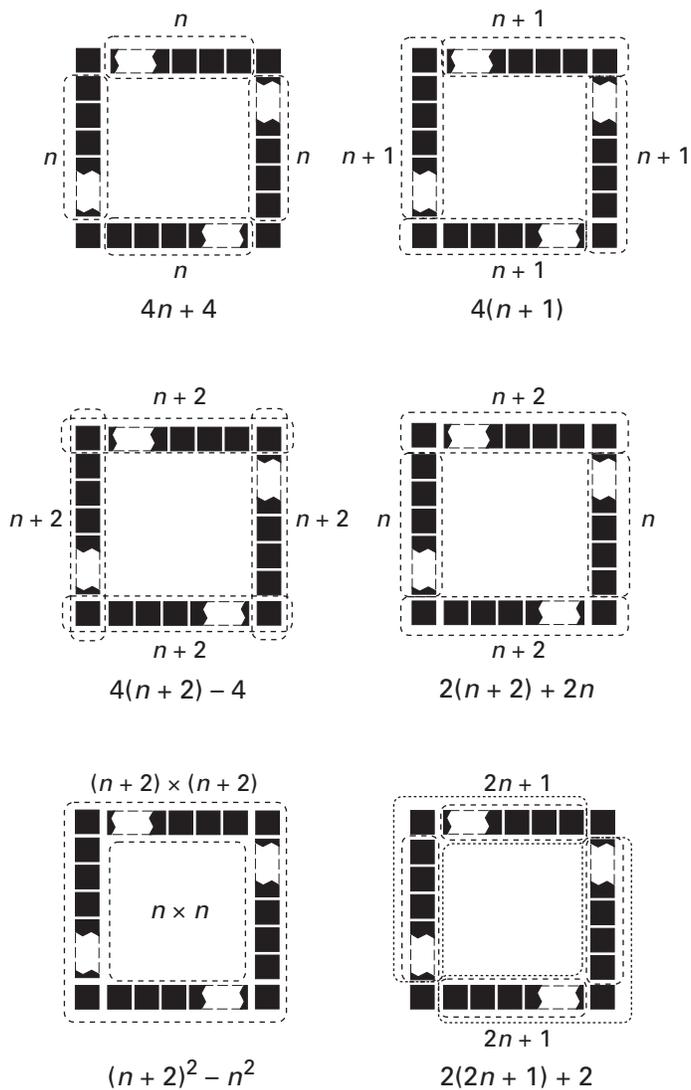
START-UP

ACTIONS

COMMENTS

TILE PATTERNS & GRAPHING
START-UP BLACKLINE MASTER 1.3 LESSON 1

5 continued



If desired, the transparency can be cut apart so that, on the overhead, an n th arrangement can be placed alongside its corresponding 20th arrangement.

START-UP

ACTIONS

6 List all the expressions obtained for the number of tile in the n th arrangement. Discuss *equivalent expressions*.

7 Tell the students that one of the arrangements requires 200 tile to build. Ask them to determine which arrangement this is. Discuss the methods the students use. Relate the students' work to solving equations. Repeat, as appropriate, for other numbers of tile.

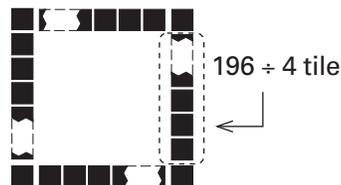
COMMENTS

6 Here are expressions for the number of tiles in the n th arrangement, as illustrated above: $4n + 4$, $4(n + 1)$, $4(n + 2) - 4$, $2(n + 2) + 2n$, $(n + 2)^2 - n^2$, $2(2n + 1) + 2$.

Expressions, such as those listed, which give the same result when evaluated for any possible value of n , are said to be *equivalent*. They are also said to be *identically equal* or, simply, *equal*.

Which form of equivalent expressions is preferable depends upon the situation. For example, when n is 99, the second of the above expressions is easy to evaluate, while when n is 98, the third may be preferable.

7 Various methods can be used. Viewing the arrangement as described in Comment 2 suggests removing the 4 corner tile and dividing the remaining 196 by 4. Thus, it is arrangement number 49 that contains 200 tile. A sketch such as the one shown below may be helpful.



The above line of thought can be given an algebraic cast. The number of tile in the n th arrangement is $4 + 4n$. Thus, one wants the value of n for which $4 + 4n = 200$. Excluding the 4 corner tiles reduces $4 + 4n$ to $4n$ and 200 to 196. Thus, $4n = 196$ and, hence, $n = 49$.

A statement of equality involving a quantity n , such as $4 + 4n = 200$ is called an *equation in n* . Determining the quantity n is called *solving the equation*.

Other ways of viewing the arrangement may lead to other methods of determining its number. For example, viewing the arrangement as described in the first method of Comment 3 may lead to dividing 200 by 4 and noting that the result, 50, is one more than the number of the arrangement. This, in effect, is solving the equation $4(n + 1) = 200$.

continued next page

START-UP

ACTIONS

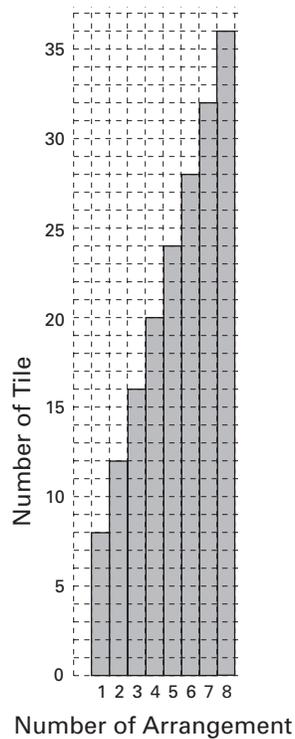
8 Ask the students to convert the sequence of 4 arrangements in Action 1 to portray a bar graph showing the numbers of tile in these 4 arrangements, as illustrated below. Then distribute one sheet of $\frac{1}{4}$ " grid paper to each student and ask the students to draw a bar graph to illustrate the number of tile in the first 8 arrangements of this sequence. Discuss the students' observations about the graph.



COMMENTS

8 By rearranging the tile to portray a bar graph, students are more likely to see the relationship between the sequence and their bar graph.

Shown below is a bar graph showing the number of tile in the first 8 arrangements.



Observations about the graph may be varied. Here are a few examples:

Each bar is 4 squares higher than the previous bar.

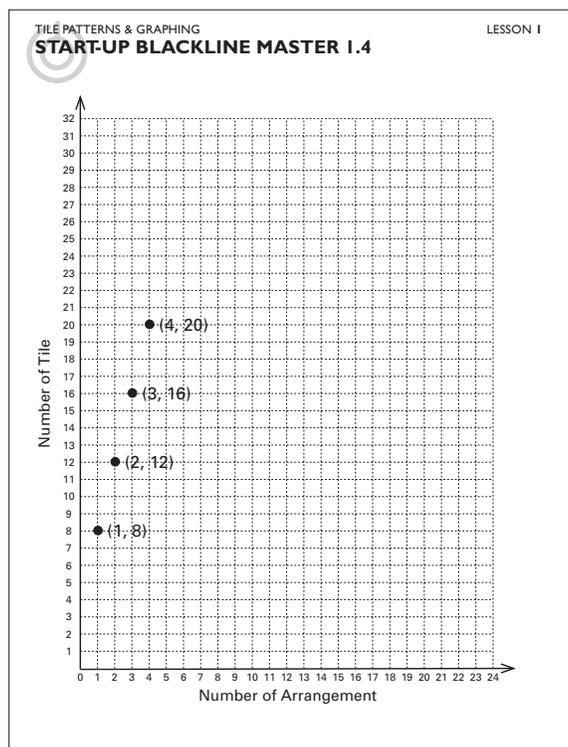
The increase from bar to bar is always the same.

The number of squares in each bar is a multiple of 4, starting with 8.

START-UP

ACTIONS

9 Give each student a copy of Start-Up Master 1.4 and tell them this is a *coordinate graph* of the first 4 arrangements of the sequence. Ask the students to discuss their ideas about how the graph was formed and where on the graph they think points for other arrangements in the sequence would lie. Have volunteers show their ideas on a transparency of Master 1.4.



COMMENTS

9 The coordinates of a point on the graph are an ordered pair of numbers, the first of which tells how many units to count from zero along the horizontal axis (in this case, how many units to count to the right of zero, identifying the arrangement number). The second coordinate in an ordered pair tells how many units to count from zero along the vertical axis (in this case, above zero, identifying the number of tile in the arrangement). It is customary to label the horizontal and vertical axes by the quantities they represent. (Note that grid lines, not spaces, are numbered.)

Some possible observations students may make about the graph include:

The points of the graph lie on a straight line.

The points are equally spaced.

To get from one point to the next, go 1 square to the right and 4 up.

The increase from point to point is always the same.

There are only points on the graph where n is an integer.

Some students may draw a line connecting the points of the graph. Note that, while doing so is okay, it does imply there are arrangements for non-integral values of n . That is, it suggests there are arrangements numbered $2\frac{1}{2}$ or $4\frac{1}{3}$, for example. The students may even suggest ways of constructing such arrangements. However, note that throughout this lesson, each graph is a set of discrete points since n is always viewed as a counting number.

START-UP BLACKLINE MASTER I.1

Arrangement

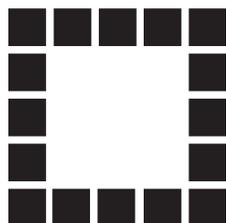
1st

2nd

3rd

4th

20th



Arrangement

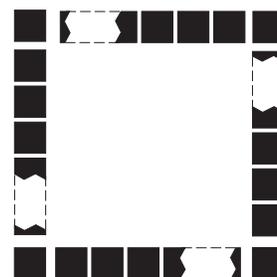
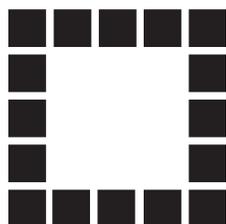
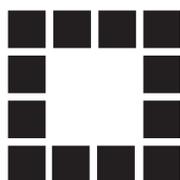
1st

2nd

3rd

4th

20th



Arrangement

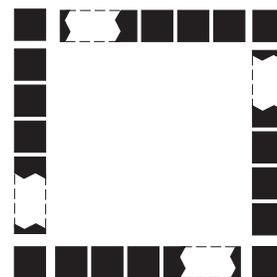
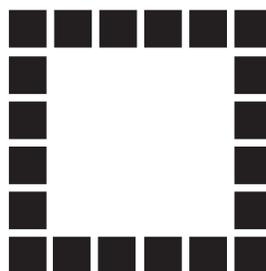
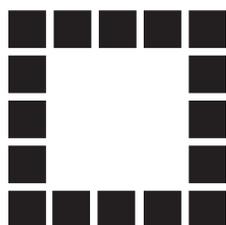
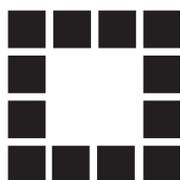
1st

2nd

3rd

4th

20th



**START-UP BLACKLINE MASTER 1.2**

Arrangement 1 contains $1 + 2$ tile in the bottom row.

Arrangement 2 contains $2 + 2$ tile in the bottom row.

Arrangement 3 contains $3 + 2$ tile in the bottom row.

Arrangement 4 contains $4 + 2$ tile in the bottom row.

Arrangement 5 contains $5 + 2$ tile in the bottom row.

Arrangement 6 contains $6 + 2$ tile in the bottom row.

Arrangement 7 contains $7 + 2$ tile in the bottom row.

Arrangement 8 contains $8 + 2$ tile in the bottom row.

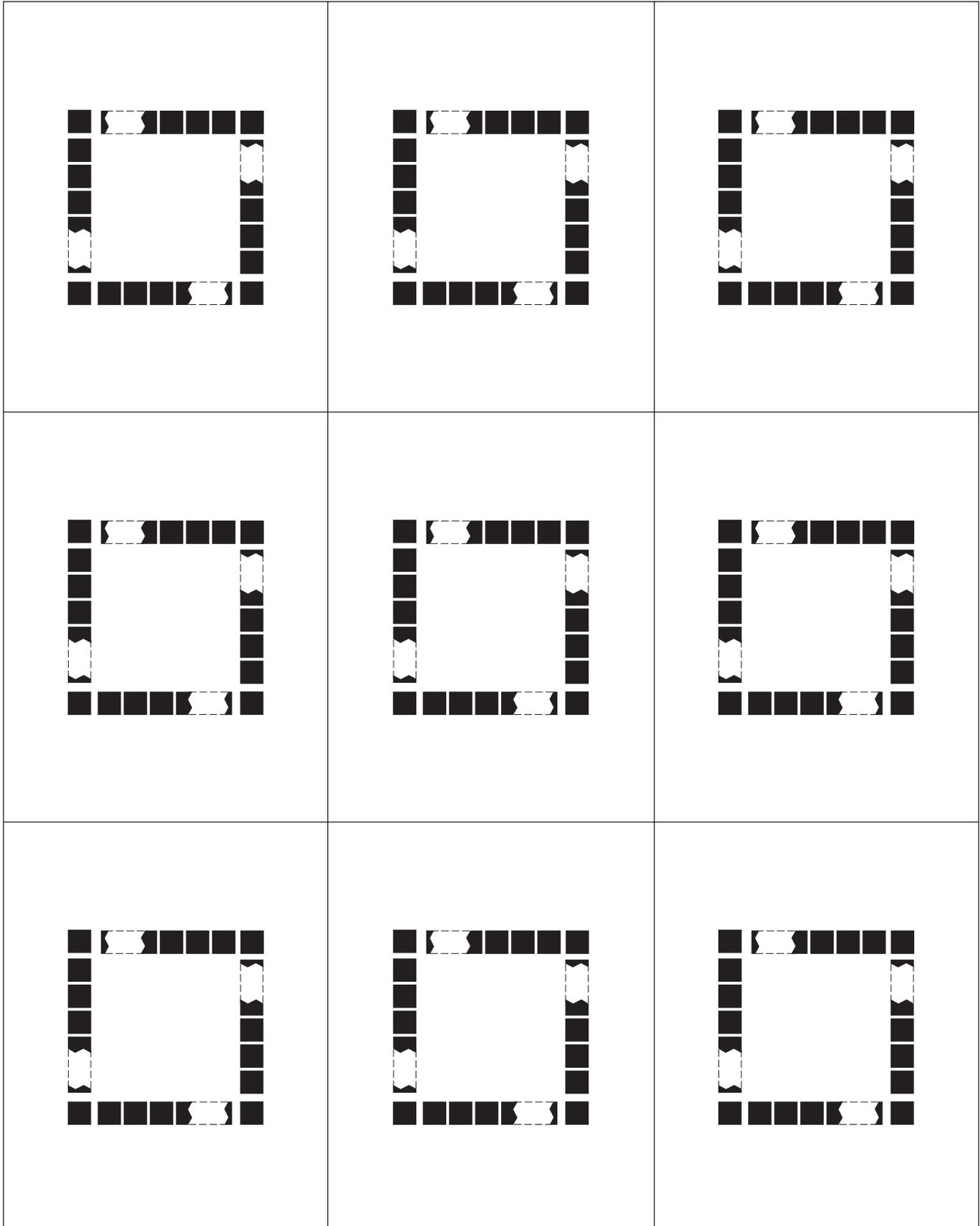
...

Arrangement ____ contains ____ + 2 tile in the bottom row.

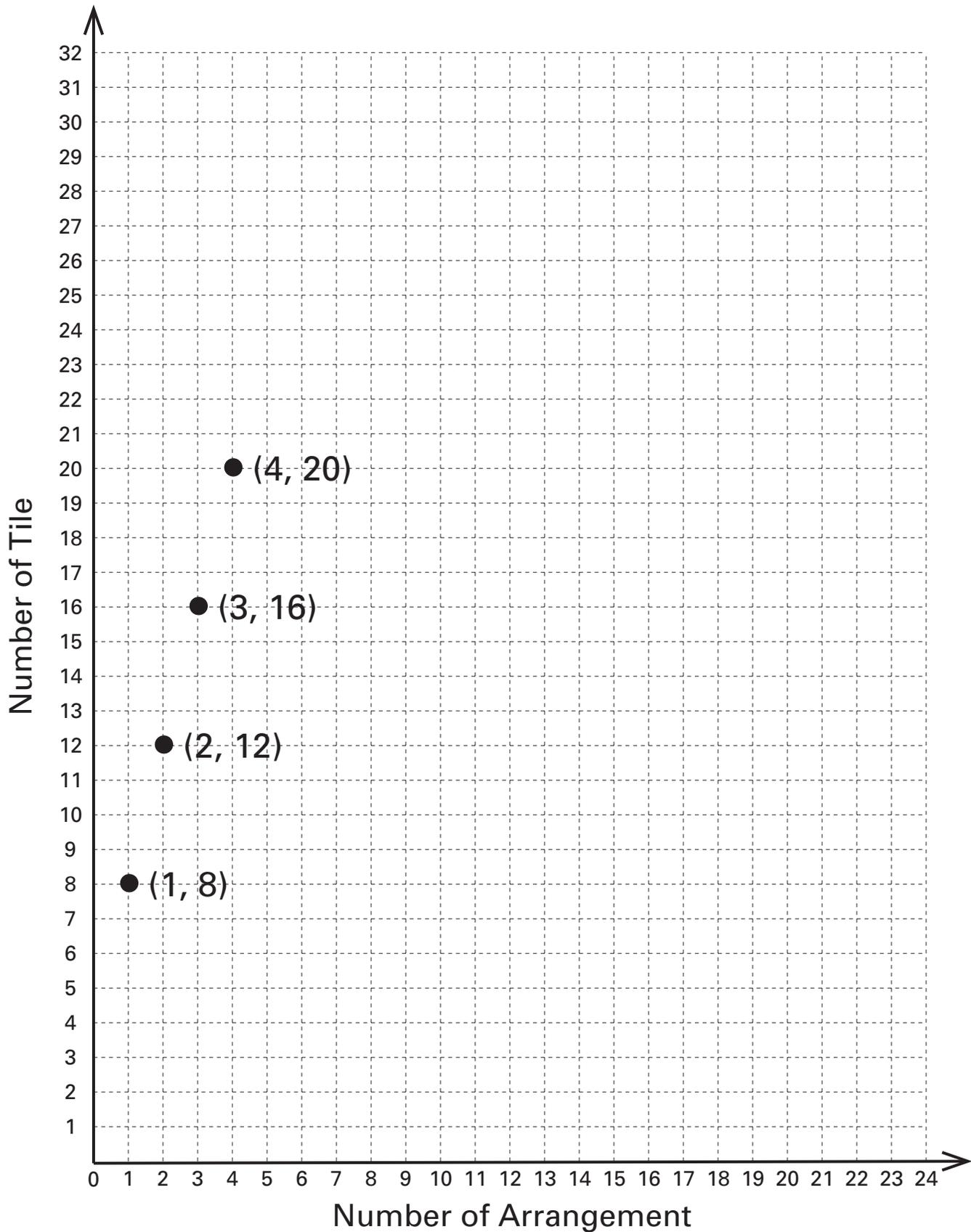
...

Arrangement n contains $n + 2$ tile in the bottom row.

START-UP BLACKLINE MASTER 1.3



START-UP BLACKLINE MASTER 1.4





FOCUS

Overview

Tile patterns are used to generate equivalent expressions, formulate equations, solve equations, and introduce coordinate graphs.

Materials

- Red and black counting pieces, 25 per student.
- $\frac{1}{4}$ " grid paper (see Appendix), 2 sheets per student and 1 transparency.
- Focus Master 1.1-1.2, 1 copy of each per student and 1 transparency.
- Focus Master 1.3, 1 copy per student.
- Black counting pieces for the overhead.

ACTIONS

1 Distribute counting pieces to each student or group of students. Display the following sequence of 4 tile arrangements on the overhead. Have the students form this sequence of arrangements. Then have them form the next arrangement in the sequence.



2 Distribute a copy of Focus Master 1.1 to the students. Ask the students to consider the sequence of arrangements in which the 5th arrangement is the one illustrated in Comment 1. Ask them to determine a variety of ways to view the 20th arrangement and to determine the number of tile required to build it, and to record their methods on Focus Master 1.1 (see following page). Place a transparency of Focus Master 1.1 on the overhead and ask for volunteers to describe their methods.

COMMENTS

1 Here is the most frequently suggested 5th arrangement. Acknowledge other ideas suggested by students.

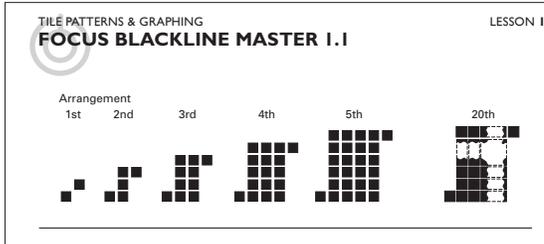


2 The 20th arrangement contains 401 tile. On the following page are 4 different methods of viewing the arrangements. Notice that in Method D some of the tile in the arrangements have been relocated. The students may devise other methods of viewing the arrangements.

continued next page

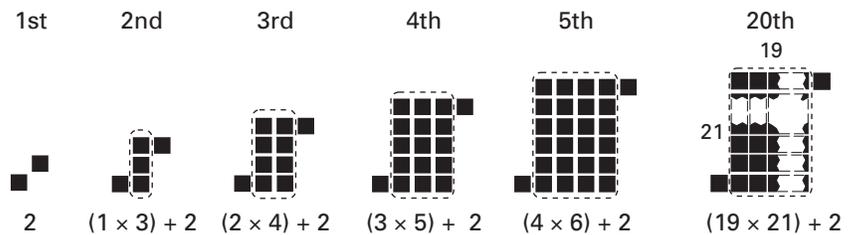
ACTIONS

COMMENTS

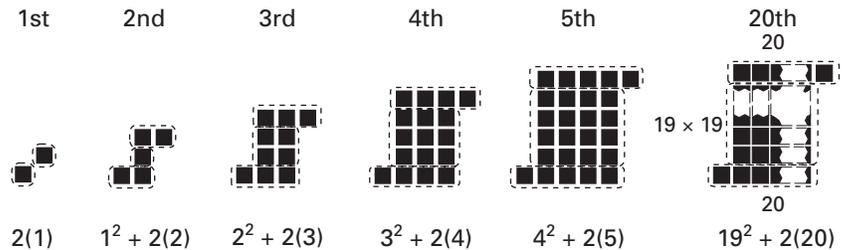


2 continued

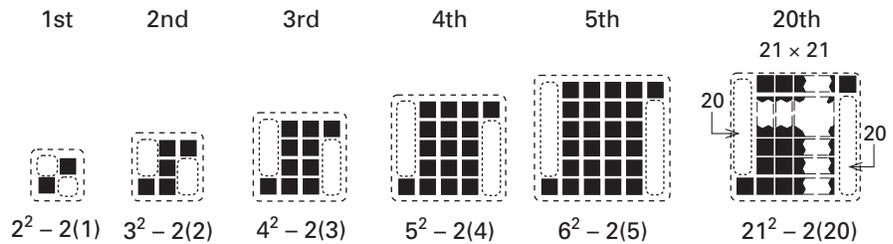
Method A: A 19×21 rectangle with a single tile attached to each of the longer sides.



Method B: A 19×19 square with a row of 20 tile on the top and another row of 20 tile on the bottom.



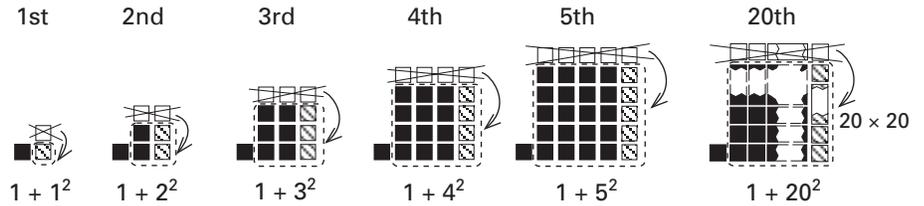
Method C: A 21×21 square with 20 tile removed from the first and last columns.



ACTIONS

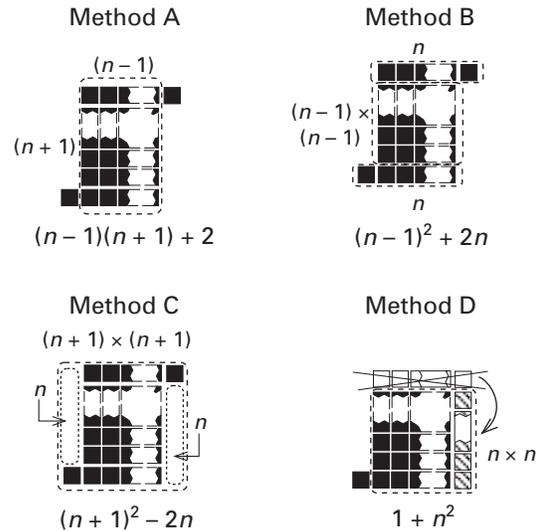
COMMENTS

Method D: A 20×20 square with a single tile attached to the lower left corner.

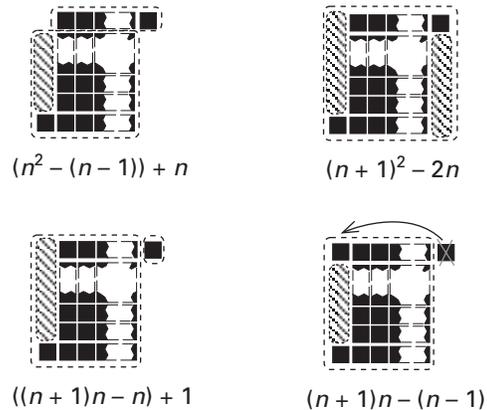


3 Distribute a copy of Focus Master 1.2 to each student. Ask the students to devise methods of viewing the n th arrangement of the sequence. For each method, ask the students to illustrate that method of viewing the arrangement and write a formula for the number of tile in the n th arrangement which reflects that method of viewing the arrangement. Place a transparency of Focus Master 1.2 on the overhead and ask for volunteers to show their methods.

3 Shown below are the n th arrangements corresponding to the 4 ways of viewing the arrangements shown above.



In addition to the methods shown above. Other methods are possible. Shown below are methods which few view the n th arrangement as a configuration from which tile have been removed. The regions from which tile have been removed are shaded.



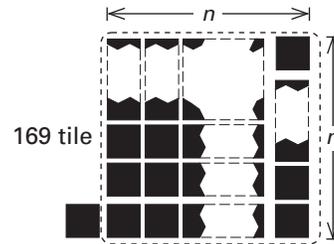
TILE PATTERNS & GRAPHING
FOCUS BLACKLINE MASTER 1.2 LESSON 1


FOCUS
ACTIONS

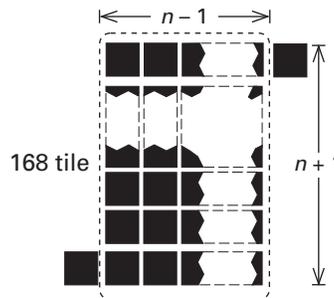
4 Ask the students to determine which arrangement contains 170 tile. Discuss the methods they use.

COMMENTS

4 If the arrangement is thought of in the manner of Method D on the previous page, one of the 170 tile would be attached to a square formed with the remaining 169. The side of this square, 13, is the number of the arrangement. A solution to the equation $n^2 + 1 = 170$ has been found.



Thinking about the arrangement in the manner of Method A on the previous page, 2 of the 170 tile are attached to a rectangle formed by the remaining 168. The dimensions of this rectangle differ by 2. Examining factors of 168, one finds the dimensions are 12 and 14. Since the number of the arrangement is 1 more than the smaller of these numbers (or 1 less than the greater), it is 13. Note that a number n has been found, namely 13, such that $(n - 1)(n + 1) + 2 = 170$.



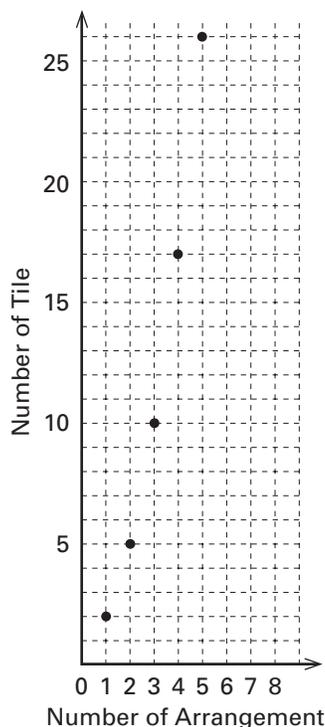

FOCUS
ACTIONS

5 Distribute $\frac{1}{4}$ " grid paper to the students. Ask them to construct and label a coordinate graph which shows the number of tile in the first 5 arrangements. Ask the students for their observations.

6 Distribute a copy of Focus Master 1.3 to each student (see following page). Ask the students to complete parts a) and b). Discuss the students' responses, in particular, ask for volunteers to show the sketches they made in part b). Then ask the students to complete the remaining parts. Discuss their results and the methods used to arrive at them.

COMMENTS

5 A coordinate graph of the first 5 arrangements is shown below. Here are some possible observations.



The points do not lie on a line.

The vertical distance between points increases as the number of arrangement increases.

The vertical distance between points goes up by 2 as we move from point to point; at first, it's 3, then it's 5, then 7, and so forth.

6 Parts of this activity could be assigned as homework.

a) Here is the most frequently suggested 4th arrangement:



continued next page

ACTIONS

COMMENTS

TILE PATTERNS & GRAPHING
FOCUS BLACKLINE MASTER 1.3 LESSON 1

a) Draw the next arrangement in the following sequence:

1st 2nd 3rd 4th

b) How many tile does the 40th arrangement contain? Draw a rough sketch or diagram that shows how you arrived at your answer.

c) Find at least 2 different expressions for the number of tile in the n th arrangement. For each expression, draw a rough sketch or diagram that shows how you arrived at that expression.

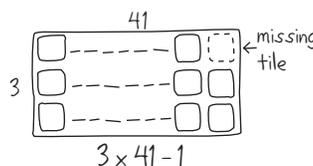
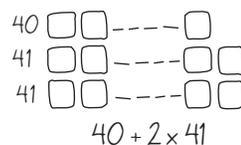
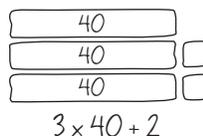
d) Which arrangement contains exactly 500 tile? Draw a rough sketch or diagram of this arrangement.

e) On a sheet of $\frac{1}{4}$ " grid paper, construct and label a coordinate graph showing the number of tile in each of the first 8 arrangements.

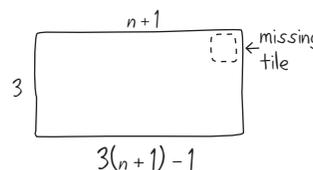
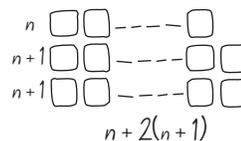
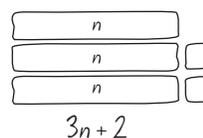
f) (Challenge) Two arrangements together contain 160 tile. One of the arrangements contains 30 more tile than the other. Draw a rough sketch or diagram of these 2 arrangements. Which arrangements are these?

