Becoming a Better Math Tutor

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Abstract

> "Tell me, and I will forget. Show me, and I may remember. Involve me, and I will understand." (Confucius; Chinese thinker and social philosopher; 551 BC – 479 BC.)

This book is about math tutoring. It is designed to help math tutors and tutees get better at their respective and mutual tasks.

Tutoring is a powerful aid to learning. Much of the power comes from the interaction between tutor and tutee. (See the quote from Confucius given above.) This interaction allows the tutor to adjust the content and nature of the instruction to specifically meet the needs of the tutee. It allows ongoing active participation of the tutee.

The intended audiences for this book include volunteer and paid tutors, preservice and inservice teachers, parents and other child caregivers, students who help other students (peer tutors), and developers of tutorial software and other materials.

The book includes two appendices. The first is for tutees, and it has a 6th grade readability level. The other is for parents, and it provides an overview of tutoring and how they can help their children who are being tutored.

An extensive References section contains links to additional resources.

About the Authors

Your authors have authored and/or co-authored nearly 90 academic books as well as hundreds of articles. They have given hundreds of conference presentations and workshops.

This is the second of their co-authored books. Their first co-authored book is book is:

> Moursund, David and Albrecht, Robert (2011). *Using math games and word problems to increase the math maturity of K-8 students.* Salem, OR: The Math Learning Center.
>
> It is available in PDF and Kindle formats. For ordering information go to http://iae-pedia.org/Moursund_and_Albrecht:_Math_games_and_word_problems.

Dr. David Moursund

After completing his undergraduate work at the University of Oregon, Dr. Moursund earned his doctorate in mathematics from the University of Wisconsin-Madison. He taught in the Mathematics Department and Computing Center at Michigan State University for four years before joining the faculty at the University of Oregon. There he had appointments in the Math Department and Computing Center, served six years as the first head of the Computer Science Department, and spent more than 20 years working in the Teacher Education component of the College of Education.

A few highlights of his professional career include founding the International Society for Technology in Education (ISTE), serving as its executive officer for 19 years, establishing ISTE's flagship publication, *Learning and Leading with Technology*, serving as the Editor in Chief for more than 25 years. He was a major professor or co-major professor for more than 75 doctoral students. Six of these were in mathematics and the rest in education. Dr. Moursund has authored or coauthored more than 50 academic books and hundreds of articles. He has presented several hundred keynote speeches, talks, and workshops around the world. More recently, he founded Information Age Education (IAE), a non-profit organization dedicated to improving teaching and learning by people of all ages and throughout the world. IAE currently provides free educational materials through its Wiki, the free IAE Newsletter published twice a month, and the IAE Blog.

For more information about David Moursund, see http://iae-pedia.org/David_Moursund. He can be contacted at moursund@uoregon.edu.

Robert Albrecht

A pioneer in the field of computers in education and use of games in education, Robert Albrecht has been a long-time supporter of computers for everyone. He was instrumental in helping bring about a public-domain version of BASIC (called Tiny BASIC) for early microcomputers. Joining forces with George Firedrake and Dennis Allison, he co-founded People's Computer Company (PCC) in 1972, and also produced and edited *People's Computer Company*, a periodical devoted to computer education, computer games, BASIC programming, and personal use of computers.

Albrecht has authored or coauthored over 30 books and more than 150 articles, including many books about BASIC and educational games. Along with Dennis Allison, he established *Dr. Dobb's Journal*, a professional journal of software tools for advanced computer programmers. He was involved in establishing organizations, publications, and events such as Portola Institute, ComputerTown USA, *Calculators/Computers Magazine*, and the Learning Fair at Peninsula School in Menlo Park, California (now called the Peninsula School Spring Fair).

Albrecht's current adventures include writing and posting instructional materials on the Internet, writing Kindle books, tutoring high school and college students in math and physics, and running HurkleQuest play-by-email games for Oregon teachers and their students.

For information about Albrecht's recent Kindle books, go to http://www.amazon.com/.

Select Kindle Store and search for **albrecht firedrake**.

For more information about Robert Albrecht, see http://iae-pedia.org/Robert_Albrecht. He can be contacted at starshipgaia1@msn.com.

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Preface

> Somebody came up to me after a talk I had given, and said, "You make mathematics seem like fun." I was inspired to reply, "If it isn't fun, why do it?" Ralph P. Boas; mathematician, math teacher, and journal editor; 1912–1992.

This book is about math tutoring. The intended audience includes preservice and inservice teachers, volunteer and paid tutors. The audience includes parents and other child caregivers, students who help other students, and developers of tutorial software and other materials.

Tutors—Both Human and Computer

A tutor works with an individual or with a small group of students. The students are called *tutees*. In this book we focus on both human and computer tutors. Nowadays, it is increasingly common that a tutee will work with a team consisting of one or more humans and a computer.

Formal tutoring within a school setting is a common practice. Formal tutoring outside of a school setting by paid professionals and/or volunteers is a large business in the United States and in many other countries.

Underlying Theory and Philosophy

Both the tutor (the “teacher”) and the tutee (the “student”) can benefit by their participation in a good one-to-one or small-group tutoring environment. Substantial research literature supports this claim (Bloom, 1984). Good tutoring can help a tutee to learn more, better, and faster. It can contribute significantly to a tutee’s self-image, attitude toward the area being studied, learning skills, and long-term retention of what is being learned.

Most people think of tutoring as an aid to learning a specific subject area such as math or reading. However, good tutoring in a discipline has three general goals:

1. Helping the tutees gain knowledge and skills in the subject area. The focus is on immediate learning needs and on building a foundation for future learning.
2. Helping the tutees to gain in math maturity. This includes learning how to learn math, learning how to think mathematically (this includes developing good math “habits of mind”), and learning to become a more responsible math student (bring necessary paper, pencil, book, etc. to class; pay attention in class; do and turn required assignments).
3. Helping tutees learn to effectively deal with the various stresses inherent to being a student in our educational system.

The third item in this list does not receive the attention it deserves. Many students find that school is stressful because of the combination of academic and social demands. Math is particularly stressful because it requires a level of precise, clear thinking and problem-solving activities quite different than in other disciplines. For example, a tiny error in spelling or pronunciation usually does not lead to misunderstanding in communication. However, a tiny error in one step of solving a math problem can lead to completely incorrect results. Being singled out to receive tutoring can be stressful. To learn more about stress in education and in math education, see Moursund and Sylwester (2011).

Some Key Features of this Book

While this book focuses specifically on math tutoring, many of the ideas are applicable to tutoring in other disciplines. A very important component in tutoring is helping the tutee become a more dedicated and efficient lifelong learner. This book emphasizes "learning to learn" and learning to take more personal responsibility for one's education. A good tutor uses each tutoring activity as an aid to helping a tutee become a lifelong, effective learner.

An important component of tutoring is helping the tutee become a more dedicated and efficient lifelong learner. This book emphasizes "learning to learn" and learning to take more personal responsibility for one's education.

The task of improving informal and formal education constitutes a very challenging task. "So much to learn … so little time." The totality of knowledge and skills that a person might learn continues to grow very rapidly.

We know much of the math that students cover in school is forgotten over time. This book includes a focus on helping students gain a type of math maturity that endures over the years.

The book makes use of a number of short "case studies" from the tutoring experience of your authors and others. Often these are composite examples designed to illustrate important ideas in tutoring, and all have been modified to protect the identity of the tutees.

Appendix 1. Advice to Tutees. This material can to be read by tutees with a 6th grade or higher reading level. Alternatively, its contents can be discussed with tutees.

Appendix 2: Some Things Parents Should Know About Tutoring. This material is designed to help parents and other caregivers gain an increased understanding of what a child who is being tutored experiences and possible expectations of having a child being tutored. Tutors may want to provide a copy of this appendix to parents and other primary caregivers of the students they are tutoring.

The book has an extensive Reference section. For the most part, the references are to materials available on the Web.

The book ends with a detailed index.

David Moursund and Robert Albrecht, September 2011

Chapter 1

Some Foundational Information

"God created the natural numbers. All the rest [of mathematics] is the work of mankind." (Leopold Kronecker; German mathematician; 1823-1891.)

All the world's a game,
And all the men and women active players:
They have their exits and their entrances;
And all people in their time play many parts. (David Moursund–Adapted from Shakespeare)

Tutors and other math teachers face a substantial challenge. Keith Devlin is one of our world's leading math education researchers. Here is a quote from his chapter in the book *Mind, brain, & education: Neuroscience implications for the classroom* (Sousa et al., 2010.)

> Mathematics teachers—at all education levels—face two significant obstacles:
>
> 1. We know almost nothing about how people do mathematics.
> 2. We know almost nothing about how people learn to do mathematics.

Math tutors and math teachers routinely grapple with these daunting challenges. Through the research and writings of Devlin and many other people, solutions are emerging. We (your authors) believe that the tide is turning, and that there is growing room for optimism. This chapter presents some foundational information that will be used throughout the book.

The Effectiveness of Tutoring

Good tutoring can help a tutee to learn more, better, and faster (Bloom, 1984). It can contribute significantly to a tutee's self-image, attitude toward the area being studied, learning skills, and long-term retention of what is being learned.

> [Research studies] began in 1980 to compare student learning under one-to-one tutoring, mastery learning [a variation on conventional whole-class group instruction], and conventional group instruction. As the results of these separate studies at different grade levels and in differing school subject areas began to unfold, **we were astonished at the consistency of the findings and the great differences in student cognitive achievement, attitudes, and self-concept under tutoring as compared with group methods of instruction** (Bloom, 1984). [Bold added for emphasis.]

Here are two key ideas emerging from research on tutoring and other methods of instruction:

1. An average student has the cognitive ability (the intelligence) to do very well in learning the content currently taught in our schools.

2. On average, good one-to-one tutoring raises a "C" student to an "A" student and a "D" student to a "B" student. Many students in the mid range of F grades see progress to the "C" level.

These are profound findings. They say most students have the innate capabilities to learn much more and much better than they currently are. This insight leads educational researchers and practitioners in their drive to develop practical, effective, and relatively low cost ways to help students achieve their potentials.

> Most students have the innate capabilities to learn both much more and much better than they currently are learning.

Math tutoring is not just for students doing poorly in learning math. For example, some students are especially gifted and talented in math. They may be capable of learning math faster and much better than average students. The math talented and gifted (TAG) students can benefit by working with a tutor who helps them move much faster and with a better sense of direction in their math studies.

What is Math?

We each have our own ideas as to what math is. One way to explore this question is to note that math is an area of study—an academic discipline. An academic discipline can be defined by a combination of general things such as:

1. The types of problems, tasks, and activities it addresses.
2. Its tools, methodologies, habits of mind, and types of evidence and arguments used in solving problems, accomplishing tasks, and recording and sharing accumulated results.
3. Its accumulated accomplishments such as results, achievements, products, performances, scope, power, uses, impact on the societies of the world, and so on. Note that uses can be within their own disciplines and/or within other disciplines. For example, reading, writing, and math are considered to be "core" disciplines because they are important disciplines in their own rights and also very important components of many other disciplines.
4. Its methods and language of communication, teaching, learning, and assessment; its lower-order and higher-order knowledge and skills; its critical thinking and understanding; and what it does to preserve and sustain its work and pass it on to future generations.
5. The knowledge and skills that separate and distinguish among: a) a novice; b) a person who has a personally useful level of competence; c) a reasonably competent person, employable in the discipline; d) a state or national expert; and e) a world-class expert.