6 continued

b) There are 122 tile in the 40th arrangement. The students may have difficulty drawing simple sketches which depict their method of viewing their arrangement. Initially, they may include more detail than necessary. Here are some possibilities:



c) Here are some sketches of the n th arrangement:

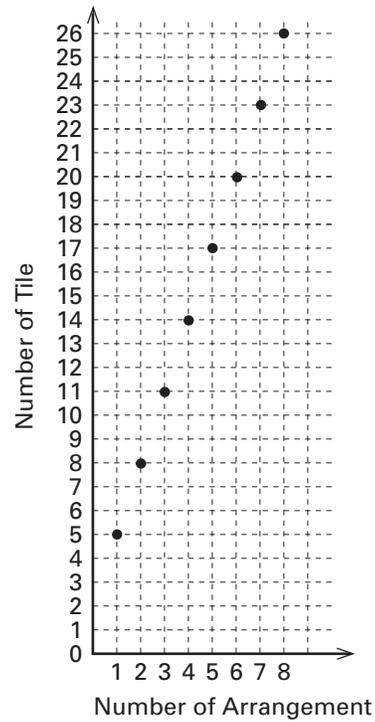


ACTIONS

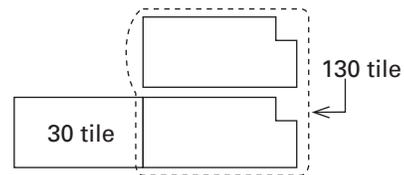
COMMENTS

d) If an arrangement contains 500 tile, the top row contains $498 \div 3$, or 166, tile. The number of tile in the top row is the same as the number of the arrangement.

e) Below is a graph showing the number of tile in each of the first 8 arrangements.



f) As shown in the figure below, the smaller arrangement contains $130 \div 2$, or 65, tile. An arrangement with 65 tile has 21 tile in the top row and hence is the 21st arrangement. The larger arrangement has $30 \div 3 = 10$ more tile in the top row and, hence, is the 31st arrangement.



FOCUS BLACKLINE MASTER I.1

Arrangement

1st

2nd

3rd

4th

5th

20th



Arrangement

1st

2nd

3rd

4th

5th

20th



Arrangement

1st

2nd

3rd

4th

5th

20th



Arrangement

1st

2nd

3rd

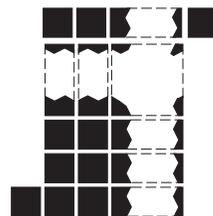
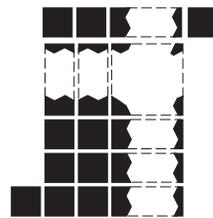
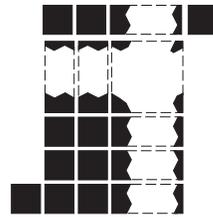
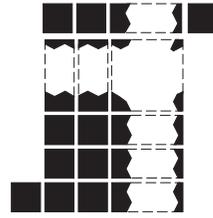
4th

5th

20th



FOCUS BLACKLINE MASTER 1.2



FOCUS BLACKLINE MASTER 1.3

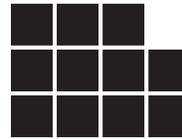
a) Draw the next arrangement in the following sequence:



1st



2nd



3rd

4th

b) How many tile does the 40th arrangement contain? Draw a rough sketch or diagram that shows how you arrived at your answer.

c) Find at least 2 different expressions for the number of tile in the n th arrangement. For each expression, draw a rough sketch or diagram that shows how you arrived at that expression.

d) Which arrangement contains exactly 500 tile? Draw a rough sketch or diagram of this arrangement.

e) On a sheet of $\frac{1}{4}$ " grid paper, construct and label a coordinate graph showing the number of tile in each of the first 8 arrangements.

f) (Challenge) Two arrangements together contain 160 tile. One of the arrangements contains 30 more tile than the other. Draw a rough sketch or diagram of these 2 arrangements. Which arrangements are these?

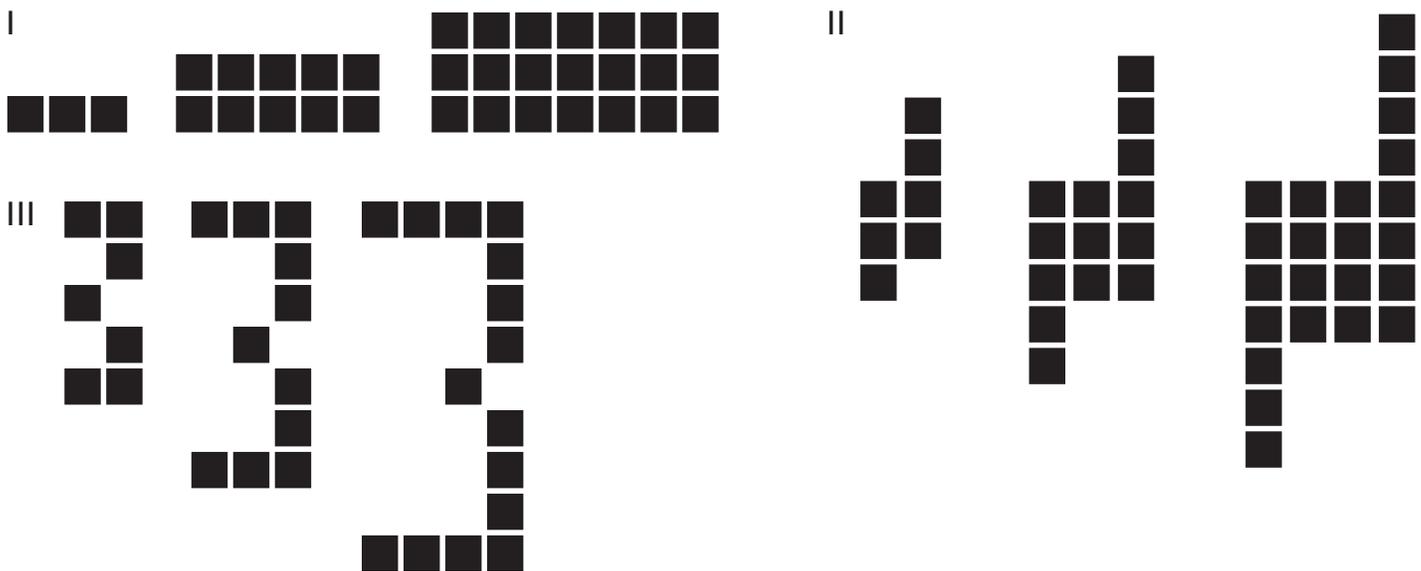
FOLLOW-UP BLACKLINE MASTER I

1 Shown here are the first 3 arrangements in a sequence of tile arrangements.



- a) Describe, in words only, the 50th arrangement so anyone who reads your description could build it.
- b) Determine the number of tile in the 50th arrangement. Draw a rough sketch or diagram that shows how you determined the number.
- c) Find at least 2 different expressions for the number of tile in the n th arrangement. Draw rough sketches or diagrams to show how you obtained these expressions.
- d) On a sheet of $\frac{1}{4}$ " grid paper, draw a graph showing the number of tile in the first several arrangements of the above sequence.

2 Repeat parts b), c), and d) above for each of the following sequences of tile arrangement.



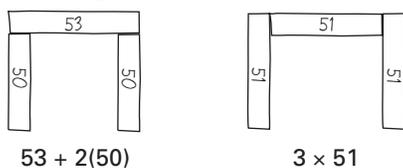
ANSWERS TO FOLLOW-UP I

1 a) Some possible descriptions of the 50th arrangement:

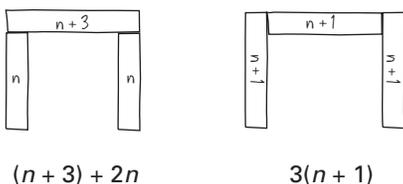
“A row of 53 tile with columns of 50 tile added under the first and last tiles in the row.”

“2 columns of 51 tile with a row of 51 tile added between the top tiles of the 2 columns.”

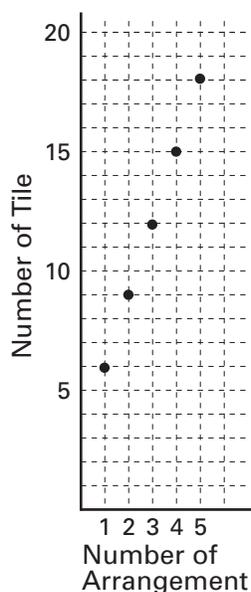
b) The 50th arrangement contains 153 tile.



c) Possible ways of viewing n th arrangement:

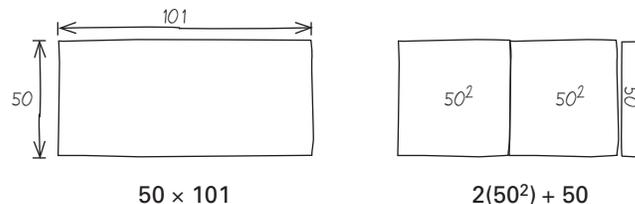


d)

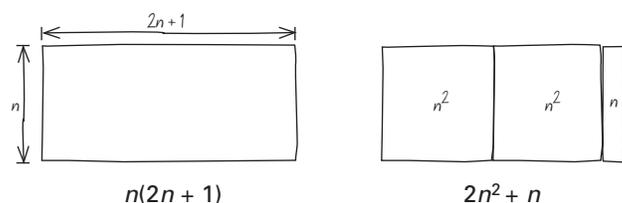


2 | The 50th arrangement contains 5050 tile.

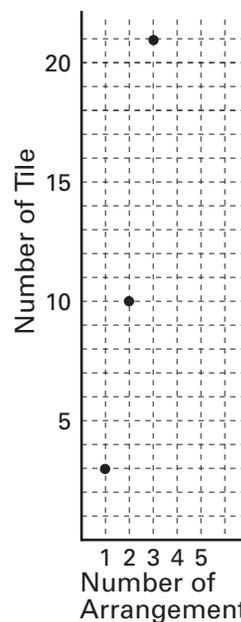
b) Possible ways of viewing 50th arrangement:



c) Possible ways of viewing n th arrangement:



d)



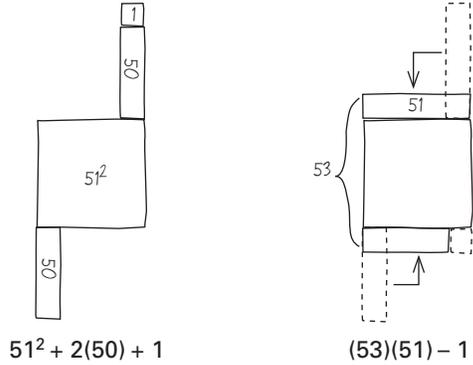
continued

ANSWERS TO FOLLOW-UP I (CONT.)

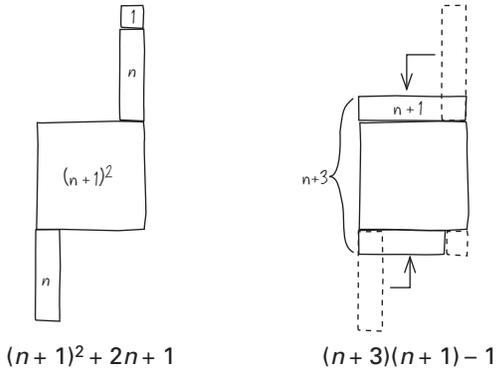
2 continued

II The 50th arrangement contains 203 tile.

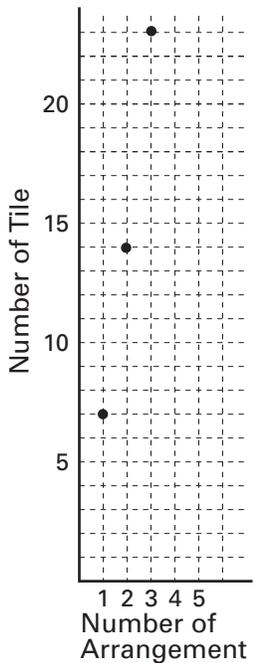
b) Possible ways of viewing 50th arrangement:



c) Possible ways of viewing n th arrangement:

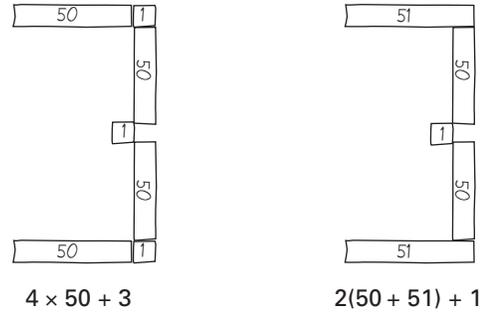


d)

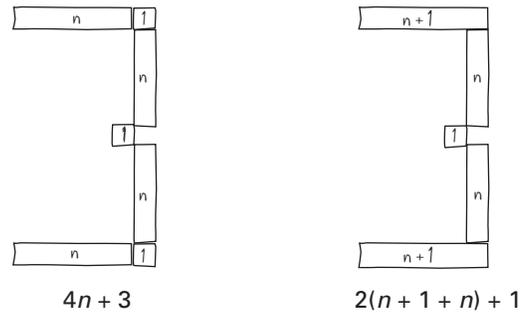


III The 50th arrangement contains 2702 tile.

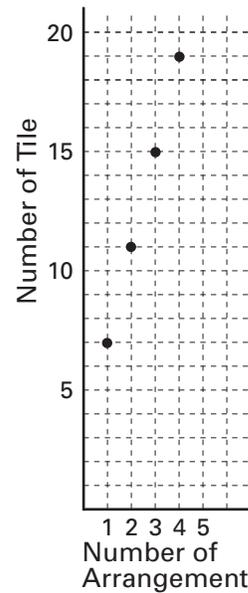
b) Possible ways of viewing 50th arrangement:



c) Possible ways of viewing n th arrangement:



d)



THE BIG IDEA

Collections of red and black counting pieces serve as a model of the integers. By combining, observing, and discussing the counting pieces, students develop mental pictures that help them to understand and retain the meaning of integers and terminology associated with integers. These experiences lay groundwork for understanding arithmetic operations with integers.

START-UP

Overview

Students write equations suggested by questions about collections of tile. They consider whether or not these equations can be solved if only counting numbers are available.

Materials

- Tile for overhead

FOCUS

Overview

Red and black counting pieces are used to introduce signed numbers and provide a model for the integers. Students use the pieces to model situations that involve integers.

Materials

- Red and black counting pieces (see Focus Comment 1), 25 per student.
- Red and black counting pieces for the overhead.
- Focus Master 2.1, 1 copy per student.
- Focus Master 2.2, 1 transparency.

FOLLOW-UP

Overview

Students form or sketch collections of bicolored counting pieces and determine their net values.

Materials

- Follow-Up 2, 2 pages run back-to-back, 1 copy per student.
- Red and black counting pieces, 25 or more per student (see Appendix; you could make a paper or card-stock copy of a quarter sheet of counting pieces for each student to cut out and keep at home).

START-UP

Overview

Students write equations suggested by questions about collections of tile. They consider whether or not these equations can be solved if only counting numbers are available.

Materials

- Tile for overhead

ACTIONS

1 Place several small collections of tile on the overhead as shown.



Count the pieces in the first collection and record the number of pieces it contains under the collection. Repeat for the remaining collections.

Tell the students that the numbers 1, 2, 3, ... we use to count the number of pieces in a collection are called the *counting numbers* or the *natural numbers*. Point out that for every counting number there corresponds a set of tile containing that number of pieces.

2 Place a row of 4 tile on the overhead followed by a row of 7 tile. Beneath the rows write the question, “How many tile must be added to the top row so that it has the same number of tile as the second row?”

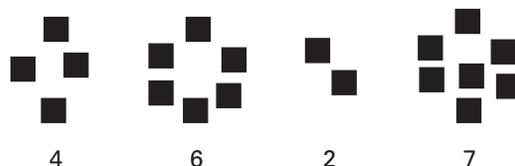


How many tile must be added to the top row so it has the same number of tile as the bottom row?

Ask the students to write an equation whose solution provides an answer to this question.

COMMENTS

1 The collections with the number of pieces in each collection:



The usage of the term “counting number” varies. Some sources include 0 in the counting numbers.

2 The question might be phrased, “What must be added to 4 to get 7?” or, in algebraic form, “What is the solution of the equation $4 + n = 7$?”

The students may suggest another equation, for example, $n = 7 - 4$.

START-UP

ACTIONS

3 Interchange the two rows of tile and raise the same question.



Ask the students to write an equation whose solution provides an answer to this question.

Solicit the students' reactions. Discuss whether or not this equation has a solution.

4 Tell the students that by introducing red tile, which have the effect of negating black tile, one can create a tile model for dealing with the equation $7 + n = 4$, and similar equations. Describe to the students how this will be done.

COMMENTS

3 The students reactions to the question may vary. Some may say it's not possible. Some may say the question doesn't make sense.

One equation is $7 + n = 4$.

Some students who are familiar with negatives, may say that a solution to the equation is -3 . However, if we limit the numbers at our disposal to the natural numbers, there is no solution to the equation.

4 This will be done by turning attention from the number of tile in a collection to the *value* of the tile in a collection, and agreeing that a red tile negates the value of a black tile and conversely. So, instead of asking, "What must be added to a collection of 7 tile to get a collection of 4 tile?" we can ask, "What must be added to a collection whose value is 7 black in order to get a collection whose value is 4 black?" The answer to the latter question is 3 red tile.

The shift from talking about the number of tile to the value of the tile is similar to changing the question "How many coins are in your pocket?" to "What is the value of the coins in your pocket?" There are 6 coins in a collection of 1 penny, 3 nickels, and 2 dimes, but the value of the collection is 36 cents.

Overview

Red and black counting pieces are used to introduce signed numbers and provide a model for the integers. Students use the pieces to model situations that involve integers.

Materials

- Red and black counting pieces (see Focus Comment 1), 25 per student.
- Red and black counting pieces for the overhead.
- Focus Master 2.1, 1 copy per student.
- Focus Master 2.2, 1 transparency.

ACTIONS

I Place the students in groups of 2-4 and give red and black counting pieces to each student. Draw a chart like the one shown below on the overhead or chalkboard. Drop a small handful of counting pieces on the overhead. Discuss the meaning of *net value* and then record the information about this collection on the first line of the chart.

Total No. of Pieces	No. of Red	No. of Black	Net Value



COMMENTS

I Counting pieces are red on one side and black on the other. They can be made from red cardstock using the masters in the Appendix (or bicolored plastic pieces can be purchased from The Math Learning Center). Copy the Counting Piece Master/Front on one side of red cardstock and the Counting Piece Master/Back on the other side; then cut on the lines. One sheet of cardstock will provide enough counting pieces for four students.

Any cardstock or plastic counting piece will appear as a black piece on the overhead; to make red overhead pieces copy the Counting Piece Master/Front on red transparency film and cut on the lines. Note that throughout this lesson a number of new terms are introduced in reference to the red and black counting pieces. It isn't intended or necessary to emphasize memorization of this vocabulary. Rather, terms will become familiar through use during this and the following lessons.

Red and black pieces are said to be of *opposite* color. The *net value* of a collection of counting pieces is the number of red or black pieces in the collection that can not be matched with a piece of the opposite color. A collection in which all pieces can be matched with a piece of the opposite color has a net value of 0.

Collection 1 at left contains 12 pieces, 5 black and 7 red. Its net value is 2 red. Collection 2 at the left contains 8 pieces, 4 red and 4 black. Its net value is 0. This information is recorded in the following table.

Total No. of Pieces	No. of Red	No. of Black	Net Value
12	7	5	2R
8	4	4	0
⋮	⋮	⋮	⋮

FOCUS

ACTIONS

2 Ask a volunteer to take a modest collection of counting pieces (a dozen or so) from their set and drop them on their desktop. Record information about this collection on the chart. Repeat this Action with different students until there are several entries on the chart.

3 Discuss the information contained in the chart. In particular, draw out the students' observations concerning net values. If discussion of each of the following questions is not initiated by students, ask the groups to explore them now:

a) What is the effect of adding/removing an equal number of red and black pieces to/from a collection?

b) For any given nonzero net value, what is the collection with the fewest pieces that has that net value?

c) What is the collection with the fewest pieces that has net value 0?

4 Discuss with the students how plus and minus signs will be used to designate net values.

Total No. of Pieces	No. of Red	No. of Black	Net Value	
12	7	5	2R	-2
8	4	4	0	0
13	4	9	5B	+5
⋮	⋮	⋮	⋮	⋮

COMMENTS

2 You could have a student report the number of pieces of each color in their collection and then ask the class for the net value of the collection.

3 To clarify the intent of these questions, it may help to begin discussion using specific examples. Then ask for generalizations. For example, before posing b), you might ask the following: What are several collections that have a net value of 2 red? What is the collection containing the fewest number of pieces that has this value? If a collection has a net value of 2 red and contains 10 black pieces, how many red pieces are in the collection? What general observations can you make about collections with net value 2?

a) Adding or removing an equal number of red and black pieces from a collection does not change its net value.

b) For a given nonzero net value, the collection with the fewest pieces that has that net value contains either all red pieces or all black pieces. For example, the collection with the fewest pieces that has a net value of 3 red is a collection of 3 red pieces.

c) The collection containing 1 black piece and 1 red piece has net value 0. It is also convenient to say that the empty collection, that is, the collection containing no pieces, has net value 0.

4 Some students may have already suggested using positive and negative numbers to indicate red and black net values. In these materials a minus sign indicates a red net value and a plus sign indicates a black net value. For example, a net value of 3 red is written -3 (read "negative three"); a net value of 2 black is written $+2$ (read "positive two"). Note that the minus and plus signs are written in superscript position.

Numbers to which a plus or minus sign are attached are called *signed numbers*. You might have a volunteer write the appropriate signed number alongside the net values in the chart developed earlier, as shown to the left.

The colors red and black were selected because of the convention in bookkeeping to refer to being "in the red" and "in the black" to mean, respectively, owing more money than one has or having more money than one owes.



ACTIONS

5 Drop a small handful of counting pieces on the overhead. Ask the students for the net value of the resulting collection. Then ask the students for the net value of the collection that would be obtained if all of the counting pieces were turned over. Repeat this Action for two or three other collections.

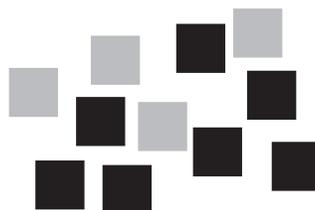
6 Referring to the results of Action 5, introduce the terms *opposite collections* and *opposite net values* to the students.

COMMENTS

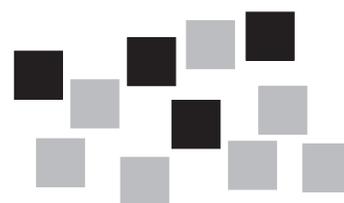
5 Turning over all pieces in a collection changes the sign (or, color) of its net value. Thus, if all pieces in a collection whose net value is +3 (or 3 black) are turned over, the resulting collection will have value -3 (or 3 red). For example, see collections A and B in Comment 6.

6 Two collections are called *opposites* of each other if one can be obtained from the other by turning over all of its pieces. The net values of opposite collections are *opposite net values*.

Collections A and B, shown below, are opposite collections. Their net values, +3 and -3, are opposite net values, that is, +3 is the opposite of -3, and -3 is the opposite of +3.



Collection A
Net Value = +3



Collection B
Net Value = -3

Note that a collection which has the same number of red and black pieces is its own opposite. The net value of such a collection is 0. Thus the opposite of 0 is 0. If this is not suggested by a student, bring it up by placing a collection with net value 0 on the overhead. Ask the students to find its opposite and to determine if there are other numbers that have the same value as their opposite (there are none).

FOCUS

ACTIONS

7 Distribute a copy of Focus Master 2.1 to each student or pair of students. Ask the students to use their counting pieces (or to imagine collections of pieces) to help them fill in the missing numbers. Discuss the methods students used to arrive at their answers.

COMMENTS

7 You may need to remind the students that a minus sign indicates a red net value and a plus sign indicates a black net value.

Some students may arrive at correct answers by imagining the pieces. Urge students who have difficulty to form each collection of pieces.

POSITIVE & NEGATIVE INTEGERS LESSON 2
FOCUS BLACKLINE MASTER 2.1

1 Fill in the missing numbers:

	Total No. of Pieces	No. of Red Pieces	No. of Black Pieces	Net Value
a)		6		+3
b)		2	7	
c)	12		5	
d)		4		0
e)	10			0
f)	7			+7
g)			8	+3
h)	13			-5

2 Suppose that:
 Collection X contains 2 red and 7 black pieces;
 Collection Y contains 8 red and 5 black pieces; and
 Collection Z contains 7 red and 3 black pieces.

a) Record the net value of collection X: ____, Y: ____, Z: ____.
 b) Record the net value if collections X and Y are combined: ____.
 c) Record the net value if collections Y and Z are combined: ____.
 d) Record the net value if collection X and the opposite of collection Y are combined: ____.

8 Point out the fact that the net values of collections of counting pieces when expressed as signed numbers serve as a model of the set of numbers called the *integers*. Discuss the students' ideas about which numbers are in the set of integers. Point out that the positive integers can be identified with the natural numbers.

8 The collection of black net values, +1, +2, +3, ..., represents the positive integers and the collection of red net values, -1, -2, -3, ..., represents the negative integers. A 0 net value represents the zero integer.

The set of positive integers, +1, +2, +3, ..., can be identified with the set of natural numbers, 1, 2, 3, ... , sometimes also referred to as the counting numbers. (In some textbooks, 0 is included in the counting numbers.) Consequently, the + sign is often omitted when referring to a positive integer, i.e., 3 is written in place of +3. The identification of the positive integers with the natural numbers works because the number of tile in an all-black collection is numerically equal to its net value.

ACTIONS

9 Place a transparency of Focus Master 2.2 on the overhead and ask the students to use red and black counting pieces to model Situation a). Have volunteers share their models at the overhead and discuss questions which could be answered using their models. Repeat for one or more of b)-d).

COMMENTS

9 Integers can be used to describe situations that involve “opposite words” such as, above and below, before and after, north and south, gaining and losing, etc. For example, instead of saying “The high temperature for the day was 18 degrees above zero and the low was 7 degrees below zero,” one can use integers and say, “The high temperature was 18 degrees and the low was -7 degrees.”

Red and black counting pieces can be used to model situations involving integers and reveal answers to many questions about those situations. For example, a model of the example given in the preceding paragraph might suggest the question, “What was the difference between the high and low temperature?”

This action and Action 10 are intended to develop readiness for addition and subtraction of integers, which are explored in Lesson 6. Some questions prompted by a model of each situation are listed below:

a) A model of this situation is shown here:



What was the net yardage gained or lost during the two plays?
How many more yards were gained than lost?

b) Does Kerry have enough to buy her new book? What was Kerry’s net gain or loss for the month of November?

c) How many floors are in the building? What floor did Mel start on? Did Mel travel farther going up or going down? How much farther?

d) Students may wish to carry out a short experiment. Some may use red and black pieces to record their points lost or gained. Other students may suggest replacing the 3 green markers with 3 red counting pieces and the other 3 markers with 3 black counting pieces, based on the points assigned to each marker. Some questions they may pose include: What is the theoretical probability Celia will lose/win one point on any single draw? What are the fewest draws it would take to earn 7 points? Is it possible to earn 7 points in an even number of draws? an odd number? (One student suggested this as a great game to give children they are baby-sitting, since theoretically the game would never end.)

POSITIVE & NEGATIVE INTEGERS LESSON 2

FOCUS BLACKLINE MASTER 2.2

Use black and/or red counting pieces to model each of the following situations:

a) During two plays, the football team gained 5 yards and lost 3 yards.

b) During the month of November, Kerry earned \$17 baby-sitting, spent \$13 on a new shirt, received a gift of \$5 from her grandma, paid \$7 for her little brother’s birthday present. The new book she wants to buy costs \$4.

c) Mel the elevator operator decided to keep track of his elevator trips during one half hour period last Monday. When he started keeping track, he wasn’t on the ground floor. Here is a list of his elevator trips that half hour: up 3 floors, down 2 floors, up 5 floors to the top floor, down 4 floors, up 2 floors, up 1 floor, down 3 floors, and down 4 more floors where he ended on the ground floor.

d) Celia invented a new game: place 3 green, 1 red, 1 blue, and 1 yellow game marker in a bag; randomly draw one marker, record its color, replace the marker, and shake the bag; repeat until you earn 7 points (for each green marker you lose 1 point and for each red, blue, or yellow marker you earn 1 point).



FOCUS BLACKLINE MASTER 2.1

1 Fill in the missing numbers:

	Total No. of Pieces	No. of Red Pieces	No. of Black Pieces	Net Value
a)		6		+3
b)		2	7	
c)	12		5	
d)		4		0
e)	10			0
f)	7			+7
g)			8	+3
h)	13			-5

2 Suppose that:

Collection X contains 2 red and 7 black pieces;

Collection Y contains 8 red and 5 black pieces; and

Collection Z contains 7 red and 3 black pieces.

- a) Record the net value of collection X: _____, Y: _____, Z: _____.
- b) Record the net value if collections X and Y are combined: _____.
- c) Record the net value if collections Y and Z are combined: _____.
- d) Record the net value if collection X and the opposite of collection Y are combined: _____.

**FOCUS BLACKLINE MASTER 2.2**

Use black and/or red counting pieces to model each of the following situations:

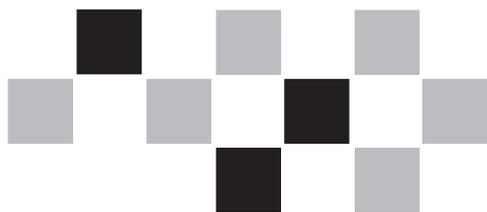
- a) During two plays, the football team gained 5 yards and lost 3 yards.
- b) During the month of November, Kerry earned \$17 baby-sitting, spent \$13 on a new shirt, received a gift of \$5 from her grandma, paid \$7 for her little brother's birthday present. The new book she wants to buy costs \$4.
- c) Mel the elevator operator decided to keep track of his elevator trips during one half hour period last Monday. When he started keeping track, he wasn't on the ground floor. Here is a list of his elevator trips that half hour: up 3 floors, down 2 floors, up 5 floors to the top floor, down 4 floors, up 2 floors, up 1 floor, down 3 floors, and down 4 more floors where he ended on the ground floor.
- d) Celia invented a new game: place 3 green, 1 red, 1 blue, and 1 yellow game marker in a bag; randomly draw one marker, record its color, replace the marker, and shake the bag; repeat until you earn 7 points (for each green marker you lose 1 point and for each red, blue, or yellow marker you earn 1 point).



FOLLOW-UP BLACKLINE MASTER 2

Counting pieces will be helpful for this activity. Remember that a minus sign indicates a red net value and a plus sign indicates a black net value.

1 Sketch the collection that is the opposite of Collection A, shown below. Write the net value of both collections.



Collection A

2 Shown below are the red pieces from a collection whose net value is 5.

Sketch the collection.



3 A certain counting piece collection has 12 pieces, 9 are red. Sketch the collection and write its net value.

4 A certain counting piece collection has 7 black pieces and a net value of 0.

Sketch the collection.

continued on back

 FOLLOW-UP BLACKLINE MASTER 2 (CONT.)

5 A collection has 13 pieces and a net value of -3 . Sketch the collection and explain how you determined the number of red pieces in the collection.

6 Sketch 3 different collections with net value -4 .

7 Suppose collections A, B, and C contain the following counting pieces:

Collection A contains 3 red and 9 black pieces.

Collection B contains 11 red and 5 black pieces.

Collection C contains 8 red and 4 black pieces.

a) What is the net value of the collection obtained by combining Collections A and B?

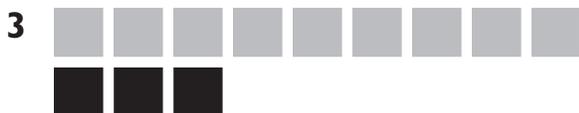
b) What is the net value of the collection obtained by combining Collection B and the opposite of Collection C?

ANSWERS TO FOLLOW-UP

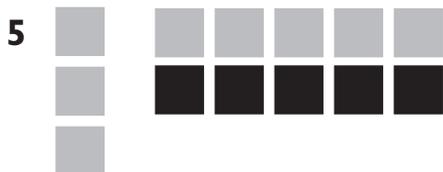
1  Collection A has net value -3 ;
its opposite has net value 3 .



Opposite of Collection A



The net value is -6 .



If 3 of the pieces are red, the remaining 10 pieces must have net value 0. Hence, 5 are red and 5 are black, so altogether there are 8 red pieces.



7 a) 0 b) -2

INTEGER ADDITION & SUBTRACTION

THE BIG IDEA

Visual models are used to embody the addition and subtraction of integers. Emphasis is placed on conceptual understanding rather than the memorization of rules.

START-UP

Overview

Students use tile to model adding and subtracting counting numbers.

Materials

- Red and black counting pieces, 25 for each student.
- Red and black counting pieces for the overhead.

FOCUS

Overview

Counting pieces are used to find sums and differences of integers and to provide insights into these operations.

Materials

- Red and black counting pieces, 25 for each student.
- Red and black counting pieces for the overhead.

FOLLOW-UP

Overview

Students use red and black counting pieces to compute sums and differences of integers and to solve problems.

Materials

- Follow-Up 3, 1 copy per student.
- Red and black counting pieces for student use at home.

START-UP

Overview

Students use tile to model adding and subtracting counting numbers.

Materials

- Red and black counting pieces, 25 for each student.
- Red and black counting pieces for the overhead.

ACTIONS

1 Distribute tile to the students. Tell the students that only black tile will be used in this and the next few actions. Write the expression “ $9 + 5$ ” on the overhead or board and ask the students to use their tile to show what this expression means to them. Discuss.

2 Write the expression “ $9 - 5$ ” on the overhead or board and ask the students to use their tile to show what this expression means to them.

3 Discuss the *take-away* and *difference* methods of subtraction. Give examples, or ask the students to supply examples, of word problems which fit these methods.

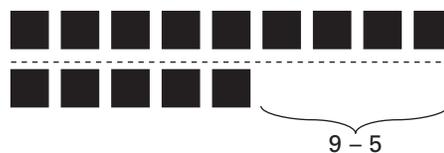
COMMENTS

1 Many students will count out a collection of 9 tile and add 5 tile to it, although some may think of “ $9 + 5$ ” as 9 added to 5, and start with a collection of 5 tile and add 9 tile to it. During the discussion you can ask the students what they would have done differently if the expression had been “ $5 + 9$ ”.

2 Many students will form a group of 9 tile and take 5 tile from this group. Other students may form groups of 9 and 5 tile and determine what must be added to the latter to obtain the former.



Take-away method



Difference method

3 Carrying out $9 - 5$ by taking 5 tile from a collection of 9 tile models the *take-away* method of subtraction. This model fits the question: “If a student has 9 pencils and gives 5 of them away, how many pencils does the student have left?”

Determining $9 - 5$ by finding how many tile must be added to a group of 5 tile to match a group of 9 tile models the *difference* method of subtraction. This model fits the question: “If a student has 9 pencils and another has 5, how many more pencils does the first student have than the second?”

START-UP

ACTIONS

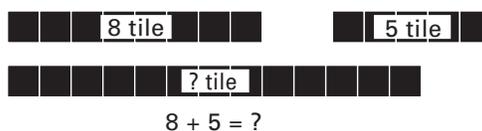
4 Discuss how the operations of addition and subtraction are related.

COMMENTS

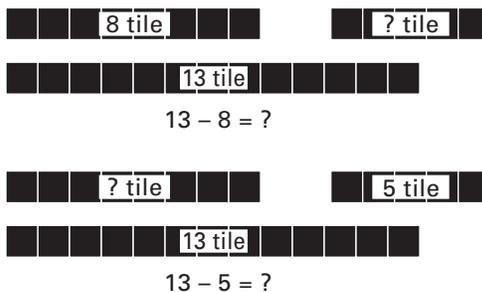
4 Suppose there are 3 groups of tile, A, B and C, such that the number of tile in A and B, when combined, equal the number of tile in C.



For addition, one knows the number of tile in A and B and seeks the number of tile in C.



For subtraction, one knows the number of tile in C and one of the groups A or B, and seeks the number of tile in the other group.



Notice that every addition statement is associated with two subtraction statements.

FOCUS

Overview

Counting pieces are used to find sums and differences of integers and to provide insights into these operations.

Materials

- Red and black counting pieces, 25 for each student.
- Red and black counting pieces for the overhead.

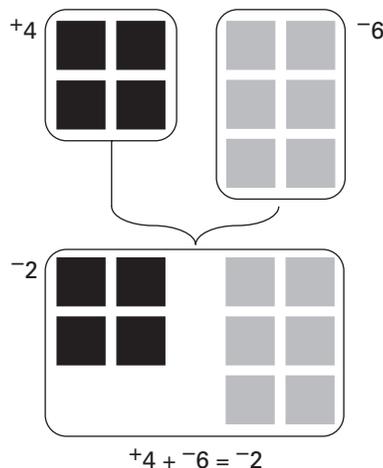
ACTIONS

1 Distribute counting pieces to each student. Ask the students to suggest ways in which counting pieces can be used to determine the sum of two integers, for example, $+4 + -6$.

COMMENTS

1 The students may have a variety of suggestions. One way to use counting pieces to determine the sum $+4 + -6$ is to combine a collection whose net value is $+4$ with a collection whose net value is -6 , and then find the net value of the combined collection.

The collections for $+4$ and -6 shown below are those that contain the fewest number of pieces. Other collections with the same net values could be used. Note that the combined collection has a net value of -2 . Hence, $+4 + -6 = -2$.



In combining collections, some students may remove pairs of red and black pieces, ending up with a collection of 2 red pieces.

2 Have the students use counting pieces to determine the following sums:

- a) $+5 + -5$
- b) $-4 + -5$
- c) $+2 + -6$
- d) $+4 + -4$

2 Some students may arrive at answers without physically manipulating counting pieces. If this happens, you can ask them how they arrived at their answers to see if they understand the counting piece model. Or, ask them to show how their methods relate to the counting piece model. If needed, have students compute other sums with the counting pieces.

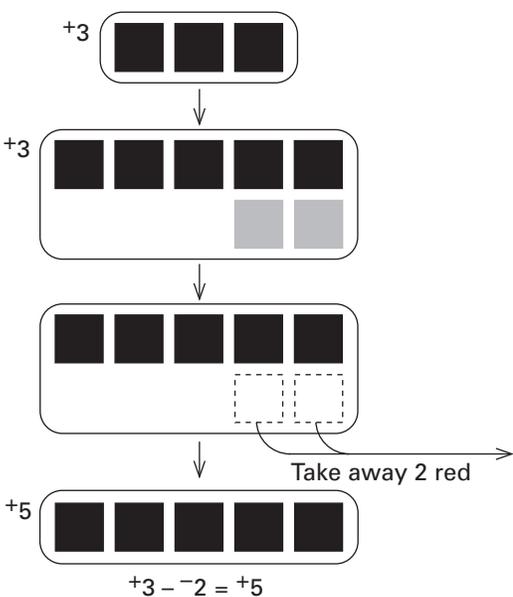
FOCUS

ACTIONS

3 Have the students imagine counting pieces to help them mentally compute the following sums:

- a) $-25 + -40$
- b) $-35 + +50$
- c) $-60 + +52$

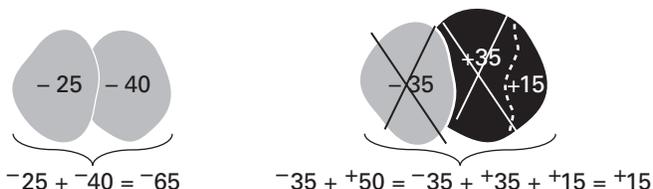
4 Write “ $+3 - -2$ ” on the overhead or chalkboard. Ask the students to suggest ways in which counting pieces can be used to compute the value of this expression.



COMMENTS

3 Because of the magnitude of the numbers, finding these sums using counting pieces is impractical. However, in finding the sums, one can think in terms of counting pieces. For example, to find the sum in b), one may think of combining a collection of 35 red pieces with a collection of 50 black pieces. The combined collection will have 15 more black pieces than red pieces. Hence, its net value is 15 black or $+15$. Thus $-35 + +50 = +15$.

When working with large numbers, some students may find making rough sketches, such as the following, helpful to their thinking:



As needed, ask the students to find the sums of additional pairs of integers. Keep emphasis on using the counting pieces to explain how to find the answers. Doing this helps students to develop intuitions about the sums of integers and later enables them to invent personally meaningful rules.

4 There are a variety of ways in which this can be done. One way is to take away a collection whose net value is -2 (i.e., 2 red pieces) from a collection whose net value is $+3$. In order to do this, one must build an appropriate collection for $+3$. The collection with the fewest pieces for $+3$ has 3 black pieces. It has no red pieces to remove. However, adding 2 black and 2 red pieces to this collection does not change its net value and results in a collection of 5 black and 2 red. Taking 2 red from this collection leaves 5 black. Hence, $+3 - -2 = +5$.

Note that adding 2 black and 2 red pieces to a collection and then removing the 2 red has the net effect of adding 2 black. Thus, subtracting -2 from $+3$ is equivalent to adding the opposite of -2 to $+3$. It isn't necessary to point this out at this time. Action 5 is intended to draw out such observations.

This computation can also be done using the difference method for subtraction. To do this, lay out collections for $+3$ and -2 and observe how the net value of the second differs from that of the first. If, as shown on the next page, $+3$ is represented by a collection of 3 black and -2 by a collection of 2 red, this difference is not immediately apparent. However, adding 2 black and 2 red to the first collection does not change its net value, and it then



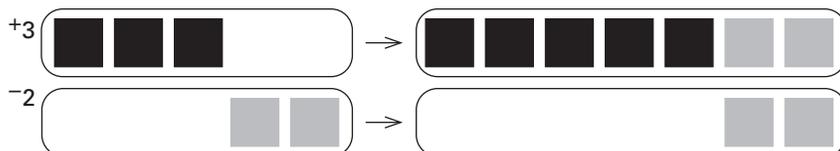
ACTIONS

COMMENTS

5 Ask the students to use counting pieces to determine the following differences. Discuss the strategies the students use and the various methods for subtracting integers.

- a) $+8 - +3$
- b) $-7 - -2$
- c) $-4 - +7$
- d) $+2 - -5$
- e) $-4 - -4$

becomes clear that the difference in the value of the second collection from the value of the first is 5 black or $+5$.



The difference is 5 black. Hence, $+3 - -2 = +5$.

A third method (the missing addends method) is to determine what must be added to the second collection in order that the 2 collections have the same value. Since adding $+5$ to -2 yields $+3$, the difference between the values of the collections is $+5$, or $+3 - -2 = +5$.

5 It is helpful to remember that there are many collections with the same value. For example, if one is going to use either the take-away or difference method to determine $+2 - -5$, it is helpful to represent $+2$ with a collection that contains 5 red:



Taking away 5 red from this collection leaves 7 black:



Or, alternatively the difference between this collection and a collection of 5 red is 7 black:



In either case, $+2 - -5 = +7$.

continued next page

FOCUS

ACTIONS

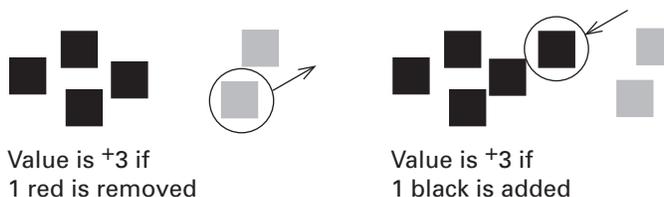
6 Have the students imagine counting pieces to help them mentally compute the following problems. Discuss their mental strategies.

- a) $-25 - -50$
- b) $+80 - +73$
- c) $+70 - -35$
- d) $-45 - -40$

COMMENTS

5 continued

Another method of subtraction in the integers is based on the observation that a change in the value of a collection brought about by removing a piece is the same as that accomplished by adding to the collection the opposite of the piece that might otherwise be removed. For example:



Thus, instead of determining $+2 - -5$ by removing 5 red pieces from a collection whose value is $+2$, $+2 - -5$ can be determined by adding 5 black to a collection whose value is $+2$, that is, $+2 - -5 = +2 + +5 = +7$. Similarly, in c), $-4 - +7 = -4 + -7 = -11$.

In general, in the integers, the subtraction $a - b$ can be accomplished by adding the opposite of b to a , that is $a - b = a + \text{opp}(b)$. Hence, in the integers, every subtraction can be turned into an addition.

6 As in Action 3, using counting pieces to perform these computations is impractical. However, it is useful to think in terms of counting pieces. For example, to compute a), one wants a collection whose net value is -25 from which one can take 50 red. One can obtain such a collection by adding 25 red and 25 black to a collection of 25 red, resulting in a collection of 50 red and 25 black. Taking 50 red from this collection leaves 25 black. Hence, $-25 - -50 = +25$.

Alternately, one could think as follows: If a first collection has 25 red and a second collection has 50 red, the second set would have the same value as the first set if 25 black were added to it. Hence, the difference *in value* of the second set from that of the first is $+25$, that is, $-25 - -50 = +25$.



**FOLLOW-UP BLACKLINE MASTER 3**

1 Use black and red pieces to find the following sums and differences.

a) $+7 + -4$

b) $-3 + -8$

c) $-9 + +5$

d) $-6 + -6$

e) $+8 - +5$

f) $-5 - +6$

g) $+4 - -7$

h) $-3 - +2$

2 For each of the following, imagine counting piece collections and compute the answers.

Write a sentence or two or draw a sketch that explains how you got your answer.

a) $+45 + -25 =$

b) $80 - +55 =$

c) $-35 - +75 =$

d) $+100 + -42 =$

3 Use black and red pieces or a sketch to model each of these situations. Determine the requested information and describe the method you used to arrive at your answer.

a) The second-grade store's profit/loss statement for last week showed the following: Monday, profit, \$3; Tuesday, loss, \$5; Wednesday, loss, \$7; Thursday, broke even; Friday, profit, \$6. The second-graders had set a goal of earning \$45 profit for the week. Determine how close they are to their goal.

b) Between midnight and noon a submarine cruised at -200 feet, then dove down 150 feet further, climbed up 115 feet, dove 180 feet, and finally climbed up 100 feet. Determine the difference between the highest and lowest positions of the submarine during this 12-hour period.

ANSWERS TO FOLLOW-UP

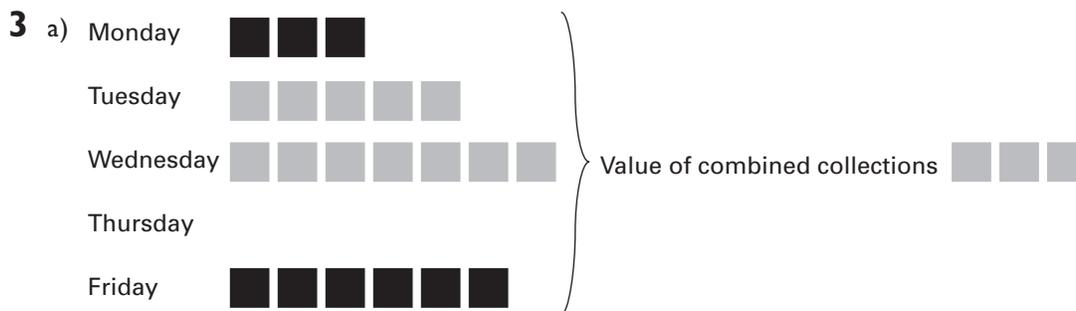
- 1 a) 3 b) -11 c) -4 d) -12 e) 3 f) -11 g) 11 h) -5

2 a) If 45 black pieces are combined with 25 red pieces and pairs of black and red pieces are removed, there will be 20 black pieces left. This shows that $+45 + -25 = 20$.

b) If 55 red pieces are removed from a collection of 80 red pieces, 25 red pieces remain. Thus, $-80 - -55 = -25$

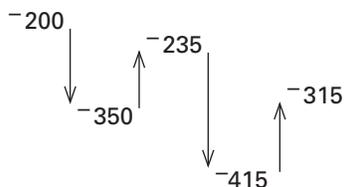
c) Begin with 35 red pieces. Add 75 red and 75 black, keeping the net value of the collection -35. Then remove 75 black to get a final collection with net value -110.

d) If collections of 100 black and 42 red are combined and pairs of black and red pieces are removed, 58 black pieces will remain. Hence, $+100 + -42 = +58$.



They lost \$3 for the week. They missed their goal by \$48.

- b) The submarine cruised at these levels:



The difference between the highest (-200) and lowest (-415) depths is $-200 - (-415) = 215$ feet.

INTEGER MULTIPLICATION & DIVISION

THE BIG IDEA

The use of red and black counting and edge pieces and the concept of net value allow the extension of the area methods of multiplication and division to integers. When students can “see” relationships between the signs of the factors and the sign of a product, or among the signs of the dividend, divisor, and quotient, they do not need to be given rules about the use of signs. Rather, they can generalize from their experiences and mental images.

START-UP

Overview

Students use tile to model multiplying and dividing counting numbers.

Materials

- Red and black counting and edge pieces. (Edge pieces are included with bicolored counting pieces.)

FOCUS

Overview

Red and black counting and edge pieces are used to extend the area model for multiplication and division of whole numbers to those operations with signed numbers.

Materials

- Red and black counting and edge pieces, 25 of each per student.
- Red and black counting and edge pieces for the overhead.

FOLLOW-UP

Overview

Students sketch diagrams of red and black counting and edge pieces to illustrate their understanding of multiplication and division of integers.

Materials

- Follow-Up 4, 1 copy per student.
- Red and black counting and edge pieces for each student (see Appendix).

START-UP

Overview

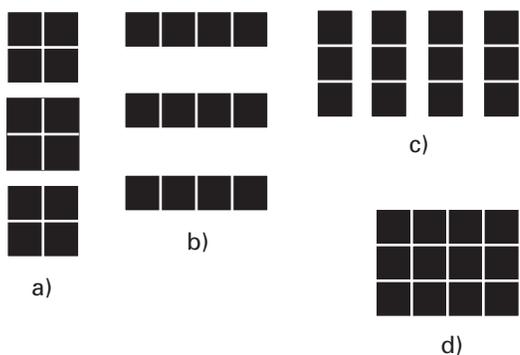
Students use tile to model multiplying and dividing counting numbers.

Materials

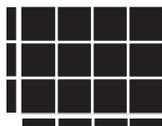
- Red and black counting and edge pieces. (Edge pieces are included with bicolored counting pieces.)

ACTIONS

1 Distribute the counting pieces to the students. Tell them only black pieces will be used in this activity. Write the expression “ 3×4 ” on the overhead or board and ask the students to use their tile to show what this expression means to them. Discuss the various representations.



2 Show model d) on the overhead. Place black edge pieces along two sides of the rectangle to indicate its dimensions, as shown below. Discuss the distinction between the *dimensions* and the *area* of a *rectangle*.



COMMENTS

1 Some students may form 3 groups of 4 as shown in a) and b). Others may form 4 groups of 3 as shown in c). These represent multiplication as *repeated addition*, that is, 3×4 is shown as $4 + 4 + 4$ or $3 + 3 + 3 + 3$.

Some students may represent the product as a rectangular array as shown in d).

Note that eliminating the spaces between the rows in model b) and the columns in model c) makes these models identical to the rectangular array model. Thus model d) can be viewed as 3 groups of 4 (its rows) or 4 groups of 3 (its columns).

2 The *area* of a rectangle is the number of square units it contains. Its dimensions are the *lengths* of its sides. Dimensions are measured in linear units. For the rectangle shown, if each tile represents 1 square unit whose edge is 1 linear unit, the area of the rectangle is 12 square units and its dimensions are 3 linear units by 4 linear units. Thus, in the *rectangular array* model of the statement $3 \times 4 = 12$, the 3 and 4 represent lengths and the 12 represents area.

START-UP

ACTIONS

3 Distribute edge pieces to the students. Tell them that each edge piece represents 1 linear unit and each tile represents 1 square unit. Ask the students to use their pieces to form a rectangular region with edges that models the statement $2 \times 6 = 12$. Ask for a volunteer to show their model on the overhead and show how the repeated addition view of 2×6 is also contained in their model.

4 Write the expression " $15 \div 3$ " on the overhead or board and ask the students to use their tile to show what this expression means to them. Discuss the various representations.

5 Discuss the *grouping* and *sharing* methods of division. Give examples, or ask the students to supply examples, of word problems which fit these methods.



$15 \div 3$: grouping method



$15 \div 3$: sharing method

COMMENTS

3 The region with edges is shown below.



The rows comprise 2 sets of 6 and the columns comprise 6 sets of 2. Thus the repeated additions $6 + 6$ and $2 + 2 + 2 + 2 + 2 + 2$ are also illustrated in the rectangle.

4 Two common models are:



5 Representing $15 \div 3$ by 5 groups of 3 is an example of the *grouping* method of division. In this case, $15 \div 3$ is thought of as the number of groups of 3 into which 15 can be divided, as in the question: "If each student is to receive 3 pencils, how many students will 15 pencils supply?"

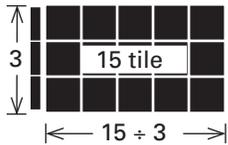
Representing $15 \div 3$ by 5 groups of 3 is an example of the *sharing* method of division. In this case, $15 \div 3$ is thought of as the number of number of objects in each set if 15 objects is divided equally into 3 sets, as in the question: "If 15 pencils are shared equally among 3 students, how many will each student get?"

The grouping method is also called the *subtractive* or *measurement* method since groups are subtracted away or measured off. The sharing method is also called the *dealing* or *partitive* method since objects are dealt or partitioned into sets.

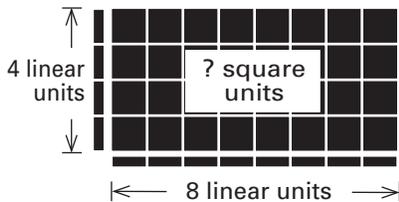
START-UP

ACTIONS

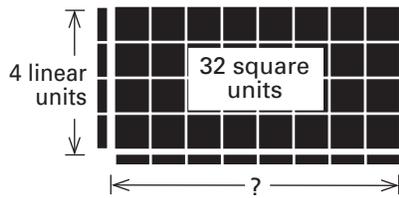
6 If it hasn't come up, show how a rectangle with dimensions can be used to show the meaning of $15 \div 3$.



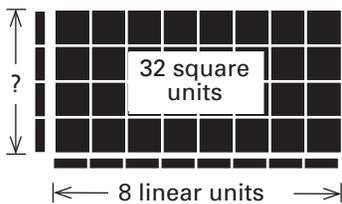
7 Discuss the relationship between the rectangular array models of multiplication and division.



4×8 is the area 32



$32 \div 4$ is the missing dimension 8



$32 \div 8$ is the missing dimension 4

COMMENTS

6 Notice that the two tile arrangements in Action 5 become the same rectangle when rows or columns are pushed together. In the *rectangular array* model of $15 \div 3$, the 15 represents an area of 15 square units and the 3 represents one dimension of 3 linear units. The quotient is the number of linear units in the other dimension.

7 In the rectangular array model, multiplication is the process of finding the area of a rectangle given its dimensions; division is the process of finding the remaining dimension of a rectangle given its area and one of its dimensions.

Notice that every multiplication statement has two division statements associated with it.

FOCUS

Overview

Red and black counting and edge pieces are used to extend the area model for multiplication and division of whole numbers to those operations with signed numbers.

Materials

- Red and black counting and edge pieces, 25 of each per student.
- Red and black counting and edge pieces for the overhead.

ACTIONS

I Tell the students that the intention of this activity is to extend the rectangular array model of multiplication and division to the integers. Then ask them to form rectangular array models, with edge pieces, of the following products:

$+3 \times +4$

$+1 \times -3$

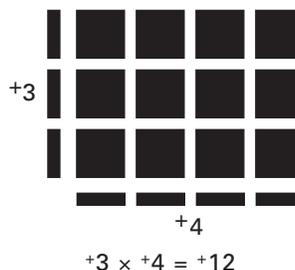
$+2 \times -3$

Ask for volunteers to show their models on the overhead and explain how they were arrived at. Discuss.

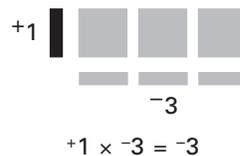
COMMENTS

I The models are shown below.

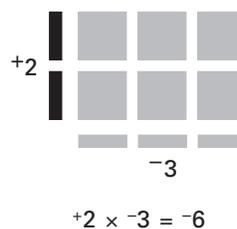
Since the positive integers have been identified with the natural numbers, the product of two positive integers should identify with the product of the corresponding counting numbers, e. g., $+3 \times +4 = +12$ since $3 \times 4 = 12$.



Also, in the counting numbers, 1 is a *multiplicative identity*, that is, the product of 1 and any counting number is that number. This property is extended to the integers by specifying that +1 is a multiplicative identity for the integers, so that +1 times any integer is that integer. Hence, $+1 \times -3 = -3$.



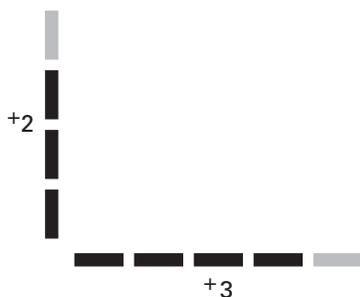
The model for $+2 \times -3$ can be obtained by combining two models of $+1 \times -3$, or, equivalently, viewing $+2 \times -3$ as the repeated addition $(+1 \times -3) + (+1 \times -3)$.



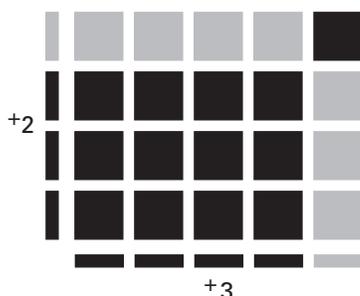


ACTIONS

2 Place the following collection of edge pieces on the overhead as shown. Tell the students these are the edges in a rectangular array model of a multiplication statement. Ask the students to complete the array and name the statement being modeled.

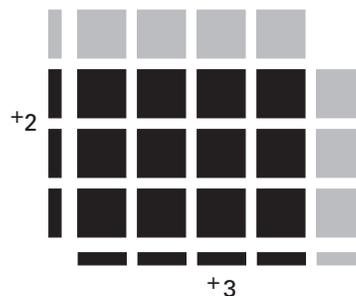


Describe the *minimal array* for the multiplication statement modeled above.



COMMENTS

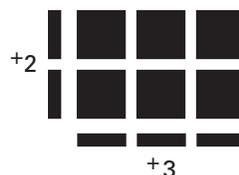
2 Since the value of the left edge of the bottom row of the rectangle is the multiplicative identity $+1$, the bottom row of the rectangle should agree in colors with the horizontal edge. Similarly for the second and third rows of the array. Since the bottom edge of the first four columns is black (and hence has value $+1$) these columns should agree in color with the vertical edge pieces. Thus, one has the following:



Since the net value of the edges are $+2$ and $+3$, the net value of the rectangle should be $+2 \times +3$, or $+6$. The net value of the pieces in the incomplete rectangle shown above is $+7$. Hence, the upper right-hand corner piece must be black.

Notice in the completed array on the left that a tile in the rectangle is black if the edge pieces of the row and column in which it lies have the same color, and it is red if they have opposite colors. In particular, if both edges are red, the tile is black, that is $-1 \times -1 = +1$.

Since there are many edges which have the same net value, there are many ways to construct a rectangular array model of $+2 \times +3$. Notice in the model on the left that the value of the last two columns of the rectangular array is 0 as is the value of the columns edge pieces. hence the two columns with edge piece can be removed without affecting the values in the array. Similarly for the first two rows. Removing these rows and columns leaves the following *minimal array*:



In a minimal array, the tile in the rectangle and the edge pieces in an edge will all be of the same color. Note that in a minimal array, if both edges are black or both are red, the rectangle will be black; if the edges have opposite colors, the rectangle will be red.



ACTIONS

3 Ask students to form minimal arrays, with edges, that model the following products.

a) $+2 \times +5$

b) -3×-3

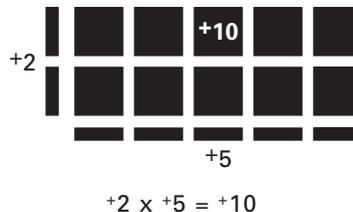
c) $+2 \times -4$

d) $-1 \times +4$

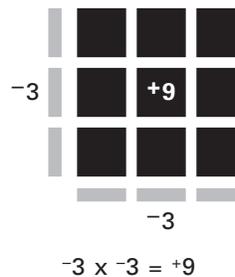
Discuss with the students the relationship between the signs of two integers and the sign of their product. Ask for volunteers to show their completed model and explain their thinking in constructing it. Discuss.

COMMENTS

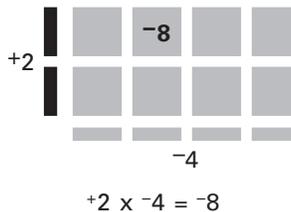
3 a)



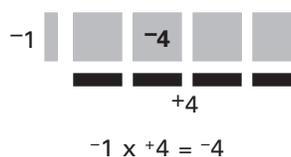
b)



c)



d)



If both edges of a minimal array have the same color, all pieces in the array will be black. Put in another way: If both edges have positive net values or both have negative net values, the array will have a positive net value. This means that the product of two positive numbers is a positive number and the product of two negative numbers is also a positive number.

If the edges of a minimal array have different colors, all pieces in the array will be red. Put in another way: If 1 edge has a positive net value and the other has a negative net value, the array will have a negative net value. This means that the product of a positive number and a negative number is a negative number.

The above can be summarized as follows: The product of two numbers with like signs is positive; the product of two numbers with unlike signs is negative.



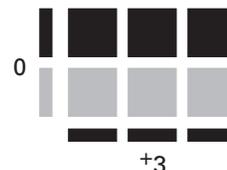
ACTIONS

4 Ask the students to form a rectangular array, with edges, so that one edge has net value 0 and the other edge has net value +3. Have them find the net value of the array. Discuss multiplication by 0.

5 Tell the students that a certain array, with edges, has net value -15 and has one edge whose net value is -3. Ask them to determine the net value of the other edge. Discuss the division statement that is being modeled.

COMMENTS

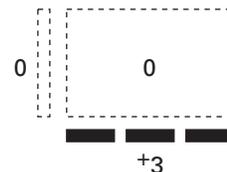
4 Below is one possible array with net value 0.



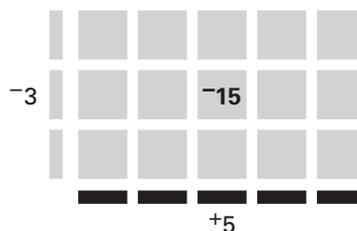
If an edge has net value 0, the edge contains an equal number of red and black pieces. If an edge contains an equal number of red and black pieces, the set of counting pieces associated with red edge pieces is the opposite of the set associated with black edge pieces. Hence the net value of the array is 0. Since this is independent of the other edge, the product of 0 and any signed number is 0.

It may be interesting to discuss with your students whether or not the array pictured above is a minimal array. The question is: Can one remove the 2 rows without affecting any net values? The answer is “Yes,” provided there is agreement that an empty set of pieces has net value 0.

Making this agreement and removing the 2 rows results in the following “minimal” array, in which one edge and the array have value zero.



5 Below is a minimal array that satisfies the conditions. Since the array has a negative net value, the 2 edges must have unlike colors. Hence, the other edge is black. Its net value is +5. The division equation being modeled is: $-15 \div -3 = +5$.



The quotient of two signed numbers can be found by forming an array, with edges, such that (1) the net value of the array is the dividend and (2) the net value of 1 edge is the divisor. The net value of the other edge is the quotient.

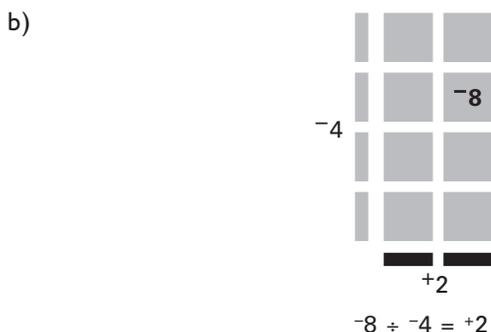
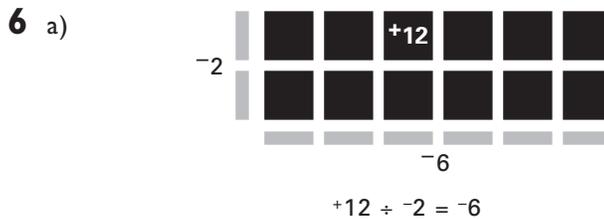


ACTIONS

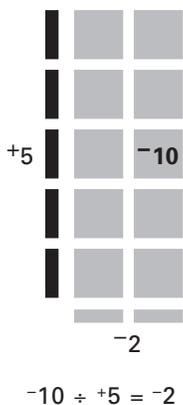
6 For each of the following, ask the students to form a minimal array, with edges, which models the quotient and then write the division equation that is being portrayed.