Thus, one way to answer the "what is math" question is to provide considerable detail in each of the numbered areas. Since math is an old, broad, deep, and widely studied discipline, each of the bulleted items has been targeted by a great many books, articles, professional talks, and academic courses. The reader is encouraged to spend a couple of minutes thinking about his or her insights into each of the numbered areas.

Humans and a number of other creatures are born with some innate ability to deal with quantity. Very young human infants can distinguish between one of something, two of that something, and three of that something. However, it is our oral and written languages that make it possible to develop and use the math students learn in school. Our successes in math depend heavily on the informal and formal education system for helping children to learn and use math.

> The language of math is a special-purpose language useful in oral and written communication. It is a powerful aid to representing, thinking about, and solving math-related problems.

Our current language of math represents thousands of years of development (Moursund and Ricketts, 2008). The language has changed and grown through the work of math researchers and math users. As an example, consider the decimal point and decimal notation. These were great human inventions made long after the first written languages were developed.

The written language of mathematics has made possible the mathematics that we use today. The discipline and language of math have been developed through the work of a large number of mathematicians over thousands of years. The written language of math has made it possible to learn math by reading math.

Math is much more than just a language. It is a way of thinking and problem solving. Here is a quote from George Polya, one of the world's leading mathematicians and math educators of the 20th century.

> **To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems.** For the higher aims about which I am now talking are some general tactics of problems—to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school. But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems. [Bold added for emphasis.]

Math educators frequently answer the "What is math?" question by discussing the processes of indentifying, classifying, and using patterns. In that sense, math is a science of patterns. However, problem solvers in all disciplines look for patterns within their disciplines. That helps to explain why math is such an interdisciplinary discipline—it can be used to help work with patterns in many different disciplines.

Other answers to the “What is math?” question are explored in Moursund (2007). The careful rigorous arguments of math proofs are a key aspect of math. The language of math and the accumulated math proofs make it possible for math researchers to build on the previous work of others. Building on the previous work of others is an essential idea in problem solving in math and other disciplines.

Helping Tutees to Become Mathematically “Mature” Adults

Our math education system places more emphasis on some of the components of the discipline of math than on others. During 2010–2011, most of the states in the United States adopted the Common Core State Standards (CCSS). These include a newly developed set of math content standards that specify what topics are to be taught at each grade level. Progress is occurring in developing assessment instruments that can be used to test how well students are learning the content standards. (CCSS, n.d.)

Students have varying levels of innate ability in math and they have varying levels of interest in math. Precollege students who have a higher level of innate ability and interest in non-math areas such as art, history, journalism, music, or psychology, may wonder why they are required to take so many math courses. They may wonder why they cannot graduate from high school without being able to show a particular level of mastery of geometry and algebra.

People who make decisions about math content standards and assessment try to think in terms of future needs of the student and future needs of the country.

Math maturity is being able to make effective use of the math that one has learned through informal and formal experiences and schooling. It is the ability to recognize, represent, clarify, and solve math-related problems using the math one has studied. Thus, we expect a student to grow in math maturity as the student grows in math content knowledge.

Mathematically mature adults have the math knowledge, skills, attitudes, perseverance, and experience to be responsible adult citizens in dealing with the types of math-related situations, problems, and tasks they encounter. In addition, a mathematically mature adult knows when and how to ask for and make appropriate use of help from other people, from books, and from tools such as computer and the Internet. One sign of an increasing level of math maturity is an increasing ability to learn math by reading math.

For students, we can talk both about their level of math maturity and their level of math education maturity. As an example, consider a student who is capable of doing math assignments, but doesn’t. Or, consider a student who does the math assignments but doesn’t turn them in. These are examples of a low level of math education maturity.

An increasing level of math maturity is evidenced by an increased understanding and ability to learn math and to relearn math that one has forgotten. Chapter 8 covers many math Habits of Mind that relate to math maturity. For example, persistence—not giving up easily when faced by challenging math problems—is an important math Habit of Mind. A growing level of persistence is an indicator of an increasing level of math maturity.

The “measure” of a math student includes both the student’s math content knowledge and skills, and the level of math development (math maturity) of the student. Chapter 7 discusses math maturity in more detail. Math tutoring helps students learn math and to gain an increasing level of math maturity.

> An increasing level of math maturity is an increasing level of being able to make effective use of one's math knowledge and skills dealing with math-related problems in one's everyday life.

The Games of Math and in Math Education

The second quote at the beginning of this chapter presents the idea that "All the world's a game…" This book on tutoring includes a major emphasis on making math learning fun and relevant to the tutee. It does this by making use of the idea that math can be considered as a type of game. Within math, there are many smaller games that can catch and hold the attention of students (Moursund and Albrecht, 2011).

You are familiar with a variety of games such as card games, board games, sports games, electronic games, and so on. Consider a child just beginning to learn a sport such as swimming, baseball, soccer, or basketball. The child can attend sporting events and/or view them on television. The child can see younger and older children participating in these sports.

Such observation of a game provides the child with some insights into the **whole game**. The child will begin to form a coherent mental image of individual actions, teamwork, scoring, and rules of the game.

Such observation does not make the child into a skilled performer. However, it provides insights into people of a variety of ages and skill levels playing the games, from those who are rank beginners to those who are professionals. It also provides a type of framework for further learning about the game and for becoming a participant in the game.

The "Whole Game" of Swimming

Consider competitive swimming as an example. You certainly know something about the "whole game" of competitive swimming, even if you have never competed. People working to become competitive swimmers study and practice a number of different elements of swimming, such as:

- Arm strokes;
- Leg kicks;
- Breathing and breathing patterns;
- The takeoff at the beginning of a race;
- Racing turns at the end of the pool;
- Pacing oneself (in a race);
- Being a member of a relay team;
- Building strength and endurance through appropriate exercise and diet.