- a) $+12 \div -2$
- b) $-8 \div -4$
- c) $-10 \div +5$
- d) $+3 \div -1$

COMMENTS



Some students may also suggest using the grouping method for $-8 \div -4$, i.e., “how many groups of value -4 are contained in a collection of value -8 ?” While this method is reasonable for this problem, it is awkward for the other three problems. For example, it doesn’t make sense to ask “how many groups of value -2 are in a collection of value $+12$?”



c) (See diagram on the left.) Some students may also suggest the sharing method to solve $-10 \div +5$, i.e., “if a collection of value -10 is divided into 5 equal groups, what is the value of each group?” However, applying this method to the other problems is awkward, e.g., “if a collection of value 12 is divided into negative 2 equal groups...”



If students don’t bring up questions or ideas regarding use of the grouping and sharing models of division, you may wish to initiate a discussion, noting that while each model is useful, the reason for the emphasis on the area model is the fact that it remains mathematically “faithful” given any situation involving integers. If the grouping or sharing models make sense and fit the context of a particular situation, it is appropriate for students to use them. Note, that given such situations, the grouping and sharing models are also visible in an area representation of the situation.



ACTIONS

7 Discuss with the students their ideas about the relationship, in general, between the signs of the dividend, the divisor, and the quotient in a division statement.

8 Ask the students to explore and discuss the possibility of the following:

a) forming an array, with edges, so the array has net value 0 and one edge has net value +4;

b) forming an array, with edges, so that the array has net value +4 and 1 edge has net value 0;

c) forming an array, with edges, so that the array has net value 0 and 1 edge also has net value 0.

COMMENTS

7 When a division statement is portrayed by a minimal array, with edges, the dividend corresponds to the net value of the array, the divisor corresponds to the net value of 1 edge and the quotient corresponds to the net value of the other edge.

If a positive number is being divided, the corresponding minimal array has a positive net value. Hence its edges' net values are both positive or both negative; that is, the quotient has the same sign as the divisor. So, if a positive number is divided by a positive number, the quotient is positive, and if a positive number is divided by a negative number, the quotient is negative.

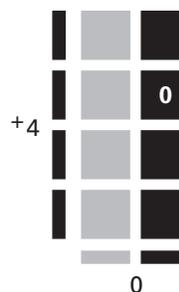
If a negative number is being divided, the corresponding minimal array has a negative net value. Hence 1 edge has a positive net value and 1 has a negative net value, that is, the quotient and divisor have opposite signs. So, if a negative number is divided by a positive number, the quotient is negative and if a negative number is divided by a negative number, the quotient is positive.

This information can be put in tabular form:

dividend ÷ divisor = quotient		
+	+	+
+	-	-
-	+	-
-	-	+

Note that if the dividend and divisor have like signs, the quotient is positive. If the dividend and divisor have unlike signs, the quotient is negative.

8 a) Here is one possibility:



b) No such array exists. If an edge has net value 0, the array will have net value 0.



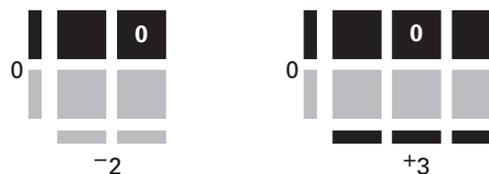
ACTIONS

Allow time for the groups to relate their findings above to divisions involving 0, in general. Ask for volunteers to report their group's conclusions.

9 Discuss notation conventions associated with the integers.

COMMENTS

c) Many such arrays exist. One is pictured in a) above. Two others are shown below.



Part a) shows that $0 \div +4 = 0$. Note that if $+4$ is replaced by any positive or negative number, the other edge would still have net value 0. Thus, 0 divided by any nonzero signed number is 0.

Parts b) and c) indicate why division by 0 is unworkable. As part b) suggests, division of a nonzero number by 0 is not possible. Attempting to divide a nonzero number by 0 is equivalent to attempting to construct an array, with edges, which has a nonzero net value, but has an edge whose net value is 0. No such array exists.

Part c) illustrates the difficulty with dividing 0 by 0. Determining a value of $0 \div 0$ is equivalent to asking for the net value of the edge of an array if the array and its other edge have 0 net values. The problem is not finding such an array; the problem is that too many exist. If the array is the one shown on the left in c) above, the answer would be -2 ; if the array is the one shown on the right, the answer would be $+3$. Thus $0 \div 0$ is an ambiguous notion or, in mathematical terms, $0 \div 0$ is *not well defined*. Since ambiguity can result in erroneous results, dividing 0 by 0 is not allowed.

9 As mentioned in Comment 8, in the Focus section of Lesson 2, Positive and Negative Integers, the $+$ sign is generally omitted when referring to a positive integer, i.e., $+5$ is written as 5.

Also when writing a signed number to represent a negative integer, the $-$ sign is usually written in normal position rather than in superscript position, i.e., -3 is written instead of -3 .

The notation “ $\text{opp}(n)$ ” is sometimes used to designate the opposite of a signed number n , e.g., $\text{opp}(-3) = +3$. More frequently, though, the opposite of n is denoted by “ $-n$,” e.g., $-+3 = -3$ and $-3 = +3$. Notice that if the conventions concerning the dropping of the $+$ sign and the location of the $-$ sign are adopted, these two statements would appear as $-3 = -3$ and $- -3 = 3$, respectively. The first of these statements, $-3 = -3$, appears to be a redundancy. However, on the left, -3 is intended to represent “the opposite of $+3$ ” and, on the right, -3 is intended to represent the negative integer “ -3 .”

continued next page

**FOCUS****ACTIONS****COMMENTS****9** continued

In standard practice, it is difficult to determine whether a $-$ sign is being used as part of the symbol for a negative integer or to designate the opposite of a positive integer. As illustrated in the previous paragraph, using standard practices, both the opposite of the positive integer $+3$ and the negative integer -3 are denoted symbolically as -3 . Since the opposite of the positive integer $+3$ is the negative integer -3 , it doesn't matter, as a rule, which of these two interpretations is given to the symbol -3 .

Notice that the $-$ sign occurs in three different ways in arithmetical notation. Besides its use in denoting the opposite of a number and its use in designating a negative number, it is also used to denote the operation of subtraction. Generally, it is clear from the context what use is intended. Nonetheless, students are apt to be confused by this variety of usage.

**FOLLOW-UP BLACKLINE MASTER 4**

1 Sketch a minimal array with edge pieces to illustrate each product and quotient below.

Label each array to show the net values of its edges.

a) -5×-4

b) 7×4

c) 2×-3

d) -6×2

e) $-20 \div 4$

f) $-20 \div -2$

g) $18 \div -9$

h) $20 \div 4$

2 Explain how the signs of two signed numbers are related to the sign of their product and the sign of their quotient.

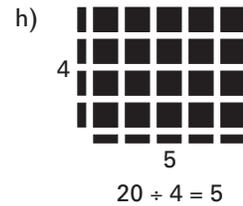
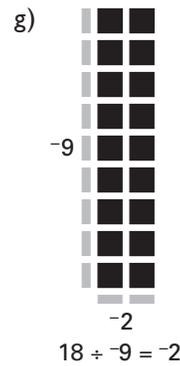
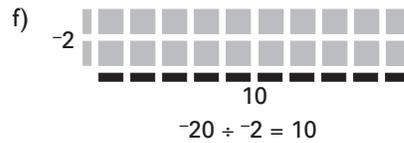
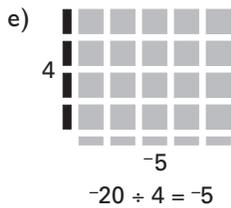
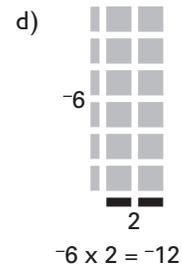
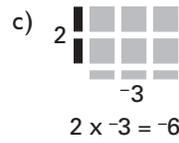
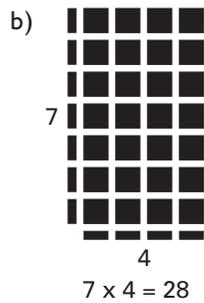
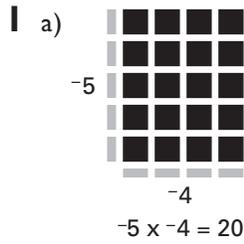
3 Use diagrams of red and black counting pieces and brief explanations to show the following:

a) $7 \times 0 = 0$,

b) $7 \div 0$ is not possible,

c) $0 \div 0$ is ambiguous (and therefore not allowed).

ANSWERS TO FOLLOW-UP

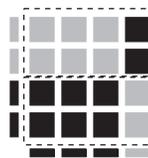


2 If the two numbers both have the same sign, their product and their quotient are both positive. If the two numbers have opposite signs, their product and their quotient are both negative.

3 a) If an array has an edge whose values are 7 and 0, the array will have value 0. Hence, $7 \times 0 = 0$.

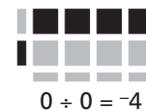
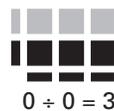
b) If an array is constructed whose value is 0 and for which one edge has value 7, the other edge has value 0. Hence, $0 \times 7 = 0$.

c) If it were possible to compute $7 \div 0$, an array of value 7 with an edge of value 0 would exist. However, if an edge of an array has value 0, the array will consist of two opposite collections and hence have value 0. Thus, $7 \div 0$ is not possible.



These two collections are opposites of each other.

d) It is possible to build many arrays which have value 0 and also have an edge whose value is 0. For example, the first array on the right would suggest that $0 \div 0 = 3$, while the second suggests $0 \div 0 = -4$. Therefore, $0 \div 0$ is ambiguous.



COUNTING PIECE PATTERNS & GRAPHS

THE BIG IDEA

Making generalizations about relationships in a sequence of counting piece arrangements deepens understanding of integer operations, the meaning of a variable, the concept of a function, algebraic notation, and order of operations. Exploring relationships between a sequence and its graph promotes insights about graphs and the data they portray.

START-UP

Overview

Students study a sequence of tile patterns which contains both black and red tile. Functional notation is used to describe net values.

Materials

- Red and black counting pieces, 25 of each per student.
- Red and black counting pieces for the overhead.

FOCUS

Overview

Students determine the net values of arrangements of red and black counting pieces and graph these net values on a coordinate system.

Materials

- Red and black counting and edge pieces, 25 of each per student.
- Focus Master 5.1, 1 copy per pair of students and 1 transparency.
- $\frac{1}{4}$ " grid paper, 2-4 sheets per pair of students and 1 transparency.
- Focus Master 5.2, 1 transparency.
- Red and black counting and edge pieces for the overhead.

FOLLOW-UP

Overview

Students find patterns in sequences of tile arrangements, determine net values of arrangements, write an algebraic expression for the net value of the n th arrangement, and sketch coordinate graphs of the sequences.

Materials

- Follow-Up 5, 2 pages run back-to-back, 1 copy per student.
- $\frac{1}{4}$ " grid paper, 5-6 sheets per student.

START-UP

Overview

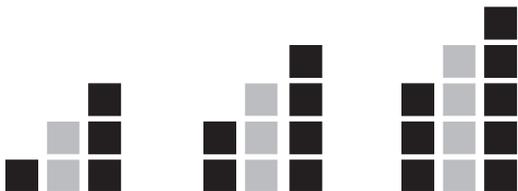
Counting pieces are used to find sums and differences of integers and to provide insights into these operations.

Materials

- Red and black counting pieces, 25 for each student.
- Red and black counting pieces for the overhead.

ACTIONS

1 Arrange the students in pairs and distribute counting and edge pieces to each student. Display the following sequence of counting piece arrangements on the overhead. Have the students form the next arrangement in the sequence.



2 Ask the students to find the net values of each of the first 4 arrangements in the above sequence.

3 Ask the students to imagine and describe the 20th arrangement of this sequence, based on the first 4 arrangements discussed above, and find its net value. Discuss.

COMMENTS

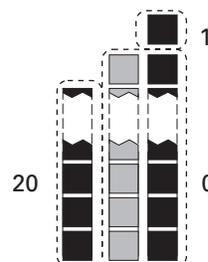
1 Here is the most frequently suggested next arrangement:



2 Using the pattern suggested in Comment 1, the net values of the first 4 arrangements are 2, 3, 4, and 5, respectively.

3 One possible description of the 20th arrangement is that it consists of a column of 21 red pieces with a column of 20 black pieces on one side and a column of 22 black pieces on the other side.

The net value of the 20th arrangement is 21, as indicated in the sketch shown below.



START-UP

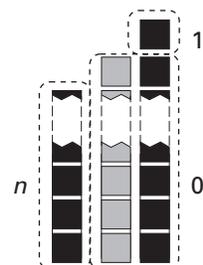
ACTIONS

4 Repeat Action 3 for the n th arrangement.

5 Tell the students that, henceforth, the “net value of an arrangement” will be referred to simply as “the value of the arrangement.” Also, the string of symbols “ $v(n)$ ” will be used as shorthand for the phrase “the value of the n th arrangement.” Discuss this convention.

COMMENTS

4 The net value of the n th arrangement is $n + 1$, as illustrated below:



n th arrangement
net value = $n + 1$

5 The statement in comment 3, “the value of the 20th arrangement is 21,” can be written, “ $v(20) = 21$.” The string of symbols “ $v(20)$ ” is often read “ v of 20.” Similarly, Comment 4 can be written “ $v(n) = n + 1$.”

The string of symbols “ $v(n)$ ” is sometimes read “ v of n .” If this reading is adopted, it is important to remember that “ v of n ” is a shortened version of the phrase “the value of the n th arrangement” and not the phrase “the value of n .” Thus the statement “ v of n is 30” means “the value of the n th arrangement is 30,” not “the value of n is 30.”

Following is some information that may be interesting to introduce informally during discussion. It is not intended for memorization by students: A *function* is a rule that relates two sets by assigning each element in the first set (called the *domain*) to one, and only one, element in the second set (called the *range*). Hence, in this lesson, $v(n)$ is a function that relates n , the number of an arrangement, to v , the value of the arrangement. The variable v is called the *dependent variable* because the value, v , of an arrangement depends on the number, n , which is called the *independent variable*. For the functions explored in this lesson, the domain is the set of all counting numbers (positive integers), and the range is the set of all positive and negative integers.

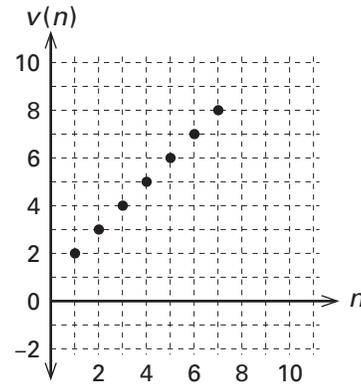
START-UP

ACTIONS

6 Again, place the 3 arrangements shown in Action 1 on the overhead. Distribute a sheet of grid paper to each student and ask them to create a graph showing the net values of the first 8 arrangements of the sequence. Discuss their observations about the graph.

COMMENTS

6 Here is a coordinate graph showing $v(n)$ for $n = 1, 2, 3, 4, 5, 6, 7,$ and 8 .



Students may notice that the points of the graph lie on a line. They might also notice that if each arrangement is replaced by the minimal collection of pieces having the same net value, these counting pieces could be placed in a column to form a bar graph illustrating the net values of the arrangements.

Overview

Students determine the net values of arrangements of red and black counting pieces and graph these net values on a coordinate system.

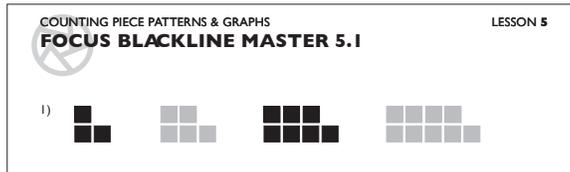
Materials

- Red and black counting and edge pieces, 25 of each per student.
- Focus Master 5.1, 1 copy per pair of students and 1 transparency.
- 1/4" grid paper, 2-4 sheets per pair of students and 1 transparency.
- Focus Master 5.2, 1 transparency.
- Red and black counting and edge pieces for the overhead.

ACTIONS

1 Give each student a copy of Focus Master 5.1 and a sheet of 1/4" grid paper. For Sequence 1), ask the students to

- a) write a formula for $v(n)$,
- b) draw a coordinate graph of $v(n)$ for several values of n .

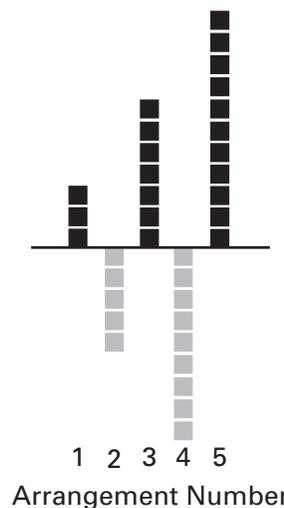


COMMENTS

1 The n th arrangement contains $2n + 1$ tile. All the tile in the odd-numbered arrangements are black and all the tile in the even-numbered arrangements are red. Hence, $v(n) = 2n + 1$ when n is odd and $v(n) = -(2n + 1)$ when n is even. This can be written as follows.

$$v(n) = \begin{cases} 2n + 1 & \text{if } n \text{ is odd} \\ -(2n + 1) & \text{if } n \text{ is even} \end{cases}$$

To assist with graphing, it may be helpful to compare the graph with the sequence obtained when the arrangements of the original sequence are rearranged into columns, as shown below. Such columns contain a minimal number of tile (so no column contains both red and black tile). Black columns extend above a base line and red columns extend below.



Note that in this and subsequent lessons – signs are being written in normal positions as described in Comment 9 of the Focus section of Lesson 4. Thus, the negative integer $-(2n + 1)$ is written $-(2n + 1)$ in the above formula for $v(n)$.

continued next page

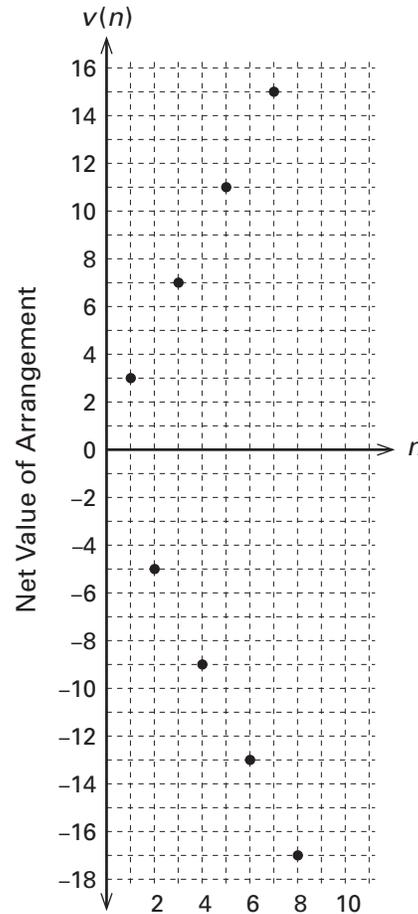


ACTIONS

COMMENTS

I continued

Ordered pairs associated with the first 8 arrangements are: (1, 3), (2, -5), (3, 7), (4, -9), (5, 11), (6, -13), (7, 15), (8, -17). Shown below is a coordinate graph of these points:



ACTIONS

2 Repeat Action 1 for other sequences selected from Focus Master 5.1.

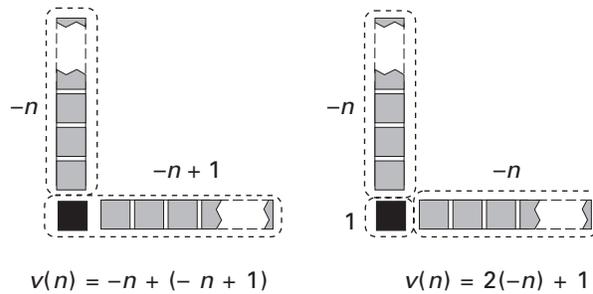
COUNTING PIECE PATTERNS & GRAPHS
FOCUS BLACKLINE MASTER 5.1 LESSON 5

COMMENTS

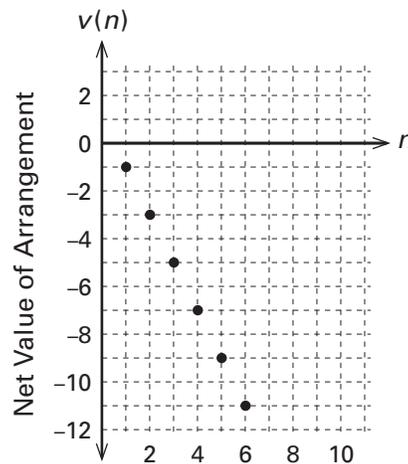
2 Additional grid paper for graphing may be needed by the students. You might ask the students to select the 3 or 4 sequences that look most interesting or challenging. The students may also be interested in constructing other sequences to add to the collection given here.

In each case, the students may find a number of equivalent expressions for $v(n)$.

2) Here are two ways of imagining the n th arrangement of Sequence 2):



Some students may benefit by forming minimal columns (see Comment 7) of tile before graphing coordinate points. The points have coordinates $(1, -1)$, $(2, -3)$, $(3, -5)$, etc. and the graph below shows these points lie on a straight line.

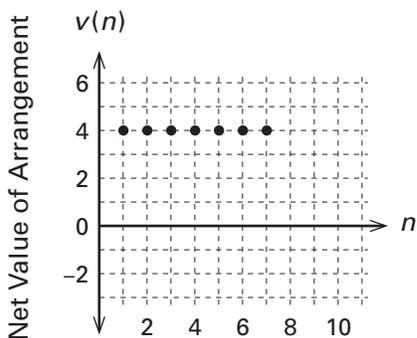


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FOCUS

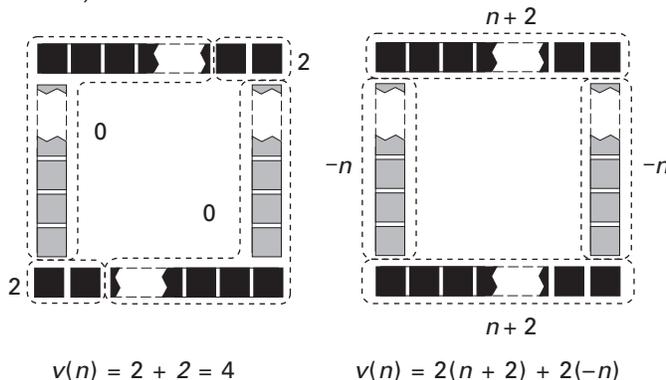
ACTIONS

COMMENTS

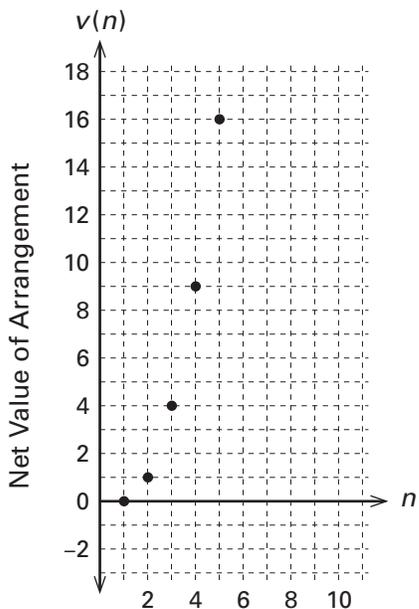


2 continued

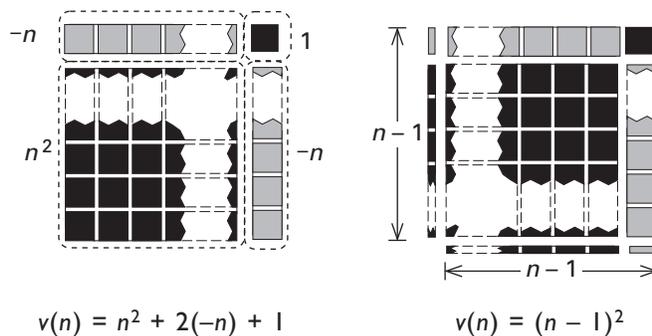
3) Here are two possibilities for “seeing” the n th arrangement of Sequence 3):



The coordinate graph for Sequence 3) is shown at left. Its points lie on a horizontal line which has a height of 4 units.



4) Here are two possible ways of imagining the n th arrangement of Sequence 4):

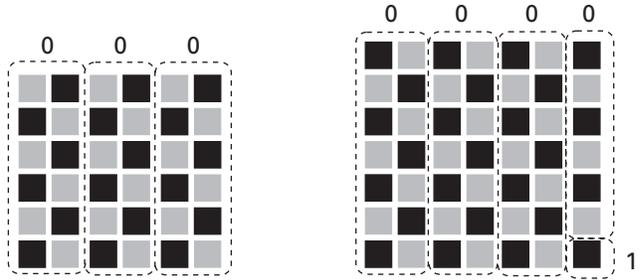


The coordinate graph for the first few arrangements of Sequence 4) is shown at the left. The students may notice the points are not linear (i.e., they do not lie on a straight line).

ACTIONS

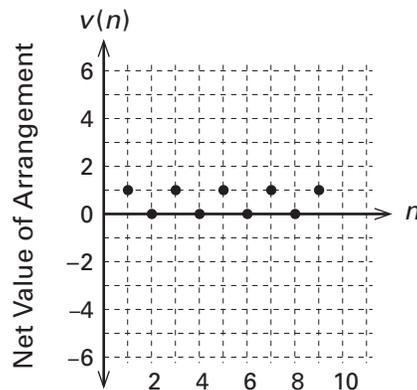
COMMENTS

5) Since two adjacent columns in the n th arrangement of Sequence 5 are opposites of each other, the net value of two adjoining columns is 0. Hence, if n is even, $v(n) = 0$. If n is odd, the remaining column, as shown in the example below, has a net value of 1. So, if n is odd, $v(n) = 1$.



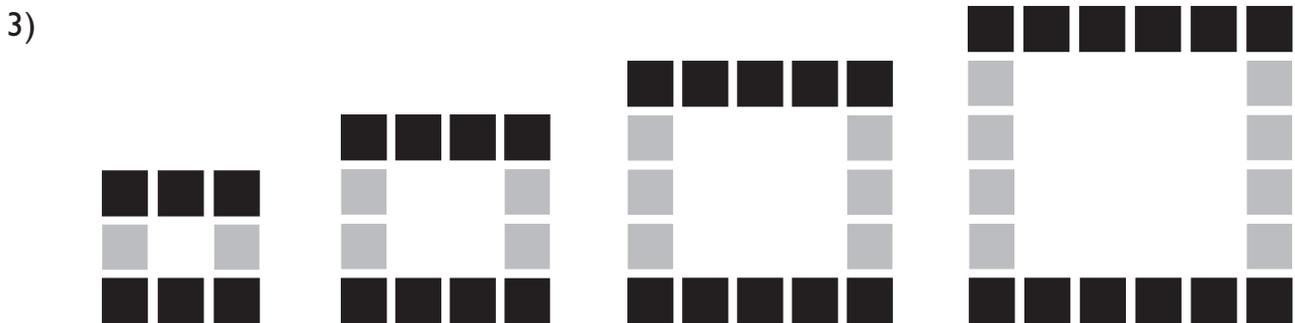
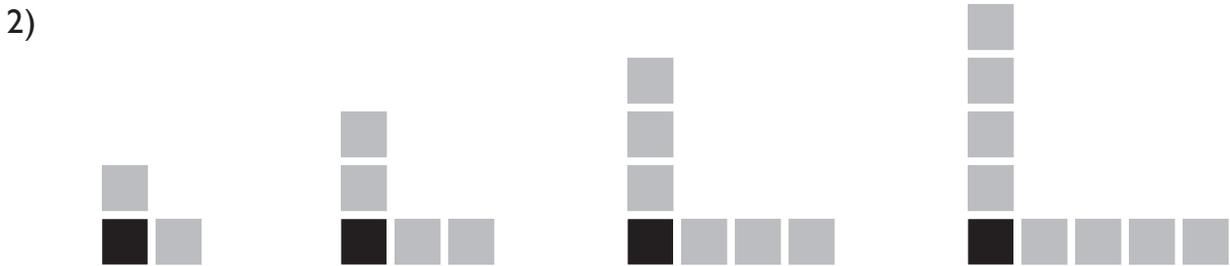
$$v(n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

The graph for this sequence is points whose heights alternate from 0 to 1, as shown below.





FOCUS BLACKLINE MASTER 5.1



FOLLOW-UP BLACKLINE MASTER 5

I For each of sequences i)-v) below, do the following on a sheet of $\frac{1}{4}$ " grid paper:

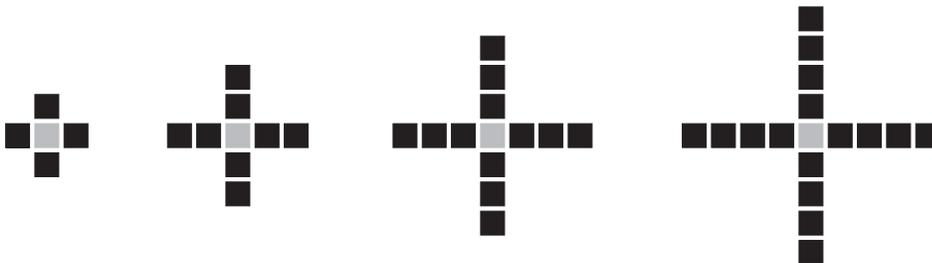
a) Sketch the 5th arrangement in the sequence.

b) Describe, in words, the 50th arrangement.

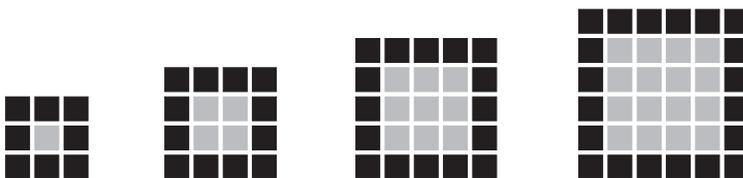
c) Write a formula for $v(n)$, the net value of the n th arrangement. Draw a diagram that illustrates your thinking..

d) Make a coordinate graph showing $v(n)$ for the first several arrangements in the sequence.

i)



ii)



iii)



iv)



continued on back

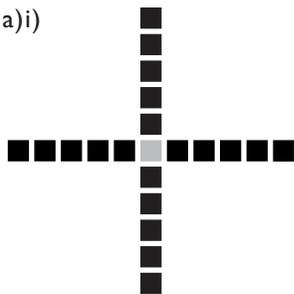
 **FOLLOW-UP BLACKLINE MASTER 5 (CONT.)**

2 Suppose the value $v(n)$ of the n th arrangement in a sequence is $5n + -3$. Do the following on a sheet of grid paper:

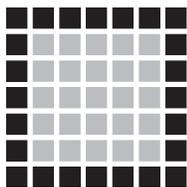
- a) Sketch what you think could be the first 4 arrangements in the sequence.
- b) Under the sequence, make a coordinate graph of $v(n) = 5n + -3$.
- c) Next to your graph, write several observations about the sequence and its graph.

ANSWERS TO FOLLOW-UP 5

I a)i)



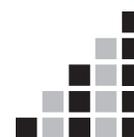
ii)



iii)



iv)



I b) Descriptions of the 50th arrangement (other descriptions are possible):

i) A red tile with rows of 50 black tile attached to each side and columns of 50 black tile attached to the top and bottom.

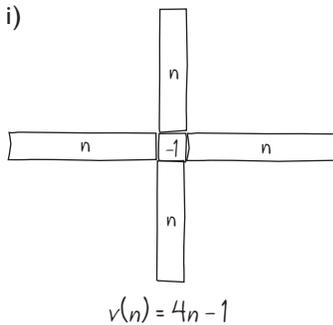
ii) A 50 x 50 square of red tile with a border of 54 black tile.

iii) Two rows of tile aligned at their left ends; the top row consisting of 2 black tile and the bottom row consisting of 99 red tile.

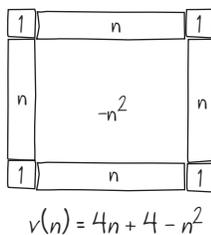
iv) Fifty columns of tile, with the bottom tiles aligned in a row. Going to the left, the columns increase in height by 1 tile and alternate between black and red tile, beginning with 1 black.

I c) Formulas for $v(n)$ (other equivalent formulas are possible):

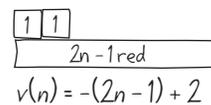
i)



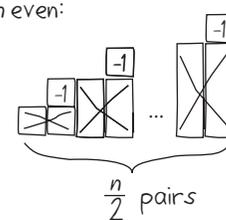
ii)



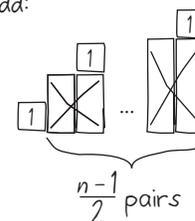
iii)



iv) n even:



n odd:

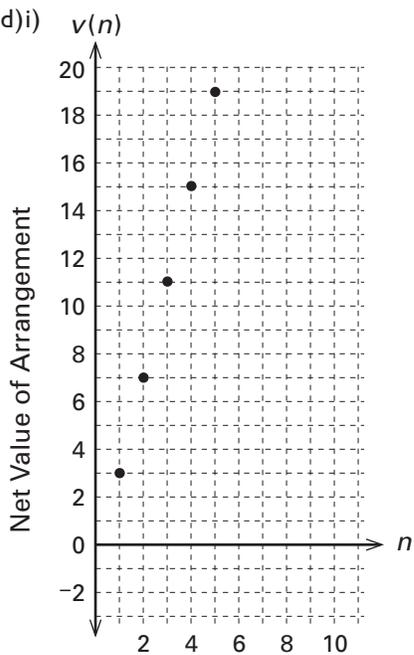


$$v(n) = \begin{cases} \frac{-n}{2}, & n \text{ even} \\ 1 + \frac{n-1}{2}, & n \text{ odd} \end{cases}$$

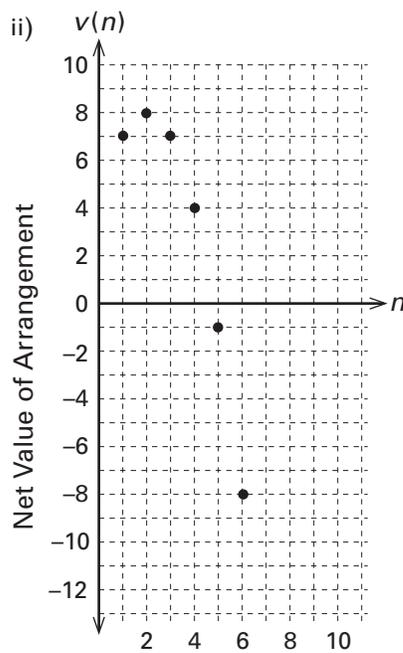
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ANSWERS TO FOLLOW-UP 5 (CONT.)

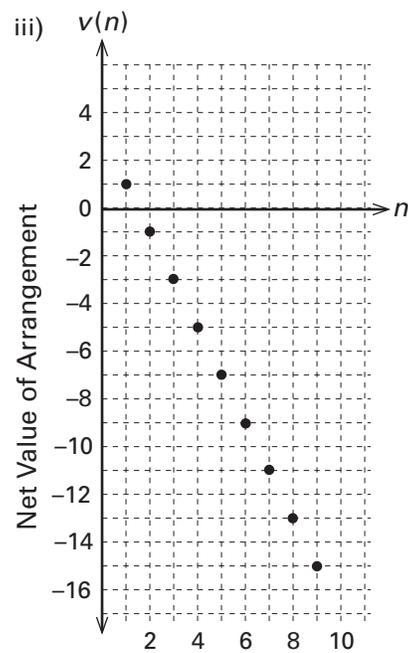
I d)i)



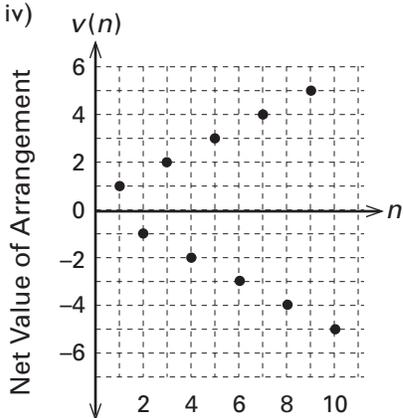
ii)



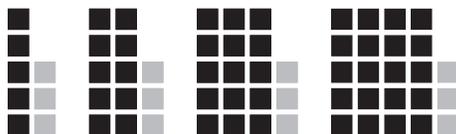
iii)



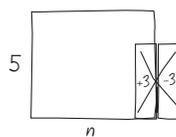
iv)



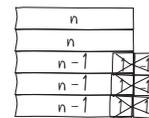
2 a)



b) Other formulas for $v(n)$:



$$v(n) = 5n - 3$$



$$v(n) = 2n + 3(n-1)$$

MODELING ALGEBRAIC EXPRESSIONS

THE BIG IDEA

By using Algebra Pieces it is possible to form a concrete representation of the n th arrangement of a sequence of arrangements. Examining these representations provides insights about the meaning of algebraic expressions and lays groundwork for solving equations.

START-UP

Overview

Students use Algebra Pieces to represent the n th arrangement of sequences of counting pieces, and reason from these representations to answer questions about sequences.

Materials

- Algebra Pieces (see Comment 1) for each student.
- Overhead Algebra Pieces.
- Start-Up Master 6.1, 1 transparency.

FOCUS

Overview

Algebraic expressions are represented as sequences of counting pieces arrangements. Students use Algebra Pieces to represent the n th arrangements of sequences, and reason from these representations to answer questions about the sequences and their graphs.

Materials

- Algebra Pieces for each student.
- Overhead Algebra Pieces.
- Red and black counting and edge pieces, 25 per student.
- Red and black counting and edge pieces for the overhead.
- Focus Masters 6.1-6.2, 1 copy of each per student.
- $\frac{1}{4}$ " grid paper.
- Blank counting pieces and n -strips (see Appendix), 1 sheet per student.

FOLLOW-UP

Overview

Given formulas for $v(n)$ for a collection of sequences of counting piece arrangements, students determine the values of certain arrangements, build Algebra Piece representations of n th arrangements, and answer questions pertaining to coordinate graphs of arrangement values of these sequences and how the arrangement values for various sequences compare to one another.

Materials

- Follow-Up 6, 1 copy per student.
- Algebra Pieces (see Appendix), paper or cardstock set for each student to use at home.

START-UP

Overview

Students use Algebra Pieces to represent the n th arrangement of sequences of counting pieces, and reason from these representations to answer questions about sequences.

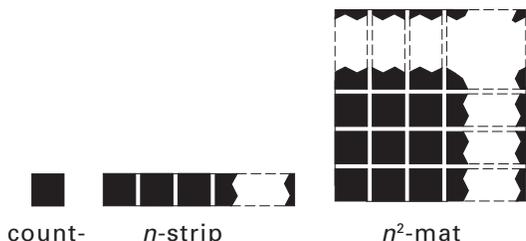
Materials

- Algebra Pieces (see Comment 1) for each student.
- Overhead Algebra Pieces.
- Start-Up Master 6.1, 1 transparency.

ACTIONS

I Give each student a set of Algebra Pieces. Discuss the pieces with the students.

Area Pieces



Edge Pieces



COMMENTS

I Algebra Pieces consist of 4 kinds of pieces: a counting piece square (a single counting piece), an n -strip (representing a strip of n counting pieces) and an n^2 -mat (representing an $n \times n$ array of counting pieces), and edge pieces (representing the length of the edge of 1 counting piece square and the length of the edge of an n -strip). These pieces are black on one side and red on the other. Black pieces have a positive value and red pieces have a negative value.

Note that the white spaces on the n -strips are intended to suggest that the strips can be mentally elongated (or shortened) to contain n counting pieces, whatever n might be. To illustrate this, Start-Up Master 6.1 contains a master for an n -strip that can be elongated as illustrated below.



Note that the n^2 -mats are designed to suggest they can simultaneously stretch or shrink in 2 directions in order to form larger or smaller squares.

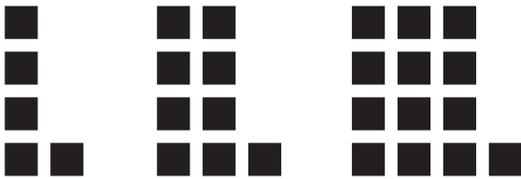
Cardstock or plastic Algebra Pieces are available from The Math Learning Center. A master for making the pieces yourself is also included in the Appendix. One master's worth of Algebra Pieces (2 n^2 -mats, 10 n -strips, 8 counting pieces) plus 17 or more additional counting pieces for each student will generally suffice for lessons that require use of the pieces. To make the Algebra Pieces, if two-colored printing is not available, pieces can be printed on white cardstock or paper. In this case, red is represented by screened gray. With reasonable care, two-sided copies can be made on standard copy machines. The two sides can be justified well enough to make usable pieces. Edge pieces can be made by cutting units and n -strips into 3rds or 4ths. Each student needs about 5 n -strip edge pieces and about 10 unit edge pieces.

Overhead Algebra Pieces can be created using clear and red transparency film (overhead pieces available from MLC are smaller than student pieces in order to better fit on the screen during demonstrations).

START-UP

ACTIONS

2 Display the following sequence of arrangements on the overhead. Ask the students to form a 4th arrangement for this sequence. Then ask them to find $v(n)$ and to use their Algebra Pieces to form a representation of the n th arrangement.



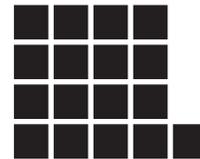
3 Ask the students to discuss ways they can use their Algebra Piece representation of $4n + 1$ from Action 2 to “see” solutions to the following questions about the sequence of counting piece arrangements for which $v(n) = 4n + 1$:

- a) What is the value of the 15th arrangement?
- b) Which arrangement has value 81? 225? 405?
- c) What are some equivalent expressions for the value of the n th arrangement?

Discuss the students’ responses.

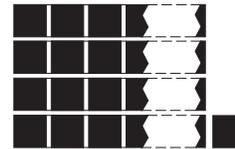
COMMENTS

2 Students will generally form the 4th arrangement as shown below.



4th arrangement

The n th arrangement, if the above method of forming arrangements is continued, will contain $4n + 1$ black pieces. Hence, $v(n) = 4n + 1$. The n th arrangement can be represented by 4 n -strips plus a single black piece:

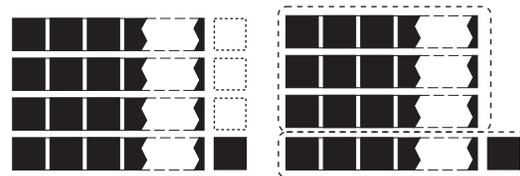


$4n + 1$

3 a) If the 15th arrangement were constructed as suggested in Comment 2, each of the 4 n -strips would be replaced by 15 black counting pieces. Hence, the value of the 15th arrangement would be $(4 \times 15) + 1 = 61$.

b) One way to “see” this is to imagine that 4 black n -strips and 1 black counting piece have a total value of 81. Hence, the 4 n -strips, in total, have value 80, so each n -strip has value $80 \div 4$, or 20. Hence, it is the 20th arrangement which has value 81 since, in this case, the number of an arrangement is equal to the value of an n -strip. Similar reasoning can be used to determine that the 56th arrangement has value 225 and the 101st arrangement has value 405.

c) Equivalent expressions for the value of the n th arrangement can be obtained by viewing Algebra Piece representations in alternate ways. Shown below are two examples.



$4(n + 1) - 3$

$3n + (n + 1)$

START-UP

ACTIONS

4 Tell the students that a certain sequence of arrangements is composed entirely of black counting pieces and that $v(n) = 2(n + 1) + 3$. Ask the students to form the first 3 arrangements of such a sequence.

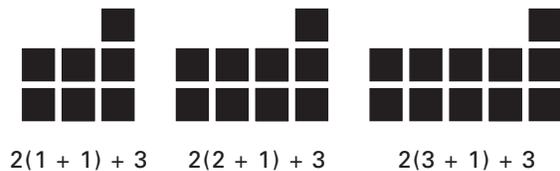
5 Ask the students to use Algebra Pieces to form a representation of the n th arrangement of the sequence of Action 4. Then ask them to use this representation to help them determine which arrangement has value 225.



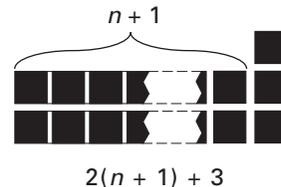
6 (Optional) Tell the students that two successive arrangements have a combined value of 400. Ask them to devise a method using Algebra Pieces to determine which arrangements these are.

COMMENTS

4 Since the arrangements contain only black pieces, the value of an arrangement will be equal to the number of counting pieces in the arrangement. Replacing n in the expression for $v(n)$ by 1, 2, and 3, respectively, we see that the first 3 arrangements contain, respectively, $2(2) + 3$, $2(3) + 3$, and $2(4) + 3$ black counting pieces. One way of arranging these pieces is shown below.

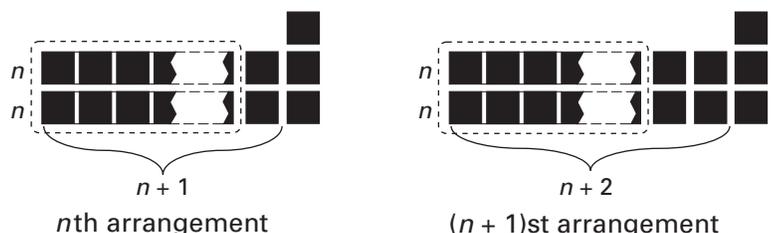


5 The n th arrangement can be described as containing 2 rows of $n + 1$ black counting pieces plus an additional 3 black counting pieces. A row of $n + 1$ black counting pieces is represented by 1 black n -strip plus an additional tile. Hence, the n th arrangement can be represented as follows:

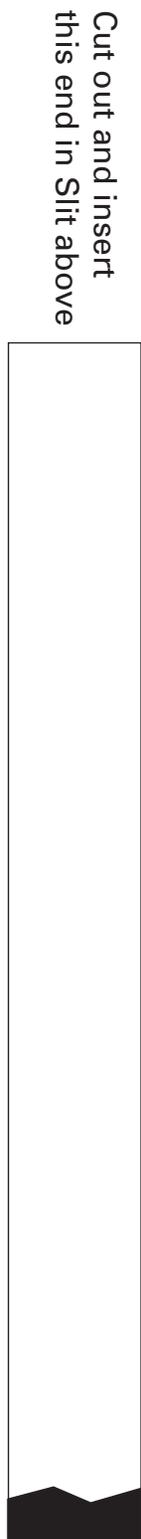


If an arrangement has value 225, then the 2 circled n -strips have a total value of 220. Hence each of these black n -strips has a value of 110. Thus $n = 110$ and it is the 110th arrangement that has value 225.

6 Action 6 can be assigned as homework. Suppose the two successive arrangements are the n th and $(n + 1)$ st arrangements. The n th arrangement (see Comment 4) is shown on the left below. The $(n + 1)$ st arrangement will contain 1 more counting piece in each row and is shown on the right below. Thus the n th and $(n + 1)$ st arrangements together contain 4 n -strips plus 12 additional black pieces. If the total value is 400, then the total value of the 4 n -strips is $400 - 12$ or 388. Hence the value of each n -strip is $388 \div 4$ or 97. Hence $n = 97$ and the two successive arrangements are the 97th and 98th.



START-UP BLACKLINE MASTER 6.1



CUT HERE 



Overview

Algebraic expressions are represented as sequences of counting pieces arrangements. Students use Algebra Pieces to represent the n th arrangements of sequences, and reason from these representations to answer questions about the sequences and their graphs.

Materials

- Algebra Pieces for each student.
- Overhead Algebra Pieces.
- Red and black counting and edge pieces, 25 per student.
- Red and black counting and edge pieces for the overhead.
- Focus Masters 6.1-6.2, 1 copy of each per student.
- 1/4" grid paper.
- Blank counting pieces and n -strips (see Appendix), 1 sheet per student.

ACTIONS

1 Show the students the following sequence of arrangements. Ask them to describe the 20th arrangement and to determine its value.



2 Ask the students to use their Algebra Pieces to build a representation of the n th arrangement and then to write an expression for $v(n)$. Discuss

3 Ask the groups to find equivalent expressions for $v(n)$. Discuss.

COMMENTS

1 There are various ways to describe the 20th arrangement. One way is to say it consists of 3 strips, each with value 20, and 2 red counting pieces. Following this description, $v(20) = 3(20) + (-2) = 58$.

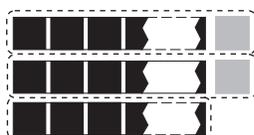
2 Here is one representation of the n th arrangement:



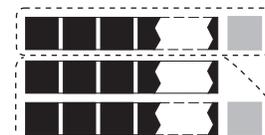
$$v(n) = 3n + (-2)$$

Since, as discussed in Comment 5 of Lesson 3, adding a second number to a first number is equivalent to subtracting the opposite of the second number from the first number, $3n + (-2)$ and $3n - 2$ are equivalent expressions. Hence, one may write $v(n) = 3n - 2$.

3 Grouping the Algebra Pieces in various ways leads to some equivalent expressions:



$$v(n) = 2[n + (-1)] + n$$



$$v(n) = [n + (-1)] + [2n + (-1)]$$

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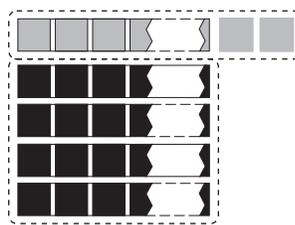
FOCUS

ACTIONS

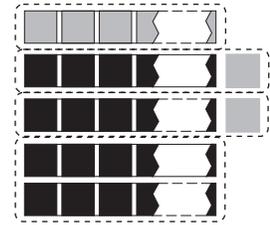
COMMENTS

3 continued

Other equivalent expressions for $v(n)$ can also be obtained by adding equal amounts of black and red pieces (a process which leaves the value of $v(n)$ unchanged) and grouping the resulting collection in various ways. In the following examples, a black and a red strip have been added.



$$v(n) = -(n + 2) + 4n$$



$$v(n) = -n + 2[n + (-1)] + 2n$$

Equivalent expressions can also be obtained by removing equal amounts of black and red pieces. In the following sketch, 2 red counting pieces are removed and 2 black counting pieces are removed by cutting them off a black n -strip.



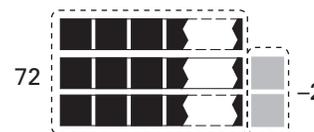
$$v(n) = 2n + (n - 2)$$

Cutting strips can be avoided by placing blank counting pieces (see Appendix) over pieces to indicate they have been removed.



4 Ask the students to show how it is possible to reason from the Algebra Pieces to determine which arrangement has a net value of 70. Discuss their methods and relate their work to the solving of an equation.

4 If the Algebra Piece representation of the n th arrangement shown below has value 70, then the 3 n -strips have a total value of 72 so each of them contains 24 black counting pieces. Since the number of counting pieces in a strip is also the number of the arrangement, it is the 24th arrangement which has value 70.



$$v(n) = 70$$

ACTIONS

5 Distribute 1 sheet of 1/4" grid paper and 1 copy of Focus Master 6.1 to each student to complete. When they have completed the activity, discuss their methods and conclusions.

MODELING ALGEBRAIC EXPRESSIONS
FOCUS BLACKLINE MASTER 6.1 LESSON 6

1 a) Shown above are the first 4 arrangements in a sequence of counting piece arrangements. Form an Algebra Piece model of the n th arrangement and list several equivalent expressions for $v(n)$.

b) Complete the following statements. Use words or a sketch to explain how you arrived at your answer.
 $v(15) = \underline{\hspace{2cm}}$
 If $v(n) = -250$, then $n = \underline{\hspace{2cm}}$.
 If $v(n) = -98$, then $v(n - 2) = \underline{\hspace{2cm}}$.

c) On a sheet of grid paper, graph the values of at least the first 5 arrangements.

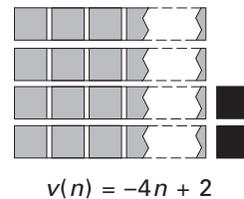
COMMENTS

Another way of stating the conclusion is that 24 is the number n for which $v(n) = 70$ or, since $v(n) = 3n - 2$, 24 is the number n for which $3n - 2 = 70$. Thus in arriving at an answer, the students have solved the equation $3n - 2 = 70$.

You can give the students other numbers and ask them to determine which arrangements, if any, have these numbers as values.

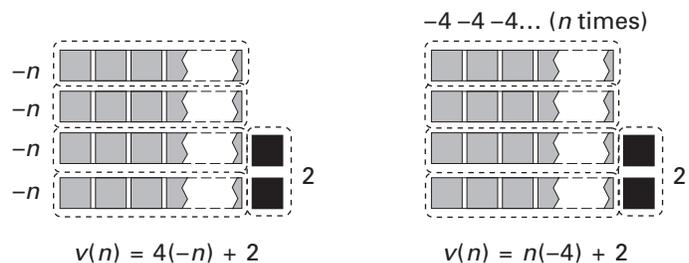
5 To begin the discussion, you can ask for volunteers to give their results and the methods they used to arrive at them.

1) Here is one Algebra Piece representation of the n th arrangement for the Sequence in 1a) on Student Activity 6.1:



The above expression for $v(n)$ was obtained by thinking of 4 red n -strips as the opposite of a collection of 4 black n -strips. Since 4 black n -strips have a value $4n$, the opposite collection has value $-(4n)$ or, dropping the parentheses, $-4n$.

Alternatively, since each red n -strip has value $-n$, the value of 4 of them can be written as $4(-n)$. The collection of 4 red n -strips can also be viewed as n collections of 4 red counting pieces each, that is, n collections each of value -4 , for a total value of $n(-4)$.



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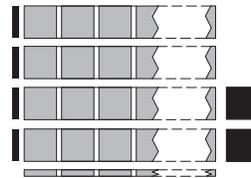
FOCUS

ACTIONS

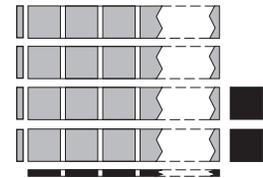
COMMENTS

5 continued

The n th arrangement of this sequence can also be viewed as 2 black counting pieces added to a rectangle with edges 4 and $-n$ or with edges -4 and n , as illustrated below.

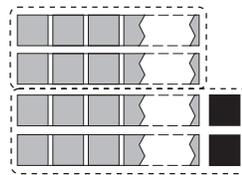


$$v(n) = 4(-n) + 2$$

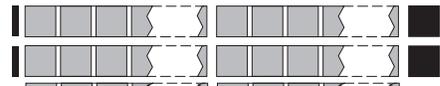


$$v(n) = (-4)n + 2$$

Below are two other variations for $v(n)$. In the sketch on the right, the pieces have been rearranged.



$$v(n) = -2n + 2(-n + 1)$$

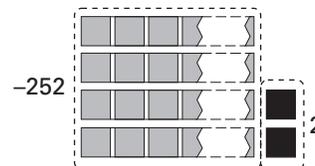


$$v(n) = 2(-2n + 1)$$

1b)i) $v(15)$ can be determined by noting that the 15th arrangement will contain 4 rows of 15 red counting pieces each and 1 column of 2 black counting pieces, so its value is $4(-15) + 2$, or -58 .

It can also be determined by evaluating any one of the above expressions for $v(n)$ when n is 15.

1b)ii) If $v(n) = -250$, then the value of the n th arrangement is -250 . Thus, the 4 red n -strips circled in the diagram at the left must have a total value of -252 . Hence, each strip must contain 63 red counting pieces. Since the number of counting pieces in a strip is the same as the number of the arrangement, $n = 63$. Note that the equation $-4n + 2 = -250$ has been solved.



Since the difference between successive arrangements is a column of 4 red counting pieces, the $(n - 2)$ nd arrangement can be obtained from the n th arrangement by removing 8 red counting pieces. If $v(n) = -98$, then $v(n - 2) = -98 - (-8) = -90$.

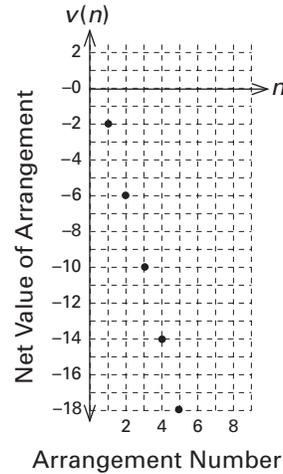
Alternately, one can first determine that if $v(n) = -98$ then $n = 25$. Thus, $n - 2$ is 23 and $v(23) = -4(23) + 2$ or -90 .

ACTIONS

COMMENTS

6 Distribute 1 sheet of 1/4" grid paper and 1 copy of Focus Master 6.2 (2 pages) to each student for completion. (See following page for Focus Master 6.2, page 2.) As appropriate, discuss the conclusions the students reach and the methods they used to arrive at these conclusions.

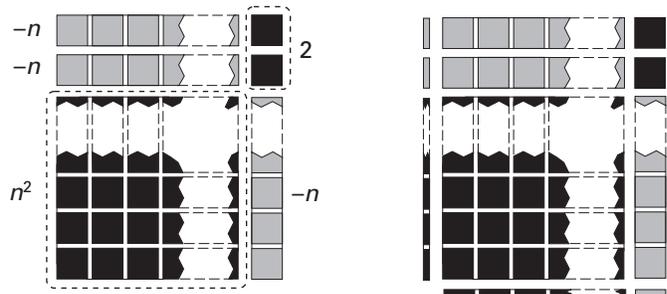
1c) A coordinate graph of $v(n)$ is shown below.



6 All or parts of the activities can be assigned as homework and then discussed in class. You may want to tell the students to be prepared to explain how they arrived at their answers.

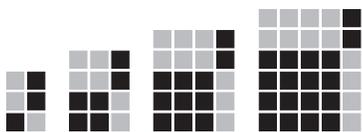
1a) In writing expressions for $v(n)$, recall that adding red (black) pieces to a collection has the same effect on its value as removing a like number of black (red) pieces. Thus, for example, $n + (-2)$ and $n - 2$ are equivalent expressions.

Below are some expressions for $v(n)$.



$$v(n) = n^2 - 3n + 2$$

$$v(n) = (n + (-2))(n + (-1)) \\ = (n - 2)(n - 1)$$



1 a) Shown above are the first 4 arrangements in a sequence of counting piece arrangements. Use Algebra Pieces to form the n th arrangement and list several equivalent expressions for $v(n)$.

b) On a sheet of grid paper, graph the values of at least 5 arrangements.

c) Challenge.

i) If $v(n) = 90$, then $n =$ _____. Explain how you arrived at your answer.

ii) If $v(n + 1) - v(n) = 50$, then $n =$ _____. Explain how you arrived at your answer.

continued next page

FOCUS

ACTIONS

COMMENTS

MODELING ALGEBRAIC EXPRESSIONS
FOCUS BLACKLINE MASTER 6.2 (CONT.) LESSON 6

2 a) Form the first 3 counting piece arrangements in a sequence for which the value of the n th arrangement is $(n + 1)(2n - 1)$.

b) Use Algebra Pieces to form a representation of the n th arrangement.

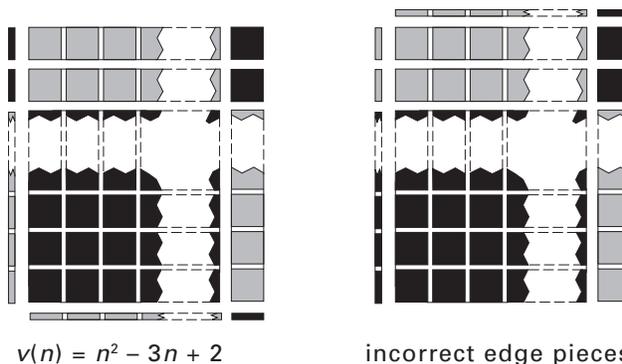
c) Which arrangement has value 1952? Explain how you decided this.

d) On a sheet of grid paper, make a coordinate graph that shows the net value of 5 or more arrangements in this sequence.

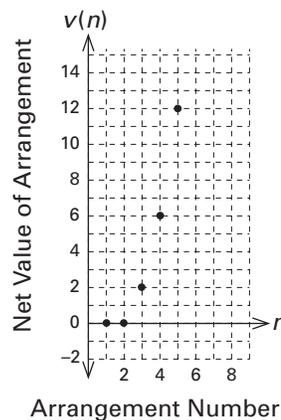
e) Challenge. The difference in value of two successive arrangements is 143. Which 2 arrangements are these? How did you decide this?

6 continued

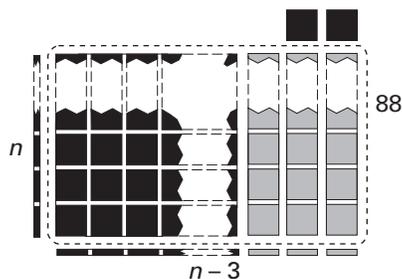
Edge pieces other than those shown above are possible. In the illustration below, the figure on the left shows a second possibility. The figure on the right shows an inappropriate selection of edge pieces. The students may need to be reminded that if edge pieces have the same color, the region associated with them is black, and if the edge pieces have opposite colors, the associated region is red.



1 b) A graph of $v(n)$ is shown below.



1 c) i) If $v(n) = 90$, the circled portion of the n th arrangement, rearranged as shown below, has value 88. This portion is a rectangle whose dimensions have value n and $n - 3$. Two numbers which differ by 3 and whose product is 88 are 11 and 8. Hence, $n = 11$.



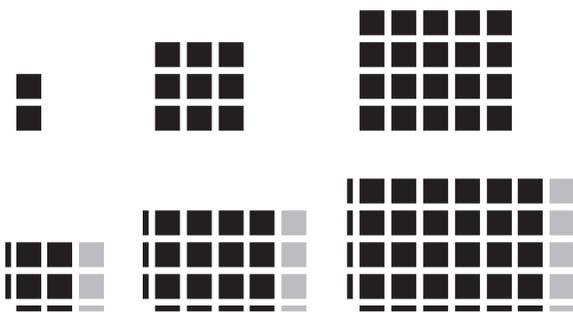
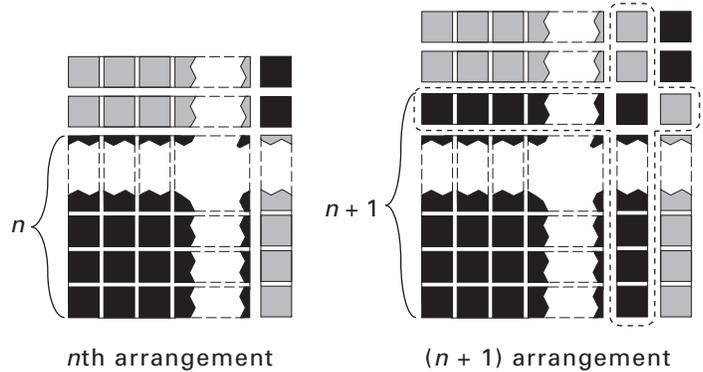
ACTIONS

COMMENTS

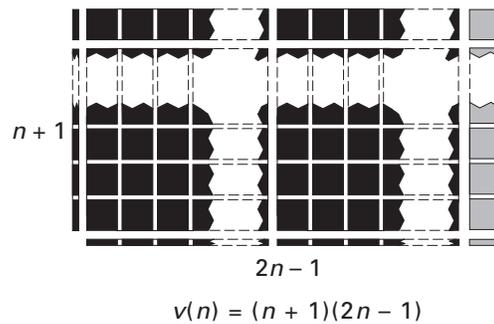
Some students may reason that since $n(n - 3) = 88$, then n^2 is somewhat greater than 88.