A swimming lesson for a person seriously interested in becoming a good swimmer will include both sustained practice on a number of different elements and practice in putting them all together to actually swim.

A student learning to swim has seen people swim, and so has some understanding of the whole game of swimming. The student gets better by studying and practicing individual components, but also by routinely integrating these components together in doing (playing) the whole game of swimming.

David Perkins' book, *Making Education Whole* (Perkins, 2010) presents the idea that much of what students learn in school can be described as "learning elements of" and "learning about." Perkins uses the words **elementitis** and **aboutitis** to describe these illnesses in our educational system.

In the swimming example, there are a great many individual elements that can be practiced and learned. These are what Perkins is referring to when he talks about elementitis.

Even if you are not a swimmer, you probably know "about" such things as the backstroke, the breaststroke, free style, racing turns, and "the thrill of victory and agony of defeat" in competitive swimming. You can enjoy watching the swimming events in the Summer Olympics, and you may remember the names of some of the super stars that have amassed many gold medals. Many of us enjoy having a certain level of aboutitis in sports and a wide variety of areas.

The "Whole Game" of Math

Most of us are not used to talking about math as a game. What is the "whole game" of math? How does our education system prepare students to "play" this game? What can be done to improve our math education system?

What is math? Each tutor and each tutee has his or her own answers. Still other answers are available from those who create the state and national math standards and tests.

Your authors enjoy talking to people of all ages to gain insights into their math education and their use of math. Here is a question for you. What is math? Before going on to the next paragraph, form some answers in your head.

Now, analyze your answers from four points of view:

1. Knowing some **elements of math.** You might have listed elements such as counting, adding, multiplication, or solving algebra equations. You may have thought about "getting right answers" and "checking your answers."
2. Knowing something **about math.** You may have listed various components of math such as arithmetic, algebra, geometry, probability, and calculus. You may have thought about names such as Euclid, Pythagoras, and Newton. You may have noted that many people find math to be a hard subject, and many people are not very good at doing math. You may have had brief thoughts

about your difficulties in working with fractions, percentages, and probability, or balancing your checkbook.

3. Knowing how to "do" and use math. This includes such things as:
    a. Knowing how to represent and solve math-related problems both in math classes and in other disciplines and everyday activities that make use of math.
    b. Knowing how to communicate with understanding in the oral and written language of math.
    c. Knowing how and when to use calculators and computers to help do math.
4. Knowing how to learn math and to relearn the math you have studied in the past but have now forgotten.

Math tutors need to have a good understanding of these four categories of answers to the question "What is math?" They need to appreciate that their own answers may be quite different than the (current) answers of their tutees. Good tutoring involves interplay between the knowledge and skills of the tutor and the tutee. The tutor needs to be "tuned" to the current knowledge and skills of the tutee, continually filling in needed prerequisites and moving the tutee toward greater math capabilities.

Junior Versions of Games

Perkins' book contains a number of examples of "junior" versions of games that can be understood and played as one makes progress toward playing the "whole game" in a particular discipline or sub discipline. This is a very important idea in learning any complex game such as the game of math.

Examples of Non-Math Junior Games

Think about the whole game of writing. A writer plays the whole game of effective communicating in writing. Now, contrast this with having a student learning some writing elements such spelling, punctuation, grammar, and penmanship. These elements are of varying importance, but no amount of skill in them makes one into an effective player of the whole game of writing.

A child can gain insight into the whole game of swimming. How does a child gain insight into the whole game of writing? Obviously this is an educational challenge.

Our language arts curriculum realizes this, and it has worked to establish an appropriate balance between learning about, learning elements of, and actually doing writing. The language arts curriculum also recognizes the close connection between writing and reading. One can think of the whole game of language arts as consisting of two overlapping games—the whole game of reading and the whole game of writing.

Even at the first grade level, a child can be playing junior versions of language arts games. For example, a child or the whole class can work together to tell a story. The teacher uses a computer and projection system to display the story as it is being orally composed. The whole class can participate in editing the story. Students can "see" the teacher playing a junior game of editing. Using their knowledge of oral language and story telling, they can participate in junior versions of writing and editing.

Of course, we don't expect first graders to write a great novel. However, they can play "junior games" of writing such as writing a paragraph describing something they know or that interests them. They can add illustrations to a short story that the students and teacher have worked together to create. They can read short stories that are appropriate to their knowledge of the world and oral vocabulary.

What does an artist do? Can a first grader learn (to play) a junior version of the game of art? What does a dancer do? Can a first grader do a junior version of various games of performance arts? Obviously yes, and such junior versions of creative and performing arts are readily integrated into a first grade curriculum.

Junior Games in Math

This book provides a number of examples of junior math-oriented games. Let's use the board game Monopoly as an example. Many readers of this book played Monopoly and/or other "money" board games when they were children. Monopoly can be thought of as a simulation of certain aspects of the whole game of business. Math and game-playing strategies are used extensively in the game.

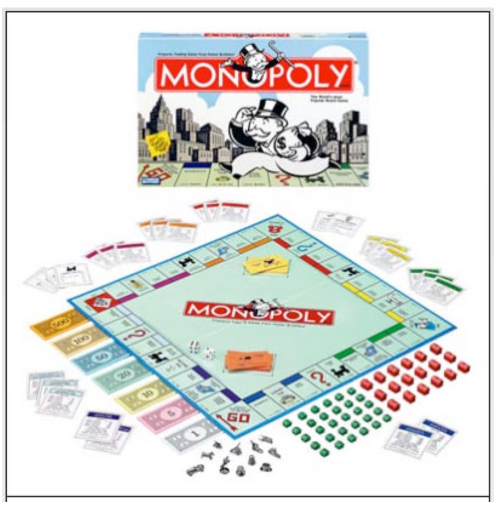

Figure 1.1. Monopoly board. Copied from http://www.hasbro.com/monopoly/.

You probably know some things "about" Monopoly even if you have never played it. If you have played Monopoly you know that there are many elements. You know that primary school students and still younger students can learn to play Monopoly. This is an excellent example of "play together, learn together."

Imagine that children were not allowed to play the whole game until they first gain appropriate knowledge of the game elements such as:

- Dice rolling, including determining the number produced by rolling a pair of dice and whether a doubles has been rolled;
- Counting and moving a marker (one's playing piece) along a board.
- Dealing with money.

- Buying property, building houses and hotels, and selling property. This includes making decisions about buying and selling.
- Making payments for landing on property owned by others.
- Collecting payments when other players land on your property.
- Checking to see that one's opponents do not make mistakes—accidently or on purpose.
- Learning and making use of various strategies relevant to playing the game well.
- Et cetera. One can break the whole game into a very large number of elements. Learning to play the game of Monopoly can degenerate into elementitis.

Now, here's the crux of the situation. In your mind, draw a parallel between learning to play the whole game of Monopoly and learning to play the whole game of math. In either case the mode of instruction could be based on **learning about** and **learning elements of**. Students could be restricted from playing the whole game or even a junior version of the game until they had mastered a large number of the elements.

We do not take this approach in the world of games—but we have a considerable tendency to take this approach in mathematics education. Your authors believe that this is a major flaw in our math education system.

> Many students never gain an overview understanding of the whole game of math. They learn math as a collection of unrelated elements. This is a major weakness in our math education system.

Fun Math, Math Games, and Math Puzzles

One unifying theme in math is finding math types of patterns, describing the patterns very accurately, identifying some characteristics of situations producing the patterns, and proving that these characteristics are sufficient (or, are not sufficient) to produce the patterns.

This combination of finding, describing, identifying, and proving is a type of math game. Junior versions of this game can be developed to challenge students at any level of their math knowledge and skill. Higher levels of such games are math research problems challenging math researchers.

Tutoring Tips, Ideas, and Suggestions

Each chapter of this book contains a section giving tutors or potential tutors specific advice on how to get better at tutoring. The example given below focuses on creating a two-way communication between tutor and tutee.

Interaction Starters and Thinking Out Loud

One of the most important aspects of math tutoring is establishing and maintaining a two-way math-related ongoing conversation between tutor and tutee. This is a good way to help a tutee learn to communicate effectively in the language of mathematics. It is a good way for the tutor both to role model math communication and to better understand the tutees math knowledge, skills, and weaknesses.

A skillful tutor is good at facilitating and encouraging a two-way math-related dialogue with the tutee. With practice, a tutee gains skill in such a dialogue and becomes more comfortable in engaging in such a dialogue. This is an important aspect of gaining in math maturity.

One approach is for the tutor to develop a list of interaction starters. As a tutee is working on a problem, a tutor's interaction starter can move the task into a math conversation. The conversation might grow to a "think out loud" conversation or to a joint tutor-tutee exploration of various points in solving a challenging problem.

Here are some interaction starters developed by the Math Learning Center (MLC, n.d.) and Mike Wong, a member of the Board of Directors of the MLC. Your authors have added a few items to the list.

- How do you know what you know? How do you know it's true? (The tutee makes an assertion. The tutor asks for evidence to back up the assertion.)
- Can you prove that? (Somewhat similar to an evidence request. A tutee solves a problem by carrying out a sequence of steps. How does the tutee know that the solution is correct?)
- What if . . .? (Conjecture. Make evidence-based guesses. Pose variations on the problem being studied.)
- Is there a different way to solve this problem? (Many problems can be solved in a variety of ways. One way to check one's understanding of a problem and increase confidence in a solution that has been produced is to solve it in a different way.)
- What did you notice about . . . ? (Indicate an aspect of what the tutee is doing.)
- What do you predict will happen if you try … ?
- Where have you seen or used this before?
- What do you think or feel about this situation?
- What parts do you agree or disagree with? Why?
- Can you name some uses of this outside the math class and/or outside of school?
- How might a calculator or computer help in solving this problem?

Final Remarks

As you read this book, think about the whole game of being a math tutor and the whole game of being a math tutee. What can you do to make yourself into a better player of the tutor game? What can you do to help your tutees become better players of the tutee game?

Use this book to learn more about the math tutor game. Determine elements of the game that are some of your relative strengths and some that are part of your relative weaknesses. Consciously think about and work to improve yourself in your areas of relative weaknesses.

Use the same approach with your tutees. Help each tutee to identify areas of relative strength and areas of relative weakness. Help each tutee work to gain greater knowledge and skill in areas of relative weaknesses.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. Imagine having individual conversations with a student you are going to tutor in math and a parent of that student. Each asks the question: "What is math and why is it important to learn math?" What answers do you give? How might your answers help to facilitate future math-related communication between the child and parent?
3. Think about games and other forms of entertainment you participated in as a child. Which (if any) contributed to your math education? Answer the same question for today's children, and then do a compare and contrast between the two answers.

Chapter 2

Introduction to Tutoring

"Knowledge is power." (Sir Francis Bacon; 1561; English philosopher, statesman, scientist, lawyer, jurist, author and father of the scientific method; 1561-1626.)

“When toys become tools, then work becomes play.” Bernie DeKoven.

Tutoring is a type of teaching. Good tutoring empowers a student with increased knowledge, skills, habits, and attitudes that can last a lifetime.