Other students may recognize that when $v(n)$ is expressed in the form $(n - 1)(n - 2)$, it is the product of two integers that differ by 1. Hence, this product will be 90 when the larger number, $n - 1$, is 10 and the smaller, $n - 2$, is 9. Then, since $n - 1 = 10$, $n = 11$.

1c)ii) If $v(n + 1) - v(n) = 50$, then the circled region of the $(n + 1)$ st arrangement, shown here, has value 50. Hence, the 2 n -strips in this region have a total value of 52 so each contains 26 counting pieces. Hence $n = 26$.



2a), b) Shown on the left are two possibilities for the first three arrangements. In the second case, edge pieces have been used and $2n - 1$ is thought of as $2n + (-1)$. When viewed in the latter way, a representation of the n th arrangement appears as shown below.



2c) From the above representation, one sees that $v(n)$ is somewhat more than twice a square. Thus, if $v(n) = 1952$, half of that, or 976, is somewhat larger than a square. Since $\sqrt{976}$ is about 31.2, this suggests that it is the 31st arrangement that has value 1952. This can be verified by computing $v(31)$.

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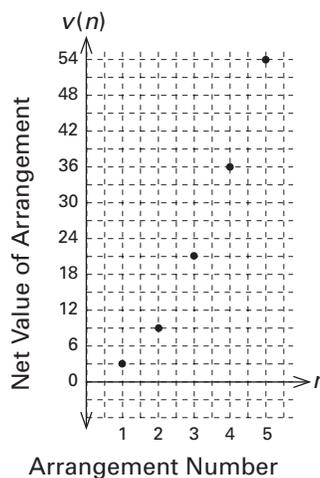
FOCUS

ACTIONS

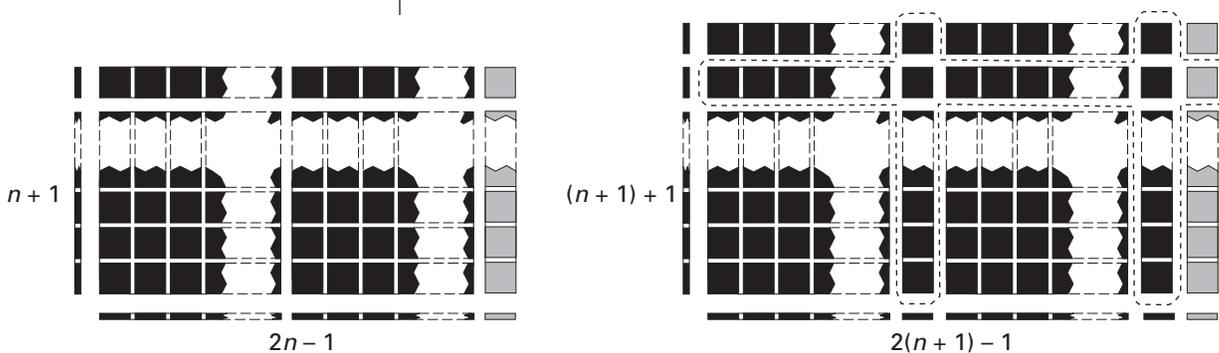
COMMENTS

6 continued

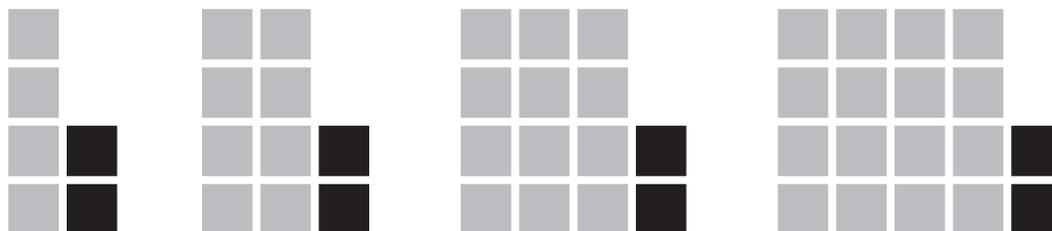
2c) A graph of $v(n)$ is shown to the left. Notice that on this graph, the scales on the n and $v(n)$ axes are different.



2d) If the n th and $(n + 1)$ st arrangements are formed as shown below, then the circled portion of the $(n + 1)$ st arrangement represents the difference, 143, in their values. Hence, the 4 encircled n -strips must have value 140, that is, $4n = 140$. Thus, $n = 35$ and the two arrangements are the 35th and 36th.



FOCUS BLACKLINE MASTER 6.1



I a) Shown above are the first 4 arrangements in a sequence of counting piece arrangements. Form an Algebra Piece model of the n th arrangement and list several equivalent expressions for $v(n)$.

b) Complete the following statements. Use words or a sketch to explain how you arrived at your answer.

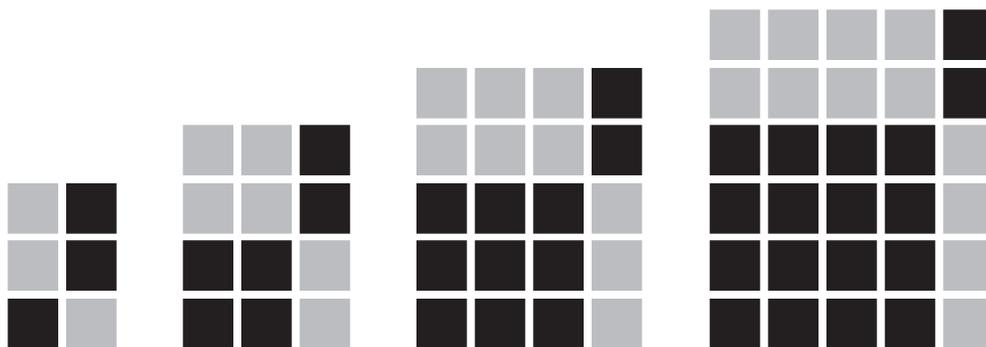
$$v(15) = \underline{\hspace{2cm}}$$

$$\text{If } v(n) = -250, \text{ then } n = \underline{\hspace{2cm}}.$$

$$\text{If } v(n) = -98, \text{ then } v(n - 2) = \underline{\hspace{2cm}}.$$

c) On a sheet of grid paper, graph the values of at least the first 5 arrangements.

FOCUS BLACKLINE MASTER 6.2



I a) Shown above are the first 4 arrangements in a sequence of counting piece arrangements. Use Algebra Pieces to form the n th arrangement and list several equivalent expressions for $v(n)$.

b) On a sheet of grid paper, graph the values of at least 5 arrangements.

c) Challenge.

i) If $v(n) = 90$, then $n = \underline{\hspace{2cm}}$. Explain how you arrived at your answer.

ii) If $v(n+1) - v(n) = 50$, then $n = \underline{\hspace{2cm}}$. Explain how you arrived at your answer.

continued

**FOCUS BLACKLINE MASTER 6.2 (CONT.)**

2 a) Form the first 3 counting piece arrangements in a sequence for which the value of the n th arrangement is $(n + 1)(2n - 1)$.

b) Use Algebra Pieces to form a representation of the n th arrangement.

c) Which arrangement has value 1952? Explain how you decided this.

d) On a sheet of grid paper, make a coordinate graph that shows the net value of 5 or more arrangements in this sequence.

e) Challenge. The difference in value of two successive arrangements is 143. Which 2 arrangements are these? How did you decide this?

FOLLOW-UP BLACKLINE MASTER 6

I The first column of the following chart contains formulas for the value of the n th arrangement of a sequence of counting piece arrangements.

- For each sequence, compute $v(3)$, the value of the 3rd arrangement, and $v(6)$, the value of the 6th arrangement, and enter these values in the chart.
- For each sequence, form a Algebra Piece representation of the n th arrangement. Then record in the chart the number of each kind of piece that occurs in your representation.

				Number of Pieces in Representation of n th Arrangement					
	$v(n)$	$v(3)$	$v(6)$	black n^2 -mats	red n^2 -mats	black n -strips	red n -strips	black counting pieces	red counting pieces
1.	$2n + 2$								
2.	$2(n + 2)$								
3.	$(n + 2)^2$								
4.	$n^2 + 2^2$								
5.	$-(n^2) + 2^2$								
6.	$(-n)^2 + 2^2$								
7.	$n^2 - 2^2$								
8.	$(n - 2)^2$								
9.	$(n - 2)(n + 2)$								
10.	$n - 2$								

a) Which, if any, of the above expressions for $v(n)$ are equivalent? _____

 **FOLLOW-UP BLACKLINE MASTER 6 (CONT.)**

- b) If, for each of the above sequences, a coordinate graph were constructed showing the values of the arrangements for that sequence, for which of these graphs would the points lie on a straight line? _____
- c) As the arrangement number gets larger, for which of the above sequences do the values of the arrangements
- i) increase most rapidly? _____
 - ii) increase least rapidly? _____
 - iii) decrease? _____
- d) For each of the sequences, determine which arrangements have negative values.

ANSWERS TO FOLLOW-UP 6

I Since Algebra Pieces representations aren't unique, other possibilities than those shown in the following chart are correct.

	$v(n)$	$v(3)$	$v(6)$	Number of Pieces in Representation of n th Arrangement					
				black n^2 -mats	red n^2 -mats	black n -strips	red n -strips	black counting pieces	red counting pieces
1.	$2n + 2$	8	14	—	—	2	—	2	—
2.	$2(n + 2)$	10	16	—	—	2	—	4	—
3.	$(n + 2)^2$	25	64	1	—	4	—	4	—
4.	$n^2 + 2^2$	13	40	1	—	—	—	4	—
5.	$-(n^2) + 2^2$	-5	-32	—	1	—	—	4	—
6.	$(-n)^2 + 2^2$	13	40	1	—	—	—	4	—
7.	$n^2 - 2^2$	5	32	1	—	—	—	—	4
8.	$(n - 2^2)^2$	1	4	1	—	—	8	16	—
9.	$(n - 2)(n + 2)$	5	32	1	—	2	2	—	4
10.	$n - 2$	1	4	—	—	1	—	—	2

- a) Expressions 4 and 6 and expressions 7 and 9 are equivalent.
- b) The points on the graphs of 1, 2, and 10 would lie on a straight line.
- c) i) The arrangement values of expression 3 increase the most rapidly.
 ii) The arrangement values of expression 10 increase the least rapidly.
 iii) The arrangement values of expression 5 decrease.
- d) Expression 5 is negative for all values of $n \geq 3$; expressions 7, 9, and 10 are negative for $n = 1$.

THE BIG IDEA

Algebra Pieces allow students to build and manipulate representations of formulas and to invent methods of solving equations. The strategies that students develop for manipulating the Algebra Pieces provide the basis for inventing procedures and strategies for manipulating algebraic symbols.

START-UP

Overview

As an introduction to solving equations, students determine a number n for which the n th arrangements of two sequences have the same value.

Materials

- Start-Up Masters 7.1-7.2, 1 copy per student and 1 transparency of each.
- Algebra Pieces for each student.
- Algebra Pieces for the overhead.
- $\frac{1}{4}$ " grid paper, 1 sheet per student.

FOCUS

Overview

Patterns of counting pieces are used to introduce equation solving involving positive and negative integers. Students use Algebra Pieces to model situations involving variables; they make observations and solve equations related to those situations.

Materials

- Algebra Pieces (see Lesson 6) for each student.
- Focus Master 7.1, 1 transparency.
- Algebra Pieces for the overhead.

FOLLOW-UP

Overview

Students solve equations and puzzle problems.

Materials

- Follow-Up 7, 2 pages run back-to-back, 1 copy per student.
- Algebra Pieces, paper or cardstock set for each student to use at home. (Note: students could complete the assignment by only sketching pieces; however, it is very helpful if they have pieces to manipulate before sketching.)

START-UP

Overview

As an introduction to solving equations, students determine a number n for which the n th arrangements of two sequences have the same value.

Materials

- Start-Up Masters 7.1-7.2, 1 copy per student and 1 transparency of each.
- Algebra Pieces for each student.
- Algebra Pieces for the overhead.
- $\frac{1}{4}$ " grid paper, 1 sheet per student.

ACTIONS

Give each student a set of Algebra Pieces and a copy of Start-Up Master 7.1. Ask them to form, first, an Algebra Piece representation of the n th arrangement of Sequence A and, second, an Algebra Piece representation of the n th arrangement of Sequence B. Have them write formulas for the value of these two arrangements. Discuss the students' results.

COMMENTS

Shown below are two possibilities for representations of the n th arrangements and formulas for their values. The students' formulas will vary depending on how they view the arrangements.

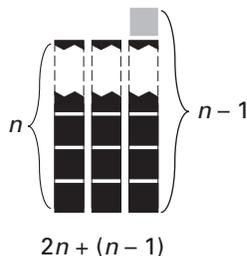
SEEING & SOLVING EQUATIONS
START-UP BLACKLINE MASTER 7.1

LESSON 7

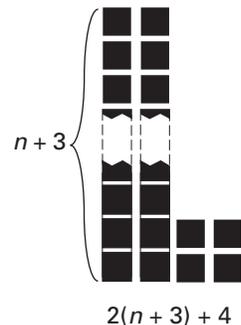
Sequence A

Sequence B

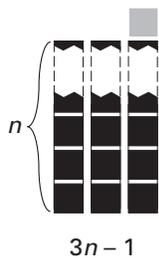
A



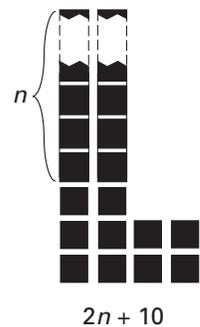
B



A



B



START-UP

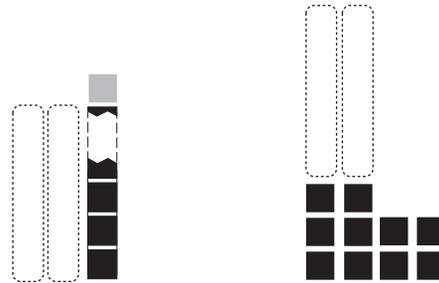
ACTIONS

2 Point out to the students that for the arrangements shown on Start-Up Master 7.1, the values of the arrangements of Sequence A are different from the corresponding arrangements of Sequence B, for example, the value of the first arrangement of Sequence A is 2 while the value of the first arrangement of Sequence B is 12. Tell the students that there is, however, a number n for which the values of the n th arrangements of Sequences A and B are equal. Ask the students to find that number. Discuss the methods the students use and call their attention to the equations they have solved.

3 Distribute a copy of Start-Up Master 7.2 to each student (see opposite page). Tell the students that these are Algebra piece representations of the n th arrangements of two sequences of tile patterns. Ask them to find the values of these arrangements and then determine the number n for which the n th arrangements of these two sequences are equal.

COMMENTS

2 If the two arrangements, in the first set of arrangements shown in Comment 1, have the same value, when 2 n -strips are removed from each, the remaining portions must have the same value. (See below.) Since the remaining portion of the arrangement on the right has value 10, the n -strip on the left must have value 11. So $n = 11$.



Some students may argue that if the remaining portions have the same value, they will continue to have the same value if 1 black is added to each remaining portion. This has the effect of eliminating the red tile on the left and one sees directly that the n -strip must consist of 11 tile. Alternatively, one can add a red and a black tile to the arrangement on the right, which leaves its value unchanged. Then a red tile can be removed from both arrangement, leaving 1 n -strip on the left and 11 black tile on the right.

The equation $2n + (n - 1) = 2(n + 3) + 4$, and any equivalent forms, such as $3n - 1 = 2n + 10$, has been solved.

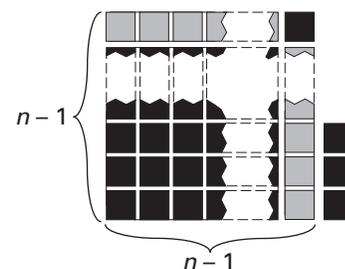
3 Subscripts have been used to distinguish $v_1(n)$ for Sequence 1 from $v_2(n)$ for Sequence 2.

There are a variety of equivalent algebraic expressions for $v_1(n)$. Here are three (edge pieces may be helpful in seeing the second of these):

$$\begin{aligned} n^2 - n - 2, \\ (n + 1)(n - 2), \\ (n^2 + n) - (2n + 2). \end{aligned}$$

Following are some expressions for $v_2(n)$. The last one is suggested by rearranging the pieces as shown.

$$\begin{aligned} n^2 - 2n + 4, \\ n(n - 2) + 4, \\ (n - 1)^2 + 3. \end{aligned}$$



ACTIONS

COMMENTS

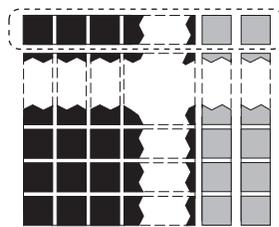
SEEING & SOLVING EQUATIONS
START-UP BLACKLINE MASTER 7.2 LESSON 7

***n*th Arrangements**

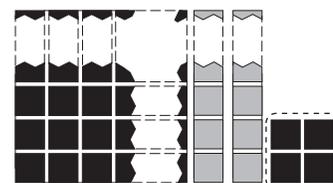
Sequence C

Sequence D

The two arrangements will have the same value if the circled portions shown to the left have the same value. This will be the case if the black *n*-strip contains 6 counting pieces. Thus, $v_1(n) = v_2(n)$ when $n = 6$. That is, 6 is the solution of $n^2 - n - 2 = n^2 - 2n + 4$.



$$v_1(n) = n^2 - n - 2$$



$$v_2(n) = n^2 - 2n + 4$$

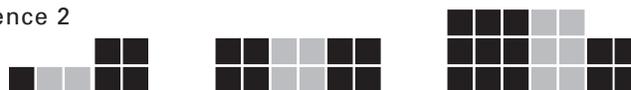
4 Ask the students to form the first 3 arrangements of each of the sequences in Action 3.

4 This Action is intended to help students visualize *n*-strips and *n*-mats for small values of *n*. In particular, they may not realize that when *n* is 1, both a *n*-strip and a *n*-mat represent a single tile.

Sequence 1



Sequence 2





START-UP BLACKLINE MASTER 7.1

Sequence A



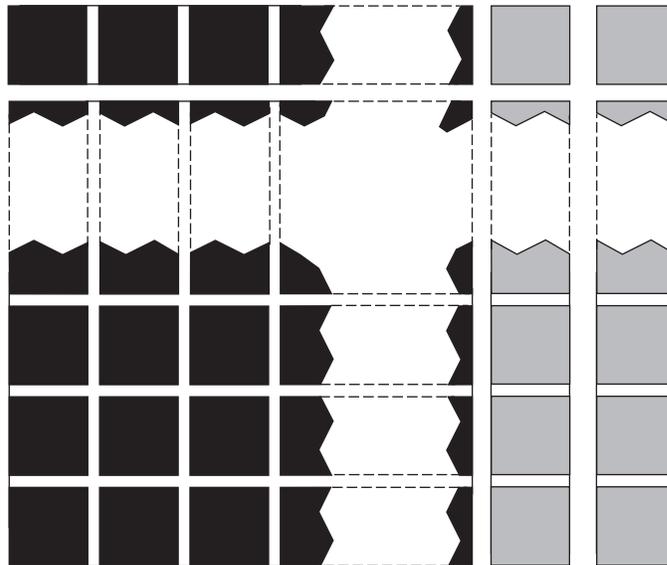
Sequence B



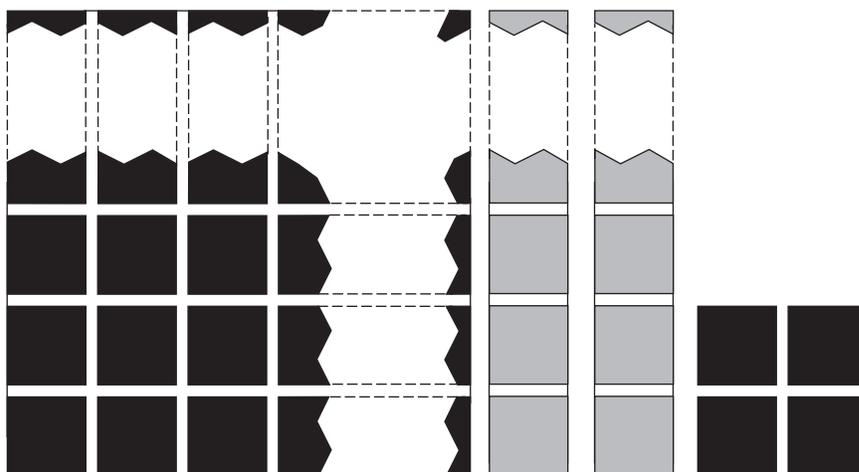
START-UP BLACKLINE MASTER 7.2

*n*th Arrangements

Sequence C



Sequence D



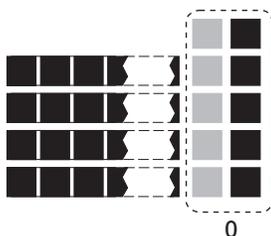
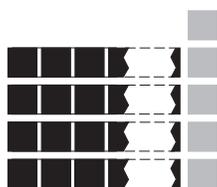
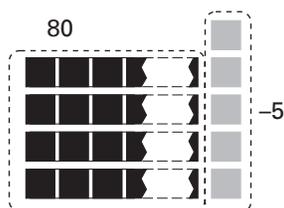
FOCUS

Overview

Patterns of counting pieces are used to introduce equation solving involving positive and negative integers. Students use Algebra Pieces to model situations involving variables; they make observations and solve equations related to those situations.

ACTIONS

Write the equation $4n - 5 = 75$ on the overhead. Ask the students to find a way to use Algebra Pieces to help solve this equation. Discuss the methods they use.



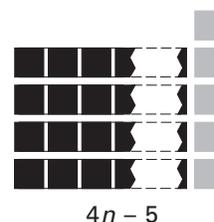
Materials

- Algebra Pieces (see Lesson 6) for each student.
- Algebra Pieces for the overhead.
- Focus Master 7.1, 1 transparency.

COMMENTS

One way to proceed is to form an Algebra Piece arrangement where value is $4n - 5$ and then determine for which n this arrangement has value 75.

Shown below is an Algebra Piece arrangement whose value is $4n - 5$.



The students will use various methods to determine for what n this arrangement has value 75. Some students may argue that if the arrangement has value 75 and the 5 red tile have value -5 , then the 4 black n -strips have value 80. Hence each has value 20, that is, $n = 20$.

Others may argue that the n th arrangement has the same value as an arrangement of 75 black tile (which might be represented by a scrap of paper with "75" recorded on it). These two arrangements will still have the same value if 5 black tile are added to both of them. This eliminates the 5 red tile in the n th arrangement leaving 4 n -strips and increases the arrangement of black tile to 80. So, again, $4n = 80$ and $n = 20$.

To introduce the student to standard algebraic notation, one can describe symbolically the thinking that is taking place. In the first example above in which 75 is partitioned into 80 and -5 , the steps in the argument might be recorded thus:

$$\begin{aligned} 4n - 5 &= 75 \\ 4n - 5 &= 80 - 5 \\ 4n &= 80 \\ n &= 20 \end{aligned}$$

continued next page


FOCUS

ACTIONS

2 Repeat Action 1 for the equation

$$(n - 1)(2n + 3) = 2n^2 + 5.$$

3 For each of the following, ask the students to use their Algebra Pieces to find a positive integer n for which it is true. Call on volunteers to explain their methods.

a) $4(n + 2) = 32$

b) $n^2 + 4 = 200$

c) $5n - 4 = 3n + 6$

d) $(n + 1)^2 = n^2 + 9$

e) $-3n + 7 = -92$

f) $n(n + 2) = 168$

g) $-4n - 1 = 2n - 49$

h) $n^2 - 4n = 140$

COMMENTS

1 continued

(In a formal development, the transition from the second to the third equation is justified by the Cancellation Law: If $a + c = b + c$, then $a = b$.)

The steps in the second example above might be recorded as follows:

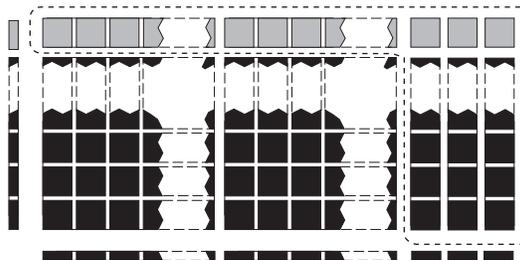
$$4n - 5 = 75$$

$$4n - 5 + 5 = 75 + 5$$

$$4n = 80$$

$$n = 20$$

2 An arrangement with value $(n - 1)(2n + 3)$ can be represented by a rectangular array whose edges have values $n - 1$ and $2n + 3$. This arrangement will have the same value as an arrangement representing $2n^2 + 5$ if the circled portion has value 5. The circled portion has value $n - 3$ which equals 5 if n is 8.



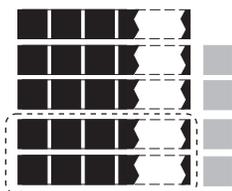
$$(n - 1)(2n + 3)$$

3 The students will use various methods to solve these equations. If a student arrives at a solution without using Algebra Pieces, you can ask them to use the pieces to illustrate their thinking.



ACTIONS

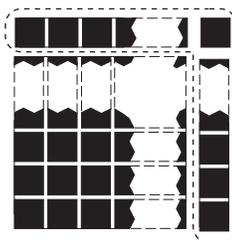
COMMENTS



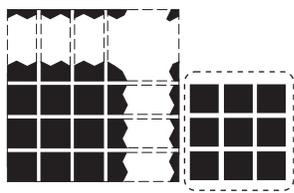
$5n - 4$



$3n + 6$



$(n + 1)^2$



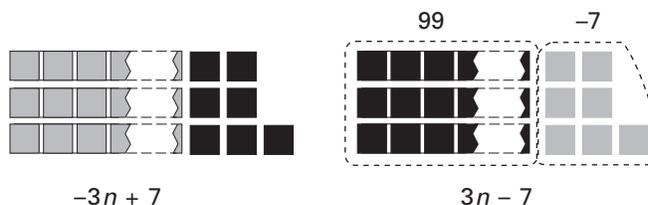
$n^2 + 9$

Possible solutions for parts c) through h) are given below.

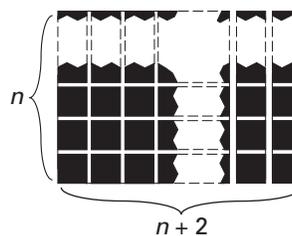
c) The arrangements shown on the left have the same value provided the circled portions have the same value. The portion on the right has value 6. The portion on the left will have value 6 if, together, the 2 n -strips have value 10. Hence, $2n = 10$ and $n = 5$.

d) The arrangements shown on the left will have the same value if the circled portions have the same value. This will be the case if each of the two n -strips in the arrangement on the left contains 4 black tile.

e) The arrangement below representing $-3n + 7$ has value -92 . The opposite of this arrangement, on the right, has value 92. Thus the 3 n -strips of black tile in the opposite collection have a total value of 99, Hence, $n = 33$.



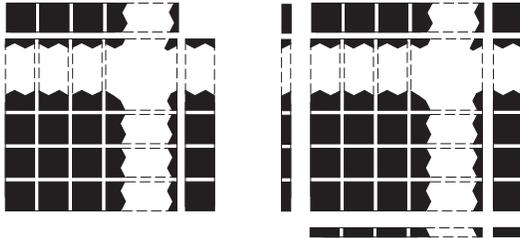
f) The edges of an arrangement for $n(n + 2)$ differ by 2. The product of the edges is the value of the array which, in this case, is 168. Two numbers which differ by 2 and whose product is 168 are 12 and 14. Thus $n = 12$.



continued next page



ACTIONS

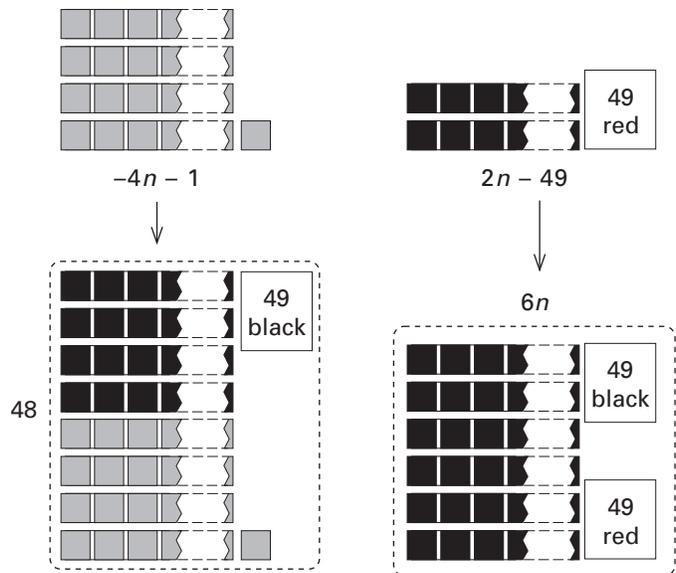


COMMENTS

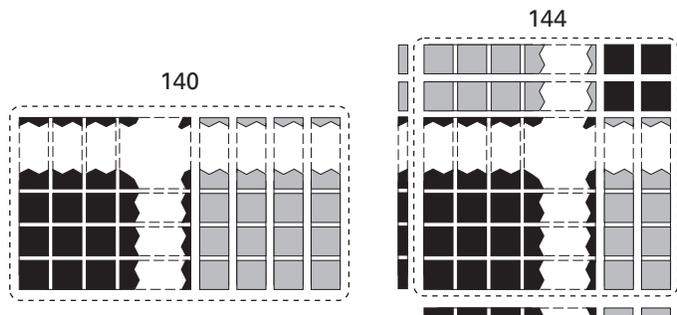
3 continued

Alternatively, the pieces can be arranged as shown on the left. Adding a black tile in the upper corner creates a $(n + 1)$ by $(n + 1)$ square whose value is 169. Hence $n + 1 = 13$ and $n = 12$. This method of solving the equation is called *completing the square*.

g) If arrangements for $-4n - 1$ and $2n - 49$ have the same value, adding 4 black n -strips and 49 black tile to both arrangements results in two arrangements which have the same value. Discarding matching red and black pieces from these arrangements leaves 48 black tile in one and 6 black n -strips in the other. Thus $6n = 48$ and $n = 8$.



h) An arrangement for $n^2 - 4n$ becomes an $n - 2$ by $n - 2$ square if 4 black tile are added. If the original arrangement has value 140, the completed square will have value 144. Hence $n - 2 = 12$ and $n = 14$.




FOCUS

ACTIONS

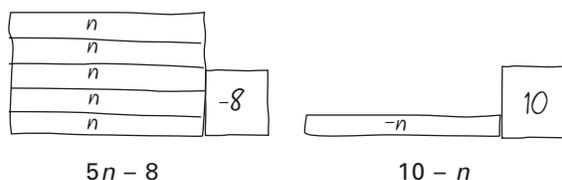
4 Ask the students to use Algebra Pieces to find a positive integer n such that $5n - 8 = 10 - n$. Then ask them to draw a sketch or sketches that show how they used the pieces to arrive at a solution. Ask for volunteers to show their sketches.

5 Repeat Action 4 for $(n + 3)(2n - 1) = 2n^2 + 12$.

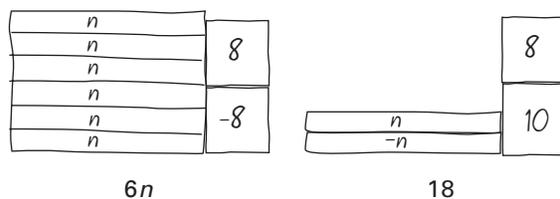
COMMENTS

4 As one student is showing their solution at the overhead, you can provide other volunteers with a blank overhead transparency and pens to prepare their sketches prior to their presentations.