This book makes use of a number of Scenarios. Each is a story drawn from the experiences of your authors and their colleagues. Some are composites created by weaving together tutoring stories about two or more tutees. All of the stories have been modified to protect the identities of the tutees and to better illustrate important tutoring ideas.

Many students have math-learning difficulties. Some have a combination of dyslexia, dysgraphia, dyscalculia, ADHD, and so on. If you do much math tutoring, you will encounter students with these and/or other learning disabilities. Learn more about the first three of these learning disabilities via a short video on dyscalculia and dysgraphia available at http://www.youtube.com/watch?v=fhpOoj7VqcE.

Special education is a complex field. All teachers and all tutors gain some “on the job” education and experience in working with students with special needs. A tutor might well specialize in tutoring students who have learning disabilities and challenges. This book does not attempt to provide the education in special education that is needed to become well qualified to tutor special education students.

During their program of study that prepares them for a teacher’s license, preservice teachers receive some introduction to special education. The regular classroom teacher is apt to have students who spend part of their school day working with tutors.

Tutoring Scenario

> In his early childhood, George was raised by a combination of his parents and two grandparents who lived near his home. George was both physically and mentally above average. He prospered under the loving care—think of this as lots of individual tutoring—provided by his parents and grandparents. He enjoyed being read to and this was a routine part of his preschool days.
>
> George was enrolled in a local neighborhood school and enjoyed school. However, his parents learned that George had a learning problem when they received his end of second grade report card. The teacher indicated that George

had made no progress in reading during that entire year and was having considerable difficulty with math word problems.

His parents were surprised by the fact that George actually passed second grade, and that the teacher had not made a major intervention sometime during the school year.

A grandparent had heard about dyslexia, and so the parents and grandparents did some reading in this area. Dyslexia is a type of brain wiring that makes it difficult to learn to read. And sometimes makes it difficult to learn arithmetic. It was obvious that George was dyslexic.

Under strong pressure from George's parents, the school tested George, and it turned out that he had severe dyslexia. With the help of an IEP (Individual Education Program) that included a substantial amount of tutoring by reading specialists for more than a year, George learned to read and more than caught up with his classmates.

This is a success story. Dyslexia is a well-known learning disability that makes it difficult to learn to read and that also can make it difficult to learn to do arithmetic. Extensive individual tutoring leads to a rewiring of the tutee's brain. This rewiring allows the reading-related structures in the tutee's brain to function much more like they do in a student that does not have dyslexia.

Many dyslexic students find the reading and writing aspects of math particularly challenging. Dyscalculia and dysgraphia are other learning disabilities that affect math learning.

Two-way Communication

Two-way communication between tutor and tutee lies at the very heart of effective tutoring. Contrast such communication with a teacher talking to a class of 30 students, with the teacher delivery of information occasionally interrupted by a little bit of student response or question asking.

Two-way communication in tutoring is especially designed to facilitate learning. Tutees who learn to effectively participate in such a communication have gained a life-long skill. The tutees learn to express (demonstrate) what they know, what they don't know, and what they want to know. To do this, they need to be actively engaged and on task. They need to learn to focus their attention. Much of the success of tutoring lies in the tutor helping the tutee gain and regularly use such communication and attention-focusing skills.

Many successful tutors stress the idea that the tutee should be actively engaged in conversation with the tutor. The tutor provides feedback based on what the tutee says and does. Tutoring is not a lecture session.

Perhaps you have heard of a general type of two-way communication that is called *active listening.* Its techniques are easily taught and are applicable in any two-way conversation. See, for example, http://www.studygs.net/listening.htm. Quoting from this Website:

> **Active listening intentionally focuses on who you are listening to,** whether in a group or one-on-one, in order to understand what he or she is saying. As the listener, you should then be able to repeat back in your own words what they have said to their satisfaction. This does not mean you agree with the person, but rather understand what they are saying.

Here is a math active listening activity that can be used over and over again in tutoring. Ask the tutee to respond to, "What did you learn in math class since the last time we got together?" If the tutee's answer is too short and/or not enlightening, the tutor can ask probing questions.

Tutors and Mentors

A mentor is an advisor, someone who helps another person adjust to a new job or situation. The mentor has much more experience in the job or task situation than does the mentee. A new mother and first-born child often have the benefit of mentoring (and some informal tutoring) from a grandmother, sister, aunt, or a friend who is an experienced mother. One of the advantages of having an extended family living in a household or near each other is mentoring and informal tutoring are available over a wide range of life activities.

Tutoring and mentoring are closely related ideas. Although this book is mainly about tutoring, mentoring will be discussed from time to time. In teaching and other work settings, a new employee is sometimes assigned a mentor who helps the mentee "learn the ropes." There has been considerable research on the value of a beginning teacher having a mentor who is an experienced and successful teacher. The same ideas can be applied to an experienced tutor mentoring a beginning tutor.

Here is a list of five key "rules" to follow in mentoring (TheHabe, n.d.).

1. Set ground rules. This can be thought of as having an informal agreement about the overall mentoring arrangement.
2. Make some quality time available. For example, agree to meet regularly at a designated time and place.
3. Share interests. Build a relationship based on multiple areas of shared interests. Include areas outside the specific area of mentorship.
4. Be available. A mentee may need some mentoring between the regularly scheduled meeting times. Email may be a good way to do this.
5. Be supportive. A mentor is "on the same side—on the same team" as the mentee.

Any long-term tutor-tutee activity will include both tutoring and mentoring. The tutor becomes a mentor—a person who supports the tutee/mentee—in learning to become a more self-sufficient, lifelong learner. Such mentoring is such an important part of long-term tutoring that we strongly recommend that such mentoring be built into any long term tutoring that a student receives.

Peer Tutoring and Mentoring

Students routinely learn from each other. Most often this is in informal conversations, interactions, and texting. However, structure can be added. For example, many schools have a variety of academic clubs such as math, science, and robotics clubs. An important aspect of these clubs is the various aspects of peer tutoring, cooperative learning, teams doing project-based learning, and other activities in which students "play together and learn together."

Such clubs often bring together students of varying ages and levels of expertise. This is an excellent environment for mentoring, with more experienced club members mentoring those just joining the club. It is delightful to create a club situation in which the members actively recruit students who will become members in the future and then help them to fit into the club activities.

> Math clubs, science clubs, and robotic clubs provide a rich environment for students to play together, learn together.

In small group project-based learning activities tend to have a strong peer-tutoring component. In forming project teams, a teacher might make sure each team includes a student with considerable experience and success in doing project-based learning. In some sense, this student serves as a mentor for others in the group. A teacher might provide specific instruction designed to help group members learn to work together and learn from each other (PBL, n.d.).

Toys

Think about the following quote given at the beginning of their chapter:

> "When toys become tools, then work becomes play." Bernie DeKoven.

Learn more about DeKoven at http://www.deepfun.com/about.php.

To a child, a new toy can be thought of as a learning challenge. The toy, the child, peers, and adults may all provide feedback in this learning process. A child immersed in learning to play with a new toy is practicing learning to learn.

A child's highly illustrated storybook is a type of educational toy. A parent and child playing together with this type of toy lay the foundations for a child learning to read.

Some toys are more challenging, open ended, and educational than others. A set of building blocks provides a wide range of creative learning opportunities. A set of dominoes or dice can serve both as building blocks and the basis for a variety of games that involve counting, arithmetic, and problem solving.

Dice

As an example, many students have played board games in which the roll of one or more 6-faced dice determines a person's move. When rolling a pair of dice, what is the most frequently occurring sum? Individual students or groups of students can do many rolls of a pair of dice, gather data on a large number of rolls, and analyze the data. They may discover that the number of outcomes of a total of seven is roughly the same as the number of doubles. How or why should that be?

In a large number of rolls of a pair of dice, the total number of rolls that sum to eight is roughly the same as the number that sum to six. How or why should that be?

It is fun to explore patterns in rolling dice. It is challenging mathematics to identify and explain the patterns. See, for example. http://mathforum.org/library/drmath/view/55804.html.

Geoboard

A wide variety of such math manipulatives are often used in elementary school math education. They can also be quite useful in working with older students. As an example, consider a 5 x 5 geoboard. A geoboard is a five-by-five grid of short, evenly space posts. Rubber bands are used to form geometric shapes on a geoboard. Two examples are shown in Figure 2.1.

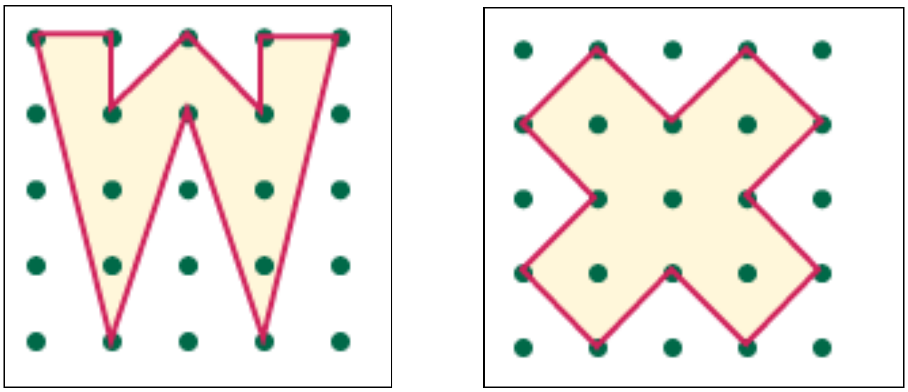

Figure 2.1. Two 5 x 5 geoboards, each showing a geometric figure.

Notice that there are exactly four posts that are completely inside the first (W-shaped) figure. Here is a simple game. Create some other geometric shapes on the geoboard that have exactly four inside posts. A much more challenging game is to determine how many geoboard-based geometric figures have exactly four inside posts.

The geometric shape on the second geoboard has five fully enclosed posts. You can see that the game given above can be extended to finding figures with one completely enclosed post, with two completely enclosed posts, and so on. One can also explore geometric shapes with specified numbers of edge posts.

What "regular" geometric shapes can one make on a geoboard? What areas can one enclose on a geoboard? What perimeter lengths can one create on a geoboard?

There are a very large number of geoboard sites on the Web, and there are many interesting and challenging geoboard activities. The Website http://www.cut-the-knot.org/ctk/Pick.shtml contains a computer-based geoboard and a discussion of some interesting math related to a geoboard.

Television

Television can be considered as a toy. Researchers indicate that it is not a good learning toy for very young children. Its use should be quite limited and carefully supervised. Passive television programming lacks the interaction and personalized feedback that is especially important for very young learners. Children have considerable inherent ability to learn by doing—to learn by being actively engaged. Passively watching television is not active engagement.

Computerized Toys

Many of today's toys are computerized. Sherry Turkle (n.d.) has spent much of her professional career doing research on how children interact with computer-based media and toys. As with TV, the nature and level of child-toy interactivity is often quite limited. Active child-to-toy engagement and interaction are essential to learning by playing with a toy.

In Summary

There are innumerable fun game-like activities that one can use to help students learn math, gain in math maturity, and develop math Habits of Mind. In analyzing a game or game-like activity for use in math education, think about:

1. What makes it attention grabbing, attention holding, and fun to play?
2. Is it cognitively challenging at a level appropriate to a tutee's math knowledge, skills, and development?
3. How does it relate to the overall "whole game" of math or a specific component of math? If you, as the tutor, cannot identify a clear area of math that is being investigated, how do you expect your tutee to gain mathematical benefit from playing the game?