Here is one possibility:

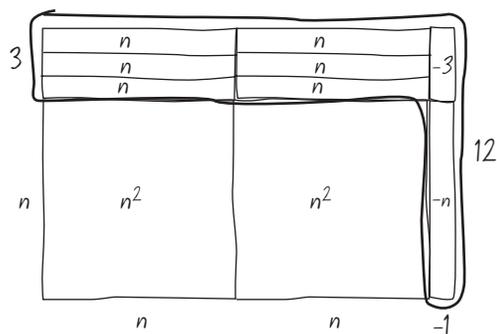


Add a black n -strip and 8 black tile to both arrangements:



Hence, $6n = 18$ so $n = 3$.

5 If $(n + 3)(2n - 1) = 2n^2 + 12$, then the circled portion of the arrangement has value 12. So the 6 black n -strips and 1 red n -strip in the circled portion have a total value of 15. Thus $5n = 15$ and $n = 3$.



FOCUS

ACTIONS

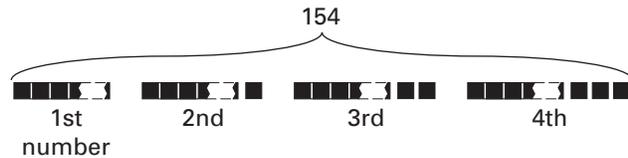
6 Place a transparency of Focus Master 7.1 on the overhead, revealing Situation a) only. Ask the students to form an Algebra Piece model of Situation a) and to make observations and conclusions about the situation. Have volunteers share their models and reasoning at the overhead. Repeat for other of Situations b) through g).

COMMENTS

6 If students focus too quickly on “getting answers” rather than on modeling the situations, you can conceal the second part of the situation until students have formed an Algebra Piece representation of the first part. For example, in a) you might conceal “have a sum of 154” until students have formed an Algebra Piece model of 4 consecutive positive integers.

You can ask students, whenever possible to write equations that correspond to their Algebra Piece models, as in the following example for Situation a)

Algebra Piece representation:



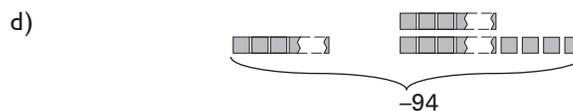
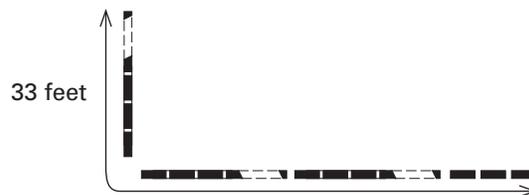
Corresponding equation:

$$n + (n + 1) + (n + 2) + (n + 3) = 154, \text{ where } n \text{ is the first number}$$

Combining the Algebra Pieces into a single collection corresponds to collecting terms in the above equation to obtain the equation $4n + 6 = 154$.



c) The edge pieces below represent half the perimeter.



SEEING & SOLVING EQUATIONS
FOCUS BLACKLINE MASTER 7.1
 LESSON 7

Situations

a) Four consecutive positive integers have a sum of 154.

b) Two pieces of rope differ in length by 8 meters. End to end, their total length is 132 meters.

c) The length of a rectangle is 3 feet more than twice its width. The perimeter of the rectangle is 66 feet.

d) The sum of 2 negative integers is -94. The second number is 4 less than twice the first.

e) Two numbers differ by 3. Their squares differ by 189.

f) Lisa high jumped twice as high as Evan. Gloria jumped 6 inches higher than Lisa. All together they jumped 161 inches.

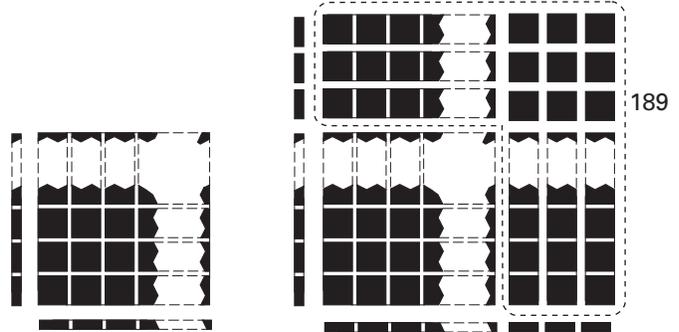
g) This number trick works every time:
 Step 1. Pick any integer.
 Step 2. Subtract 3 from the number you picked.
 Step 3. Double your answer from Step 2.
 Step 4. Add 8 to your Step 3 answer.
 Step 5. Divide by 2.
 Step 6. Add your original number.
 The answer is one more than twice your original number.



ACTIONS

COMMENTS

e)



f)



g) Below is an Algebra Piece model of the number trick. In Case 1, the number chosen is positive, in Case 2 it is negative. In both cases, the result is one more than twice the original number. Note that the trick also works if the number chosen is 0.

		Case 1	Case 2
Step 1.	Pick any integer.		
Step 2.	Subtract 3.		
Step 3.	Double it.		
Step 4.	Add 8.		
Step 5.	Divide by 2.		
Step 6.	Add original number.		

**FOCUS BLACKLINE MASTER 7.1****Situations**

- a) Four consecutive positive integers have a sum of 154.
- b) Two pieces of rope differ in length by 8 meters. End to end, their total length is 132 meters.
- c) The length of a rectangle is 3 feet more than twice its width. The perimeter of the rectangle is 66 feet.
- d) The sum of 2 negative integers is -94 . The second number is 4 less than twice the first.
- e) Two numbers differ by 3. Their squares differ by 189.
- f) Lisa high jumped twice as high as Evan. Gloria jumped 6 inches higher than Lisa. All together they jumped 161 inches.
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- Step 1. Pick any integer.
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 - Step 3. Double your answer from Step 2.
 - Step 4. Add 8 to your Step 3 answer.
 - Step 5. Divide by 2.
 - Step 6. Add your original number.
- The answer is one more than twice your original number.

 FOLLOW-UP BLACKLINE MASTER 7

1 Find a positive integer n for which each of the following is true. Describe how you arrived at your answer.

a) $2n - 8 = 4$

b) $7 - 3n = 5n - 9$

c) $n^2 + 3n - 2 = 23 + 3n$

d) $(4n - 2)(n - 1) = 4n^2 + 3n - 16$

2 Solve the following puzzle problems. Make sketches that illustrate your thinking.

a) The sum of three numbers is 118. The middle number is 5 more than the smallest and 3 less than the greatest. What are the numbers?

b) The length of a rectangle is 3 meters longer than its width. The perimeter of the rectangle is 114 meters. What are the dimensions of the rectangle?

c) Nifty Tractor Rental charges a \$15 fee plus \$2 for every mile a tractor is driven. If Jon paid \$101 to rent a tractor, how far did he drive it?

d) Jessica is 3 years older than Dylan. Jaime is twice as old as Jessica. When Jessica's, Dylan's, and Jaime's ages are added together, the total is 81. How old is each person?

e) Two consecutive square numbers have a difference of 39. What are the numbers?

continued on back

 FOLLOW-UP BLACKLINE MASTER 7 (CONT.)

3 Use diagrams of Algebra Pieces to explain why the following number trick works:

Step 1: Pick a number.

Step 2: Triple your number.

Step 3: Subtract 3 from your answer in Step 2.

Step 4: Add 9 to your answer in Step 3.

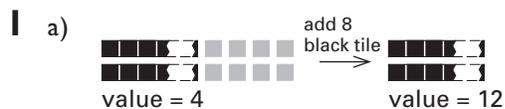
Step 5: Divide your answer in Step 4 by 3.

Step 6: Add 2 less than your original number.

The answer will always be double your original number!

4 Devise your own number trick.

ANSWERS TO FOLLOW-UP 7



If a collection of 2 black n -strips and 8 red counting pieces has value 4, then 2 n -strips have value 12 and each has value 6. So, $n = 6$.

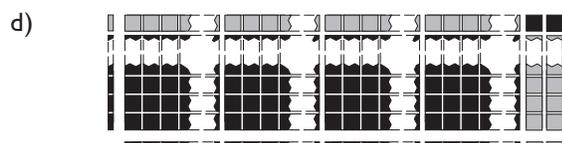
b) If 7 black counting pieces and 3 $-n$ -strips have the same value as 9 red counting pieces and 3 n -strips, then adding 3 n -strips and 9 black counting pieces to each of these collections, one sees that 16 black counting pieces and 8 n -strips have the same value. So, $n = 2$.

In symbols:

$$\begin{aligned} 7 - 3n &= 5n - 9, \\ (7 - 3n) + (3n + 9) &= (5n - 9) + (3n + 9) \\ 16 &= 8n \\ 2 &= n \end{aligned}$$



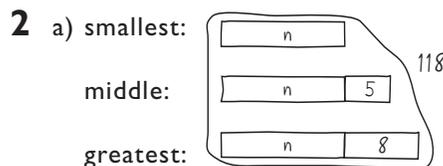
If 3 n -strips are removed from both collections and 2 black counting pieces are added to both collections, the n^2 -mat has value 25. So, $n = 5$.



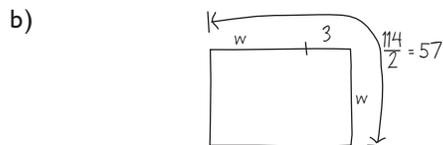
An array whose edges have values $4n - 2$ and $n - 1$ contains 4 n^2 -mats, 6 $-n$ -strips and 2 black counting pieces. If this collection has the same value as a collection of 4 n^2 -mats, 3 n -strips and 16 red counting pieces, then 9 n -strips have value 18 and $n = 2$.

In symbols:

$$\begin{aligned} (4n - 2)(n - 1) &= 4n^2 + 3n - 16 \\ 4n^2 - 6n + 2 &= 4n^2 + 3n - 16 \\ -6n + 2 &= 3n - 16 \\ (-6n + 2) + (6n + 16) &= (3n - 16) + (6n + 16) \\ 18 &= 9n \\ 2 &= n \end{aligned}$$



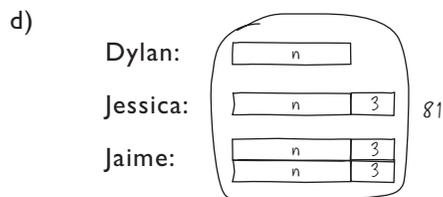
The 3 n -strips total 105, so each is 35. Thus the 3 numbers are 35, 40, and 43.



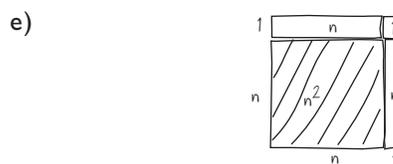
Since $2w + 3 = 57$, $2w = 54$. So $w = 27$ and $w + 3 = 30$. Thus, the dimensions are 27×30 .



If the number of miles is represented by an n -strip, $2n + 15$ is 101. So, $2n = 86$ and $n = 43$.



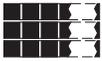
The 4 n -strips total 72, so each is 18. Thus, Dylan is 18, Jessica is 21, and Jaime is 42.



If the 2 square numbers are n^2 and $(n + 1)^2$, then their difference $2n + 1$ has value 39 and $n = 19$. So the numbers are 19^2 and 20^2 , that is, 361 and 400.

continued

ANSWERS TO FOLLOW-UP 7 (CONT.)

- 3** Step 1  n
- Step 2  $3n$
- Step 3  $3n - 3$
- Step 4  $3n + 6$
- Step 5  $n + 2$
- Step 6  $(n + 2)$
 $+ (n - 2)$
-  $2n$

EXTENDED COUNTING PIECE PATTERNS

THE BIG IDEA

Counting piece patterns whose arrangement numbers include both the negative and positive integers provide a meaningful context in which to develop strategies for representing equations whose solutions are integers. Using Algebra Pieces to represent and solve such equations builds intuitions about the meaning of a variable and promotes the invention of general strategies for solving equations and systems of equations.

START-UP

Overview

Counting piece patterns are extended to include arrangements corresponding to non-positive integers (i.e., negative integers and zero) as well as positive integers.

Materials

- Bicolored Counting Pieces, 75 per student or group of students.
- Start-Up Master 8.1, 1 transparency.

FOCUS

Overview

Counting piece patterns are used to introduce the students to equations which may have negative, as well as positive, integral solutions.

Materials

- Algebra Pieces, 1 set per student.
- Algebra Piece *frames* (see Appendix), 8 per student.
- Focus Masters 8.1-8.4; 1 copy of each per student and one transparency of each.

FOLLOW-UP

Overview

Students sketch Algebra Piece representations of the n th arrangement of sequences and use those representations as a basis for solving linear and quadratic equations.

Materials

- Follow-Up 8, 2 pages run back-to-back, 1 copy per student.

START-UP

Overview

Counting piece patterns are extended to include arrangements corresponding to non-positive integers (i.e., negative integers and zero) as well as positive, integers.

Materials

- Bicolored Counting Pieces, 75 per student or group of students.
- Start-Up Master 8.1, 1 transparency.

ACTIONS

1 Distribute Counting Pieces to each student or group of students. Form the following collection of counting piece arrangements on the overhead. Ask the students to form the same arrangements. Then ask them to form additional arrangements which maintain the pattern of the collection.

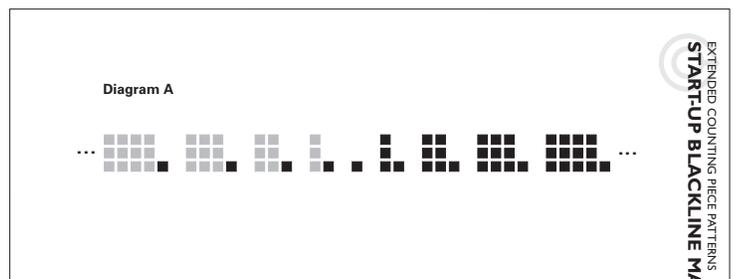


2 If it hasn't come up, discuss the students' ideas about ways the collection might be extended indefinitely in two directions.

COMMENTS

1 The students may add arrangements that extend the pattern in one direction only (left or right). If so, ask them to extend the pattern in the other direction also.

2 One way to extend the collection is shown in Diagram A on Start-Up Master 8.1. Going to the right, a column of 3 black tile is added to an arrangement to get the next arrangement. Going to the left, a column of 3 red tile is added.



START-UP

ACTIONS

3 Place a transparency of Start-Up Master 8.1 on the overhead, revealing Diagram A only (see previous page). Discuss the students' ideas about ways the arrangements in the extended sequence shown in Diagram A might be numbered.

4 Ask the students to number the arrangements in Diagram A from Start-up Master A as suggested in Comment 3 and then write an expression for $v(n)$, the value of the arrangement numbered n . Repeat for several different numberings.

COMMENTS

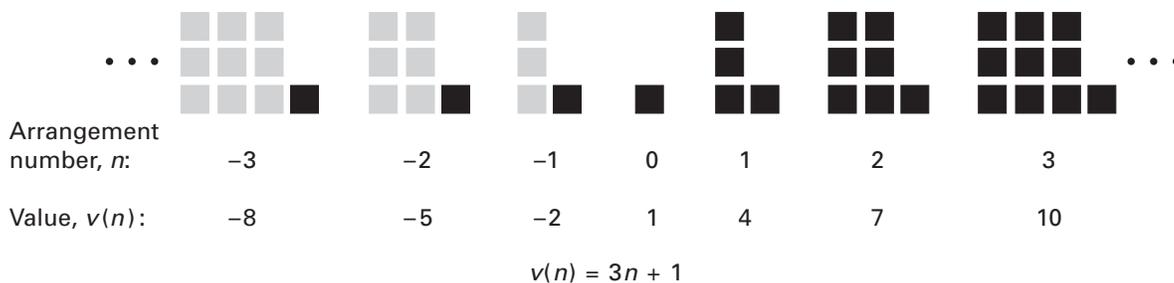
3 You may want to cut the transparency in half, separating Diagrams A and B. Diagram B is referred to in Action 6.

One way of numbering the arrangements is to select one of them and number it 0. Arrangements to the right of this arrangement are successively numbered 1, 2, 3, etc. Those to the left are successively numbered $-1, -2, -3$, etc.

A collection of arrangements which extends indefinitely in two directions and is numbered so there is an arrangement which corresponds to every integer, positive, negative, and zero, will be called an *extended sequence*.

(Mathematically speaking, a set of arrangement numbers is called an *index set* and an individual arrangement number is called an *index*. Thus, an arrangement whose number is -3 could be referred to as "the arrangement whose index is -3 ." Instead of using this language, we shall refer to this arrangement as "arrangement number -3 " or "the -3 rd arrangement." In the language of index sets, a *sequence* is a collection of arrangements whose index set is the set of positive integers and an *extended sequence*—a phrase coined for our purposes—is a set of arrangements whose index set is the set of all integers. On occasion, once a set of arrangements has been determined to be an extended sequence, it will be referred to simply as a sequence, the word "extended" being understood.)

4 The expression $v(n)$ will depend on the choice of numbering. If the arrangement consisting of a single black tile is numbered 0, then $v(n) = 3n + 1$. Notice this formula holds for all integers n , positive, negative, or zero.

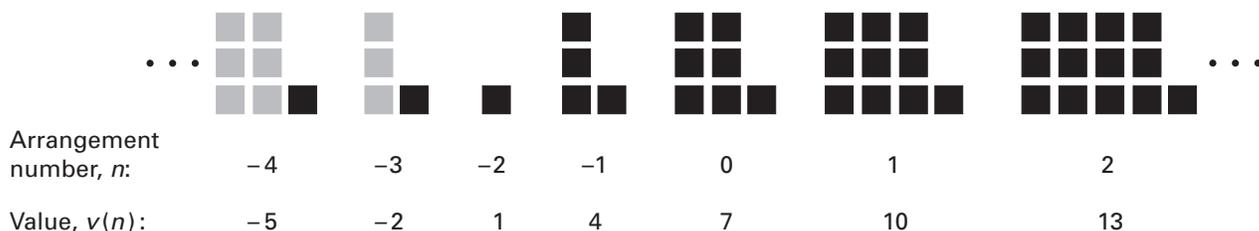


START-UP

ACTIONS

COMMENTS

A different numbering will result in a different expression for $v(n)$. For example, if the arrangement which consists of 7 black tile is numbered 0, one could view the n th arrangement as 1 black unit added to 3 strips with value $(n + 2)$. Hence, the n th arrangement has value $v(n) = 3n + 7$.

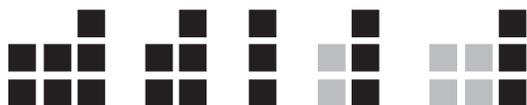


$$v(n) = 3(n + 2) + 1 \text{ or } v(n) = 3n + 7$$

5 Ask the students to assume the arrangements in Diagram A on Start-Up Master 8.1 are numbered so the 0th arrangement contains the single black counting piece (and no other pieces). Ask the students to describe

- a) the 50th arrangement
- b) the -100th arrangement

6 Form the following collection of counting piece arrangements and ask the students to do the same.



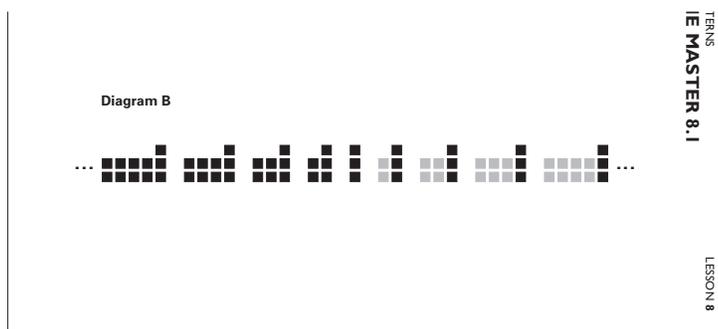
Have the students do the following:

- a) Form further arrangements that maintain the pattern and extend the collection in both directions.

5 The 50th arrangement has 50 columns with 3 black tile per column and an adjoining black tile. Alternatively, it can be described as 3 rows of 50 black tile with an adjoining black tile. Other descriptions are possible.

The -100th arrangement (that is, the arrangement numbered -100) has 3 rows of 100 red tile each and an adjoining black tile. Note that the number of red tile in each row is the opposite of the number of the arrangement.

6 a) The collection can be extended to the right by adding a column of 2 red tile to an arrangement to get the next arrangement. It can be extended to the left by adding a column of 2 black tile to successive arrangements. This is illustrated in Diagram B on Start-Up Master 8.1, shown below.



continued next page

START-UP

ACTIONS

b) Select a numbering for the extended sequence, and write an expression for $v(n)$, the value of the n th arrangement.



Arrangement number, n :	-3	-2	-1	0	1	2	3
Value, $v(n)$:	$2(3) + 3$	$2(2) + 3$	$2(1) + 3$	3	$2(-1) + 3$	$2(-2) + 3$	$2(-3) + 3$

c) Assuming the single column of 3 black counting pieces is designated the 0th arrangement, describe the 50th arrangement and the -100th arrangement.

COMMENTS

6 continued

b) If the single column of 3 black tile is designated the 0th arrangement, then $v(n) = 2(-n) + 3 = -2n + 3$. Other numberings will result in different expressions.

c) Using the given numbering, the 50th arrangement has 2 rows of 50 red tile each and a column of 3 adjoining black tile. The -100th arrangement has 2 rows of 100 black tile each and an adjoining column of 3 black tile. Other descriptions are possible.

START-UP BLACKLINE MASTER 8.1

Diagram A



Diagram B





Overview

Counting piece patterns are used to introduce the students to equations which may have negative, as well as positive, integral solutions.

Materials

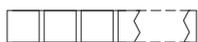
- Algebra Pieces, 1 set per student.
- Algebra Piece frames (see Appendix), 8 per student.
- Focus Masters 8.1-8.4; 1 copy per student and one transparency of each.

ACTIONS

1 Distribute Algebra Pieces, without frames, to each student and a copy of Focus Master 8.1 (see following page). For each of the extended sequences on Focus Master 8.1, ask the students to:

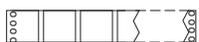
- a) build an Algebra Piece representation of the n th arrangement for n positive,
- b) build an Algebra Piece representation of the n th arrangement for n negative.

2 Distribute and discuss the role of frames. Then ask the students to use them to construct representations of the n th arrangements of the sequences of Action 1.



n -frame

Has value n for all n , positive, negative, or zero.



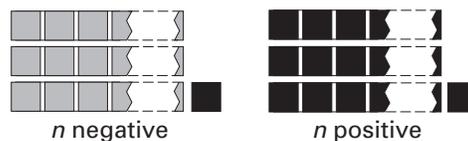
$-n$ -frame

Has value $-n$ for all n , positive, negative, or zero.

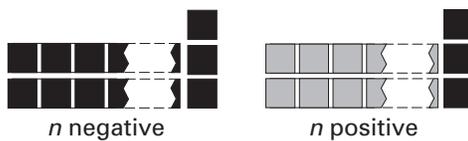
COMMENTS

1 The sequences are those introduced in the Start-Up Activity, numbered as shown on Focus Master 8.1.

One way of forming the arrangements is shown below.



$$v(n) = 3n + 1$$



$$v(n) = -2n + 3$$

2 Frames are introduced to provide pieces that can be either black or red, depending on the value of n . They enable one to build one algebra piece representation for the n th arrangement in an extended sequence, eliminating the need to build separate arrangements for positive n and negative n .

There are two types of frames: n -frames and $-n$ -frames, which are opposites of each other. The value of an n -frame is always n and the value of a $-n$ -frame is always $-n$, regardless of whether n is positive, negative, or zero. Thus if n is positive, an n -frame contains black tile and a $-n$ -frame contains red tile, if n is negative, an n -frame contains red tile and a $-n$ -frame contains black tile and if n is zero, neither frame contains tile.

For example, if $n = 50$, an n -frame contains 50 black tile and a $-n$ -frame contains 50 red tile. If $n = -50$, an n -frame contains 50 red tile and a $-n$ -frame contains 50 black tile.

A $-n$ -frame is distinguished from an n -frame by the small o's on each end.

continued next page

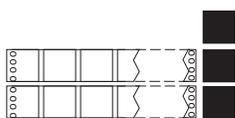
FOCUS

ACTIONS

COMMENTS



$$v(n) = 3n + 1$$



$$v(n) = -2n + 3$$

2 continued

Masters for n -frames and $-n$ -frames are included in the Appendix (n -frames are included in each set of plastic or cardstock Algebra Pieces available from The Math Learning Center). They are intended to be printed back-to-back so that turning over an n -frame results in a $-n$ -frame, and conversely.

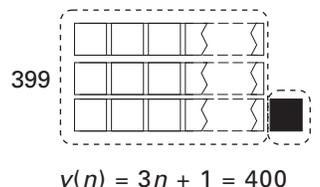
Representations of the n th arrangements of the two sequences are shown on the left.

Students may question whether there are frames for squares. Since n^2 is positive for all positive and negative values of n and $-n^2$ is negative for all positive and negative values of n , square frames are not needed.

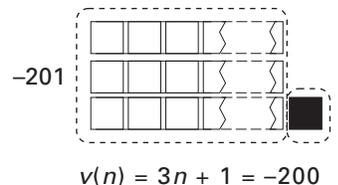
3 Ask the students to determine, for Extended Sequence A on Focus Master 8.1, the number of the arrangement which has value a) 400, b) -200 .

Then, for Extended Sequence B, ask the students to determine the number of the arrangement which has value a) 165, b) -75 .

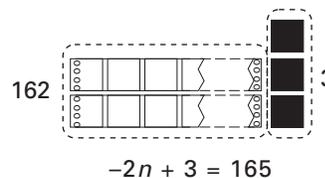
3 If an arrangement in Extended Sequence A has value 400, the 3 n -frames in the n th arrangement shown below have a total value of 399. Hence, each has value 133. Thus, $n = 133$ and it is the 133rd arrangement which has value 400.



If an n th arrangement has value -200 , the 3 n -frames in the figure have a total value of -201 , as illustrated in the diagram below. Hence, each has value -67 and thus $n = -67$.



If the value of the n th arrangement of Extended Sequence B is 165, as shown below, then each of the 2 $-n$ -frames in the figure has a value of 81. Hence, $-n = 81$ in which case, $n = -81$. So, it is the -81 st arrangement which has value 165.



Extended Sequence A									
...									...
Arrangement number, n :	-4	-3	-2	-1	0	1	2	3	4
Value, $v(n)$:	-11	-8	-5	-2	1	4	7	10	13
	$v(n) = 3n + 1$								
Extended Sequence B									
...									...
Arrangement number, n :	-4	-3	-2	-1	0	1	2	3	4
Value, $v(n)$:	11	9	7	5	3	1	-1	-3	-5
	$v(n) = -2n + 3$								

EXTENDED COUNTING PIECE PATTERNS
FOCUS BLACKLINE MASTER 8.1

LESSON 8

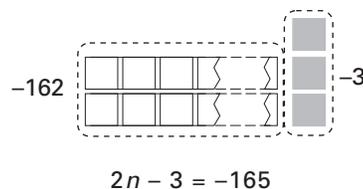


ACTIONS

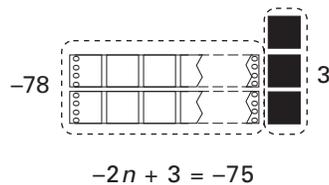
COMMENTS

4 Ask the students to build the -2 nd, -1 st, 0 th, 1 st, and 2 nd arrangements of an extended sequence for which $v(n) = -3n - 2$. Have volunteers demonstrate their results. Then have the students form a single arrangement of Algebra Pieces that represents the n th arrangement of this extended sequence.

Alternatively, one can turn over the pieces in the above arrangement to obtain the opposite arrangement, shown below. It will have value -165 and each of the n -frames will have value -81 . Hence $n = -81$.



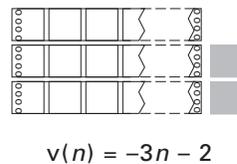
If the value of the n th arrangement shown below is -75 , each $-n$ -frame in the figure has value -39 . Hence, $-n = -39$. Thus, $n = 39$ and it is the 39 th arrangement which has value -75 .



4 Here is one possibility for the requested arrangements:

Arrangement number, n :	-2	-1	0	1	2
Value, $v(n)$:	$-3(-2) - 2$	$-3(-1) - 2$	-2	$-3(1) - 2$	$-3(2) - 2$

Since $v(n) = -3n - 2 = 3(-n) - 2$, one possible representation of the n th arrangement is the following:



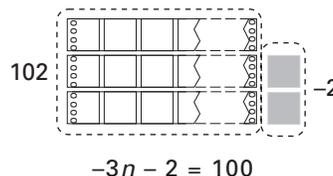
FOCUS

ACTIONS

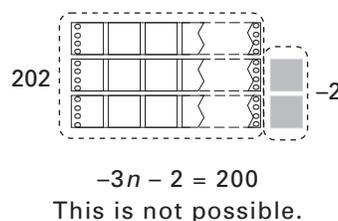
5 Ask the students to determine which arrangement, if any, of the extended sequence in Action 4 has value a) 100, b) 200, c) -200. Ask the students to record the equations that have been solved.

COMMENTS

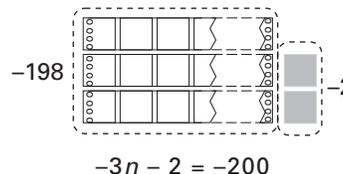
5 a) If an n th arrangement has value 100, each $-n$ -frame is $102 \div 3$ or 34. Thus $n = -34$.



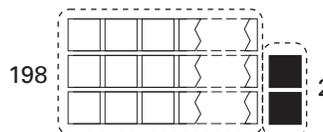
b) No arrangement of this extended sequence has value 200.



c) In this case, $-n = -198 \div 3 = -66$. Hence $n = 66$.



Some students may prefer to work with the opposite arrangements. For example, turning over the arrangement in c) above, one has, as shown below, $3n = 198$ and thus $n = 66$.





ACTIONS

6 Place a transparency of Focus Master 8.2 on the overhead and ask the students to build an Algebra Piece representation of the n th arrangement of each sequence. Then ask them to determine for which n these two arrangements have the same value. Discuss the equation that has been solved.

EXTENDED COUNTING PIECE PATTERNS
FOCUS BLACKLINE MASTER 8.2 LESSON 8

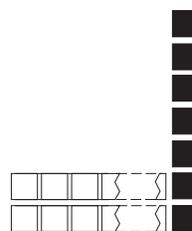
Arrangement number, n :	-2	-1	0	1	2
Sequence A					
Sequence B					

COMMENTS

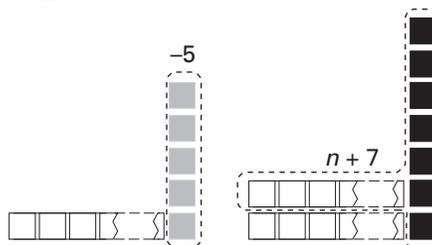
6 A representation for the n th arrangement of A:



A representation for the n th arrangement of B:



The two arrangements have the same value if the circled portions shown below have the same value. The portion on the left has value -5 , and the portion on the right has value $n + 7$. The portion on the right will have value -5 if the enclosed n -frame has value -12 , i.e., if $n = -12$. Thus, the solution to the equation $n - 5 = 2n + 7$ is $n = -12$.



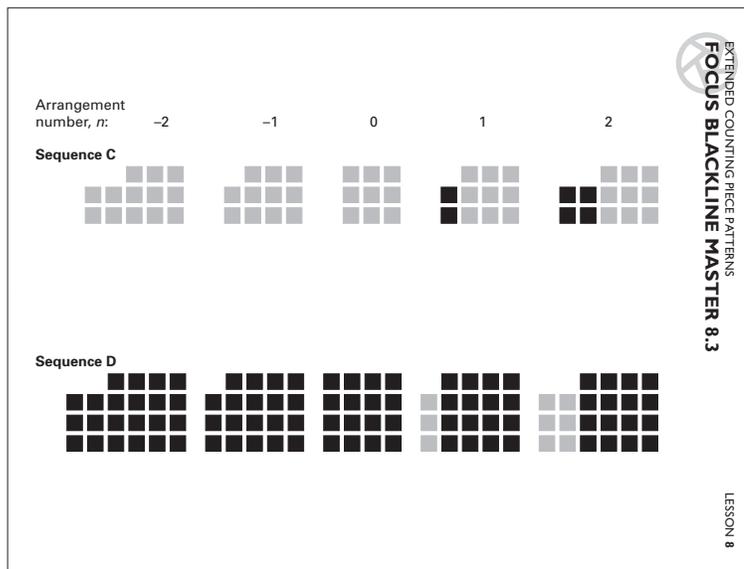
FOCUS

ACTIONS

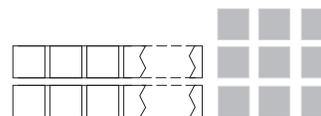
7 Repeat Action 6 for Sequences C and D on Focus Master 8.3.

COMMENTS

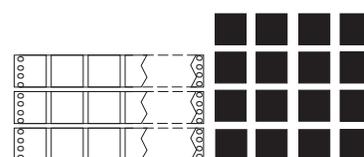
7 The n th arrangements are shown below.



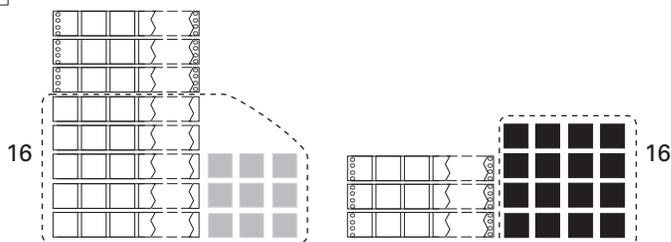
Sequence C



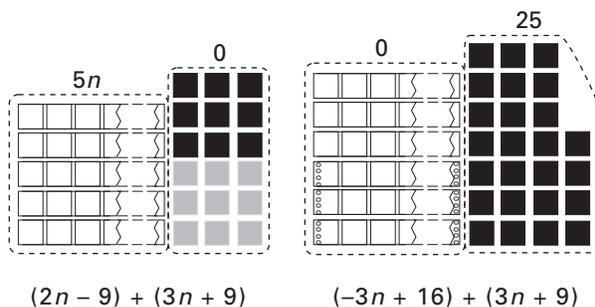
Sequence D



The values of the arrangements, as they appear above, are difficult to compare. However, adding 3 n -frames and 3 $-n$ -frames to the n th arrangement for Sequence C, as shown below, does not change its value. The two arrangements have the same value if the circled portions have the same value, i.e., if $5n - 9 = 16$. Hence, $5n = 25$ and $n = 5$.



An alternative solution is based on the observation that if the same value is added to two arrangements that are equal in value, the resulting arrangements will have equal values. Shown here, values $3n + 9$ have been added to arrangements with values $2n - 9$ and $-3n + 16$. The resulting arrangements have equal values provided $5n = 25$ or, simply, $n = 5$.



The equation $2n - 9 = -3n + 16$ has been solved.



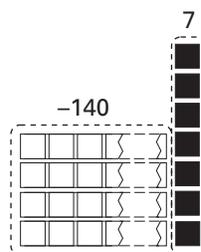
ACTIONS

8 Ask the students to use Algebra Pieces or sketches to help them solve the following equations. Ask for volunteers to show their solutions.

- a) $4n + 7 = -133$
- b) $8 - 5n = -142$
- c) $4n + 5 = 3n - 8$
- d) $8n - 4 = 6n + 10$

COMMENTS

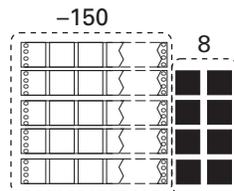
8 a) An arrangement whose value is $4n + 7$ has value -133 if the 4 circled n -frames shown below have total value -140 . This is the case if $n = -35$:



If, $4n + 7 = -133$, then $4n = -140$,
and $n = (-140) \div 4 = -35$

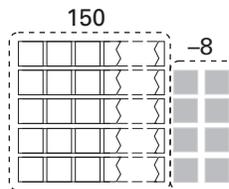
A solution to an equation can be verified by substituting the solution in the original equation to see if the result is a true statement. For example, substituting $n = -35$ in the equation $4n + 7 = -133$ produces the true statement, $4(-35) + 7 = -133$. Substituting any other value for n produces a statement that is not true.

b) An arrangement with value $8 - 5n$ has value -142 when each $-n$ -frame has value -30 , that is, when $n = 30$:



$8 - 5n = -142$,
so, $-5n = -150$
and $-n = -150/5 = -30$.
Thus, $n = 30$.

Alternatively, an arrangement with value $8 - 5n$ has value -142 when the opposite arrangement has value 142 , as shown here:

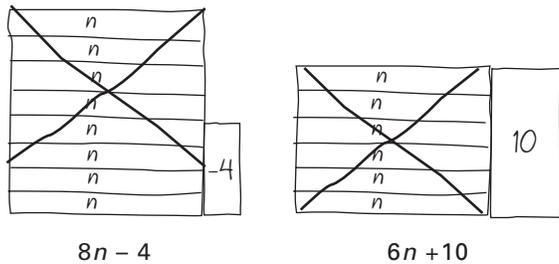
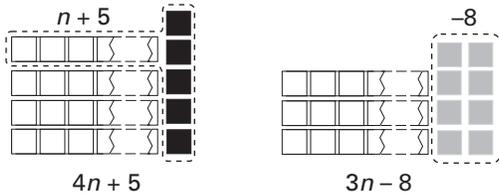


$5n - 8 = 142$,
so, $5n = 150$
and $n = 150/5 = 30$.

continued next page

FOCUS

ACTIONS



COMMENTS

8 continued

c) Arrangements with values $4n + 5$ and $3n - 8$ have the same value when the two circled portions in the diagram at the left each have value -8 , i.e., when $n + 5 = -8$, in which case $n = -13$.

d) As the sketches at the left show, $8n - 4$ and $6n + 10$ have the same value when $2n - 4$ has value 10 . This is so if $2n$ has value 14 , in which case $n = 7$.

9 Give each student a copy of Focus Master 8.4 to complete. Discuss, encouraging students to make observations about similarities and differences they notice in the strategies. Ask them to discuss the strategies they prefer and why.

9 The intent here is to relate algebraic statements to procedures with Algebra Pieces.

EXTENDED COUNTING PIECE PATTERNS LESSON 8
FOCUS BLACKLINE MASTER 8.4

Three students used Algebra Pieces to solve the equation $3n - 3 = 5n + 5$. Then they wrote the following to represent each step of their thoughts and actions with the Algebra Pieces. What Algebra Piece thoughts or actions do you think are represented by the algebra statement.

<p>Student 1s Method</p> $3n - 3 = 5n + 5$ $3n - 3 - 3n = 5n + 5 - 3n$ $-3 = 2n + 5$ $-3 - 5 = 2n + 5 - 5$ $-8 = 2n$ $-4 = n$	<p>Student 3s Method</p> $3n - 3 = 5n + 5$ $3n - 3 - 3n - 5 = 5n + 5 - 3n - 5$ $-8 = 2n$ $\frac{1}{2}(-8) = \frac{1}{2}(2n)$ $-4 = n$
--	--

Student 2s Method

$$3n - 3 = 5n + 5$$

$$3n - 3 + 5n - 5n = 5n + 5$$

$$3n - 3 - 5n = 5$$

$$-2n - 3 = 5$$

$$-2n - 3 = 5 + 3 - 3$$

$$-2n = 8$$

$$-2n \div 2 = 8 \div 2$$

$$-n = 4$$

$$-(-n) = -4$$

$$n = -4$$



ACTIONS

10 Write equation a) below on the overhead or board. Ask the students to use their Algebra Pieces, or sketches of pieces, to solve the equation. Then have them use algebraic statements (numbers and algebraic symbols only, no words or pictures) to communicate each step of their Algebra Piece methods. Discuss and repeat for b)-d).

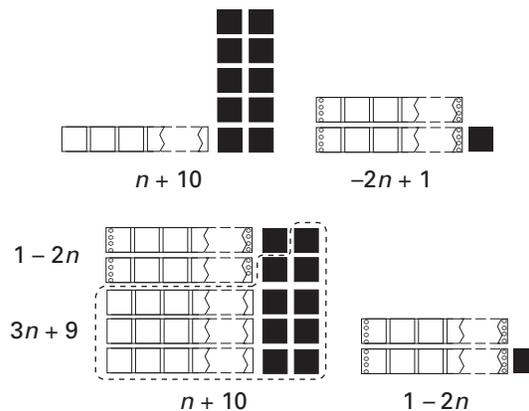
- a) $n + 10 = 1 - 2n$
- b) $6n - 64 = 2n$
- c) $3n - 81 = 6n + 84$
- d) $5n - 170 = 190 - 4n$

COMMENTS

10 Note that students' recordings will vary and some students will include more detail in their recordings than others. It is important not to be rigid or suggest "rules" for recording students' thoughts. It is hoped that students develop a view that the purpose of symbols is to provide a "shorthand" way of recording thought processes and carrying out actions with Algebra Pieces mentally (e.g., the variable n is associated with a mental image of an n -frame). You can ask for volunteers to write their algebraic statements on the board and ask the class to determine from these statements the actions carried out to solve the equations.

a) Note that solving $n + 10 = 1 - 2n$ is equivalent to finding the value of n for which the arrangements of sequences with net values $v_1(n) = n + 10$ and $v_2(n) = 1 - 2n$ are equal in value.

Adding 2 $-n$ -frames and 2 n -frames to an arrangement with value $n + 10$ doesn't change the value of the arrangement. Comparing the resulting arrangements shows they have the same value if $3n + 9 = 0$ or $n = -3$.



Here is one student's recording, representing their thoughts when solving $n + 10 = 1 - 2n$:

$$\begin{aligned}
 n + 10 &= 1 - 2n \\
 n + 10 + 2n + (-2n) &= 1 - 2n, \text{ add zero to } n + 10 \\
 3n + 10 &= 1 \text{ (remove 2 } -n\text{-frames from each } n\text{th arrangement)} \\
 3n + 10 - 1 &= 1 - 1, \text{ add } -1 \text{ to both } n\text{th arrangements} \\
 3n + 9 &= 0 \\
 3n + 9 &= 0 + 9 - 9, \text{ add zero to zero} \\
 3n &= -9 \\
 3n/3 &= -9/3 \\
 n &= -3
 \end{aligned}$$

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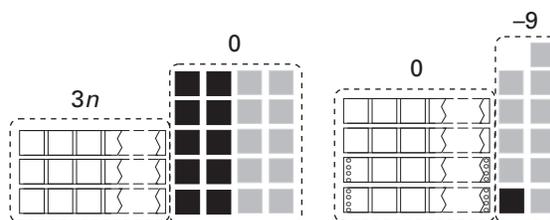


ACTIONS

COMMENTS

10 continued

An alternative solution is illustrated below. Starting with collections with values $n + 10$ and $1 - 2n$, respectively, and then adding $2n$ -frames and 10 red tile to each collection results in two collections with values $3n$ and -9 , respectively. Hence $n + 10 = 1 - 2n$ provided $3n = -9$. This is so if $n = -3$.

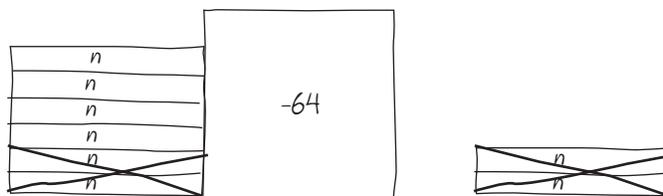


A recording of the these procedures might look like the following:

$$\begin{aligned} n + 10 &= 1 - 2n \\ n + 10 + 2n + (-10) &= 1 - 2n + 2n + (-10) \\ 3n &= -9 \\ n &= -3 \end{aligned}$$

Some students may include the statement $3n/3 = -9/3$ prior to stating $n = -3$ in the sequence of steps shown above.

b) As illustrated below, one sees that sketches for $6n - 64$ and $2n$ have the same value if $4n - 64 = 0$, which happens when $n = 16$.



Symbolically, students may record such thought processes as follows:

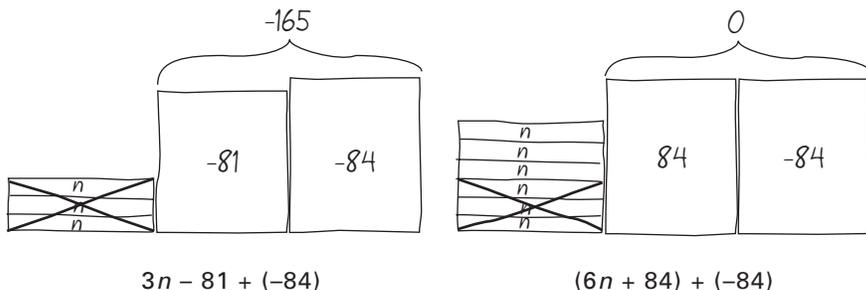
$$\begin{aligned} 6n - 64 &= 2n \\ 4n - 64 &= 0 \\ 4n &= 64 \\ n &= 16 \end{aligned}$$



ACTIONS

COMMENTS

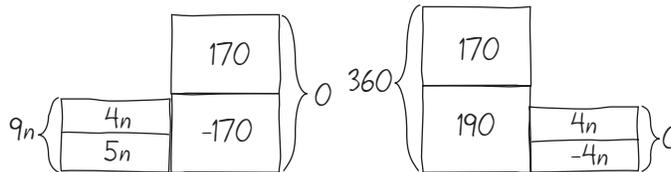
c) In the 2 diagrams below a section representing -84 has been added to sketches for $3n - 81$ and $6n + 84$.



The sketches have equal values if $3n = -165$ or $n = (-165) \div 3 = -55$. A symbolic representation of this line of thinking might be:

$$\begin{aligned} 3n - 81 &= 6n + 84 \\ 3n - 81 + (-84) &= 6n + 84 + (-84) \\ 3n - 165 &= 6n \\ -165 &= 3n \\ -165/3 &= n \end{aligned}$$

d) As shown in the sketch below, if $4n + 170$ is added to $5n - 170$ and $190 - 4n$, the results have equal values provided $n = 40$.

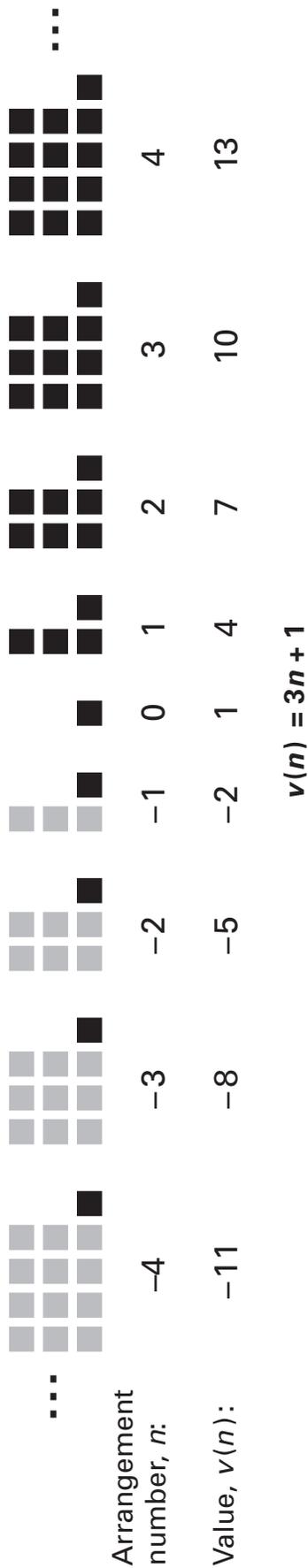


Symbolically, such reasoning could be represented as follows:

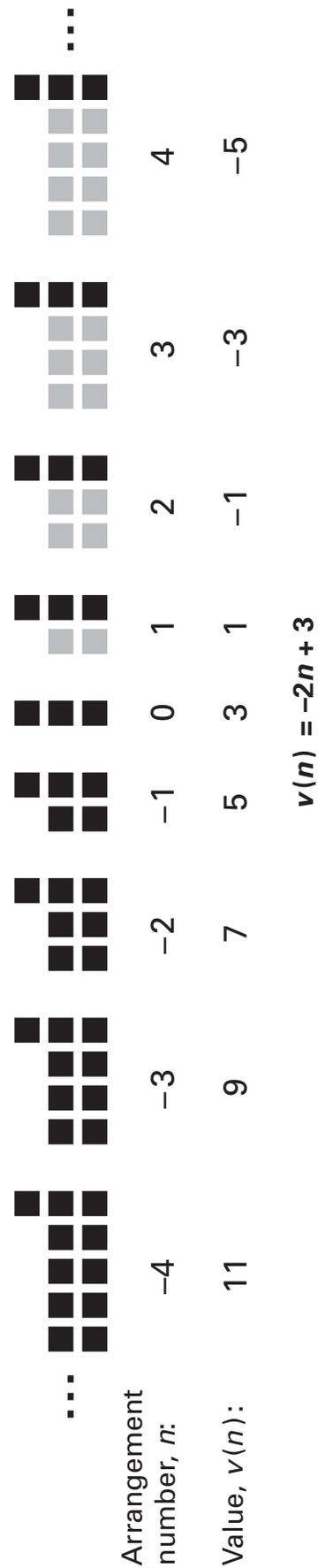
$$\begin{aligned} 5n - 170 &= 190 - 4n \\ 5n - 170 + (4n + 170) &= 190 - 4n + (4n + 170) \\ 9n &= 360 \\ n &= 360/9 = 40 \end{aligned}$$

FOCUS BLACKLINE MASTER 8.1

Extended Sequence A



Extended Sequence B



FOCUS BLACKLINE MASTER 8.2

Arrangement
number, n :

-2

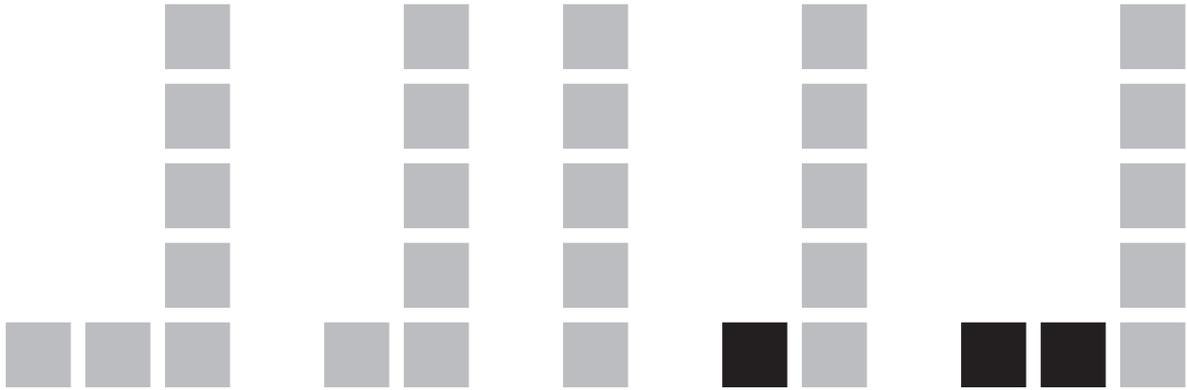
-1

0

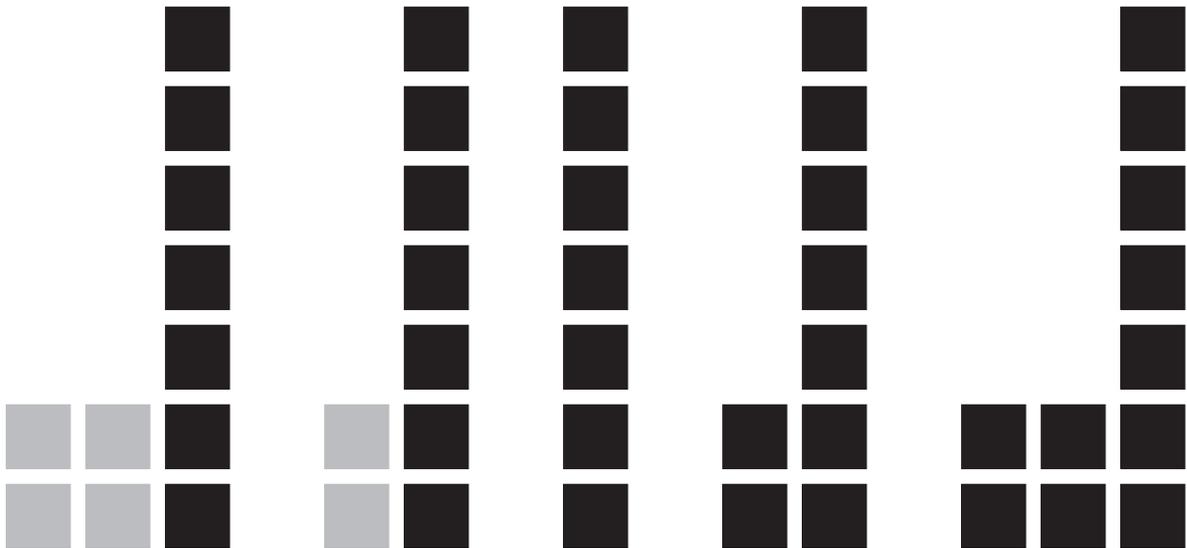
1

2

Sequence A

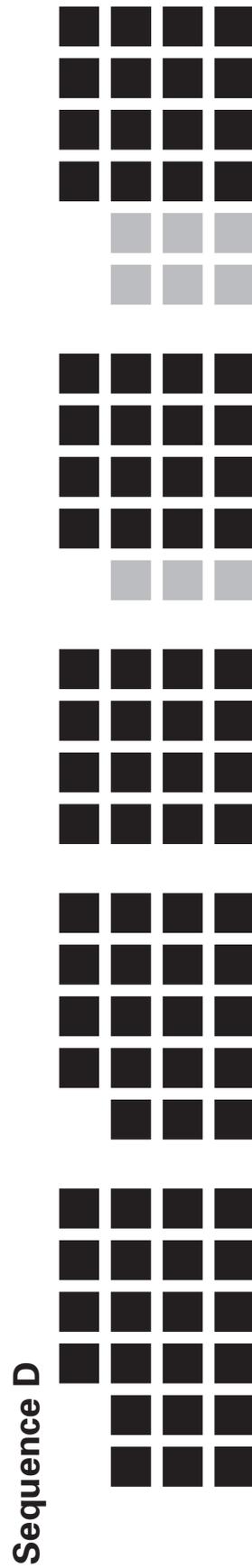
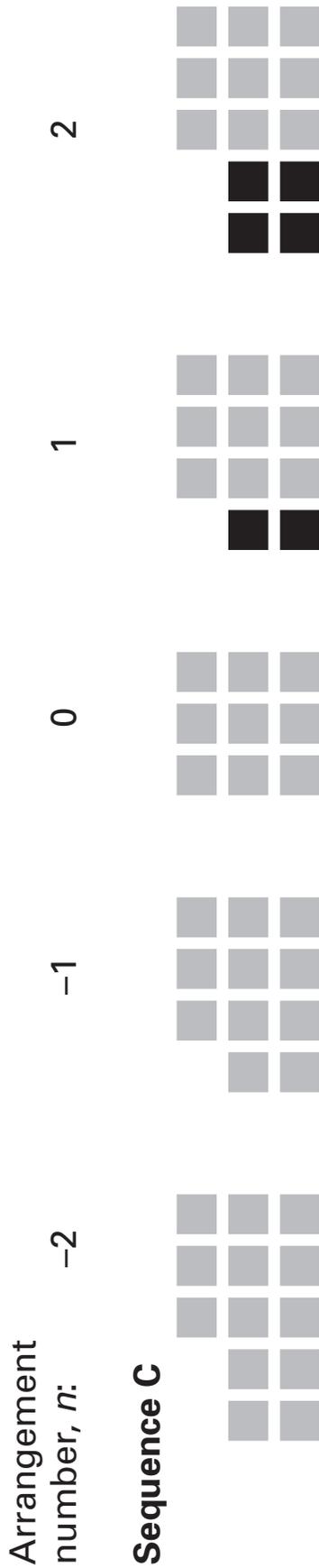


Sequence B





FOCUS BLACKLINE MASTER 8.3




FOCUS BLACKLINE MASTER 8.4

Three students used Algebra Pieces to solve the equation $3n - 3 = 5n + 5$. Then they wrote the following to represent each step of their thoughts and actions with the Algebra Pieces.

What Algebra Piece thoughts or actions do you think are represented by the algebra statement.

Student 1's Method

$$3n - 3 = 5n + 5$$

$$3n - 3 - 3n = 5n + 5 - 3n$$

$$-3 = 2n + 5$$

$$-3 - 5 = 2n + 5 - 5$$

$$-8 = 2n$$

$$-4 = n$$

Student 3's Method

$$3n - 3 = 5n + 5$$

$$3n - 3 - 3n - 5 = 5n + 5 - 3n - 5$$

$$-8 = 2n$$

$$\frac{1}{2}(-8) = \frac{1}{2}(2n)$$

$$-4 = n$$

Student 2's Method

$$3n - 3 = 5n + 5$$

$$3n - 3 + 5n - 5n = 5n + 5$$

$$3n - 3 - 5n = 5$$

$$-2n - 3 = 5$$

$$-2n - 3 = 5 + 3 - 3$$

$$-2n = 8$$

$$-2n \div 2 = 8 \div 2$$

$$-n = 4$$

$$-(-n) = -4$$

$$n = -4$$

FOLLOW-UP BLACKLINE MASTER 8

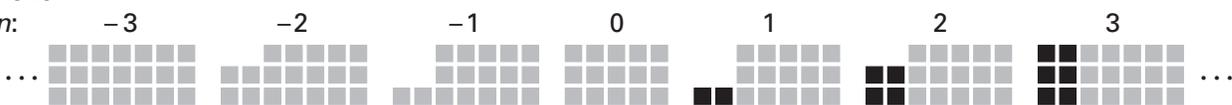
Sequence A

Arrangement
number, n :



Sequence B

Arrangement
number, n :



- I a) Sketch Algebra Piece representations of the n th arrangement of Sequence A and the n th arrangement of Sequence B.
 - b) Find the value of n for which Sequences A and B have the same net value. Draw diagrams to show how you arrived at your answer.
 - c) Tell what equation you solved in b).
- 2 Sketch the -3 rd through 3 rd arrangements of a sequence of counting piece arrangements with net value $v(n) = 3n + 4$. Then determine the value of n for which $v(n) = 190$. Describe how you arrived at your answer.
 - 3 Draw diagrams to show how Algebra Pieces can be used to solve the following equations. Write brief comments to explain what you do in each step.
 - a) $7n + 2 = 8n - 4$
 - b) $3(2n - 3) = 9n + 6$
 - c) $4n^2 + 3n - 5 = (2n + 1)^2 + 8$
 - d) $-16 + 24n = 272$

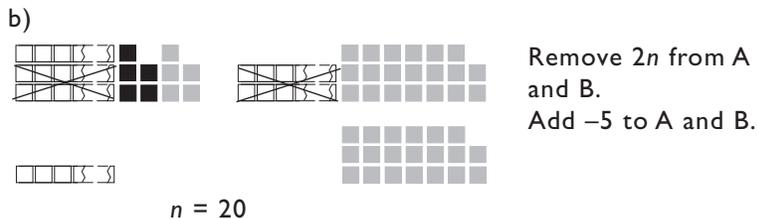
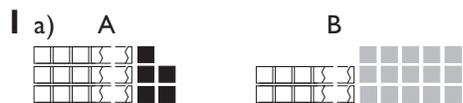
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**FOLLOW-UP BLACKLINE MASTER 8 (CONT.)**

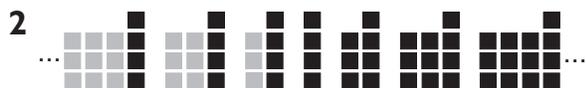
4 Use Algebra Pieces to solve the equation $8n + 12 = 4(n + 1)$. Then, using numbers and algebraic symbols, write a sequence of statements which represents the steps in your solution.

5 Solve the equation $7(n + 3) = 5(n - 3) + 6$ using whatever methods you choose. Explain or illustrate each step of your thought processes and actions. Then tell how you can be sure that your solution is correct.

ANSWERS TO FOLLOW-UP 8



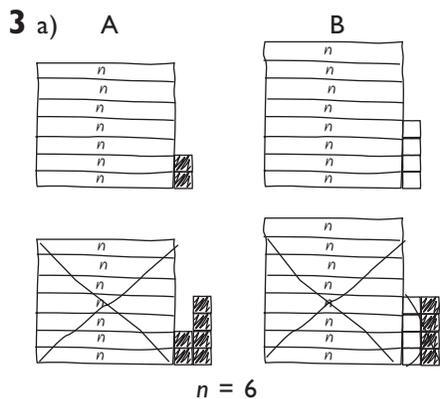
c) $3n + 5 = 2n - 15$ or $3(n + 1) + 2 = 2n - 15$
Other formulations are possible.



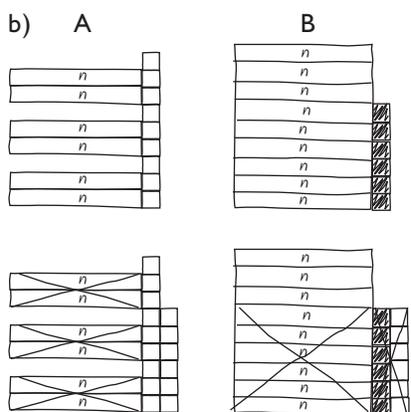
$$3n + 4 = 190$$

$$3n = 186$$

$$n = 186/3 = 62$$



Remove $7n$ from A and B.
Add 4 to A and B.



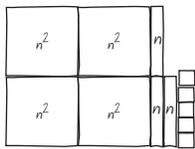
Remove $6n$ from A and B.
Add -6 to A and B.

$$3n = -15; n = -5$$

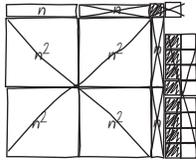
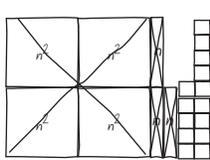
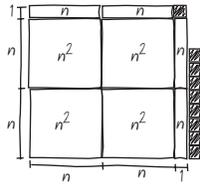
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ANSWERS TO FOLLOW-UP 8 (CONT.)

3 c) A



B



Remove $4n^2$ and $3n$
from A and B.
Add -9 to A and B,
 $-14 = n$

d)

A

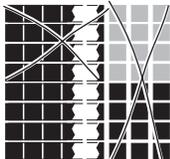
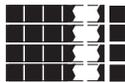
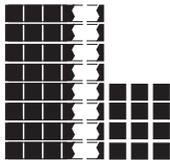


B

~~272~~
288

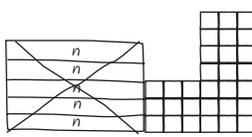
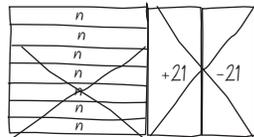
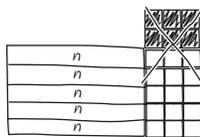
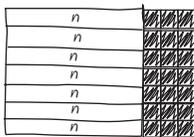
Add 16 to A and B: $24n = 288$
 $n = 12$

4



$$\begin{aligned} 8n + 12 &= 4(n + 1) \\ 8n + 12 &= 4n + 4 \\ 8n + (12 - 12) &= 4n + 4 - 12 \\ 8n &= 4n - 8 \\ 4n &= -8 \\ n &= -2 \end{aligned}$$

5



$$\begin{aligned} 7(n + 3) &= 5(n - 3) + 6 \\ 7n + 21 &= 5n - 9 \\ 2n &= -30 \\ n &= -15 \end{aligned}$$