Computer-as-Tutor

Computer-assisted instruction (now usually called computer-assisted learning or CAL) has been steadily growing in use over the past 50 years. Quite early on in the development of CAL it became obvious that:

1. A computer can be used as an automated "flash card" aid to learning. A computer presents a simple problem or question, the computer user enters or indicates an answer, and the computer provides feedback on the correctness of the answer.
2. A computer can be used to simulate complex problem-solving situations, and the user can practice problem solving in this environment. Nowadays, such CAL is a common aid in car driver training and airplane pilot training, and in such diverse areas as business education and medical education. Many computer applications and computer games include built-in instructional modules.

One of the characteristics of a good CAL system is that it keeps detailed records of a student's work—perhaps even at the level of capturing every keystroke. If the CAL is being used in an online mode, the company that produced the CAL can analyze this data and use it to improve the product. Very roughly speaking, it costs about $5 million for a company to develop a high quality yearlong CAL course and $1 million a year to improve it and keep it up to date. Over the years, this level of investment has led to increasing quality of commercially produced CAL materials. This high developmental cost means that the leading edge CAL is not apt to be available free on the Web unless its development was paid for by Federal or other grants.

The US Federal Government has funded a variety of CAL research and development projects. In recent years, this has led to the development of the Cognitive Tutor CAL by

Carnegie Mellon University , and a variety of pieces of software called Highly Interactive Intelligent Computer-Assisted Learning (HIICAL) systems.

Such systems are taking on more of the characteristics of an individual tutor. They are not yet as effective as a good human tutor, but for many students they are better than large group (conventional) classroom instruction. In this book, we use the term "computer tutor" to refer to computer-as-tutor, in the same way that we use the term human tutor to refer to human-as- tutor.

See https://mathtutor.web.cmu.edu/ for some of Carnegie Mellon's Cognitive Tutor middle school math materials. It is targeted at students who are reasonably good at math. Recently Carnegie Mellon sold much of their Cognitive Tutor materials and business for $75 million to the corporation that owns and runs Phoenix University—one of the largest distance education intuitions in the world.

Computer tutors can be used in conjunction with human tutors and/or conventional classroom instruction. The computer tutor, human tutor, and conventional group instruction combine to provide a better education.

Tutoring Tips, Ideas, and Suggestions: Every Number is a Story

Each chapter of this book contains a **Tutoring Tips** example. Most experienced tutors develop a large repertoire of such examples that they can draw upon as needed. Nowadays, it is convenient to collect and organize such examples in a Digital Filing Cabinet. See details at http://iae-pedia.org/Math_Education_Digital_Filing_Cabinet.

When you think about the number 13, what thoughts come to mind? Perhaps for you the number 13 is an unlucky number or a lucky number. Perhaps you remember that 13 is a prime number.

Robert Albrecht, one of your authors, has written an entire book telling part of the story of each of the positive integers 1-99. The 99-cent book is one of a number of books Albrecht is making available in Kindle format. (Remember, there is free software that makes it possible to read Kindle-formatted books on Macintosh and PC computers, on the iPad, and on Android phones. For information about downloading these free applications, see http://iae-pedia.org/IAE_Kindle_Books.

Albrecht, Robert (2011). Mathemagical numbers 1 to 99. Retrieved 6/3/2011 from http://www.amazon.com/s/ref=nb_sb_noss?url=search-alias%3Ddigital-text&field-keywords=Bob+Albrecht&x=0&y=0. Price: $.99. Other Kindle books by Albrecht are available at the same location.

Here is a short activity that you might want to try out with a math tutee. In this example, we use the number 13. Pick a number and ask your tutee to say some of the things they know or believe about that number. The idea is to engage your tutee in a conversation about a particular natural number.

The natural number 13 might be a good choice. Here is Robert Albrecht's story about 13.

13 (thirteen)

13 is a natural number.

13 is the successor of 12.

13 is the predecessor of 14.

13 is a prime number.

13 is an emirp. (31 is a prime number.)

Factors of 13: 1, 13

Proper factor of 13: 1

Sum of factors of 13 = 14

Sum of proper factors of 13 = 1

13 is a deficient number.

13 is a Fibonacci number.

Triskaidekaphobia is the fear of 13.

Triskaidekaphilia is the love of 13.

An aluminum (Al) atom has 13 protons.

Notice that this "story" includes quite a few words from the language of math. Albrecht's book contains a glossary defining these words. Here is a suggestion. One of your goals as a math tutor could be to help your tutee learn to make use of the Web to find math-related information. For example, what is a natural number? What is a prime number and why is it important in math? Who is Fibonacci and why is a certain type of number named after him? Do some very tall buildings not have a 13^{th} floor? How can that be possible? Are there widely used words that have exactly 13 letters?

What is a proton? Is there an atom that has exactly 12 protons, and is there an atom that has exactly 14 protons? Why and how is math used in sciences such as biology, chemistry, and physics?

What can one learn about the number 13 through use of the Web? A recent Google search using the term *13* produced over 20 billion hits! Suppose a person spent just 10 seconds looking at a hit to see if it relevant to their interests? How long would it take to process 20 billion hits?

A Google search of the word *thirteen* produced a little over 72 million hits. Why do you suppose that the math notation *13* produced so many more hits than the written word *thirteen?*

Final Remarks

In some sense, each person is a lifelong student and a lifelong teacher. In our day-to-day lives we learn from other people and we help other people to learn. Using broad definitions of tutor and tutee, each of us is both a tutor and a tutee in our routine, everyday lives. As both tutor and tutee, our lives are full of learning and helping others to learn.

Most of us now make routine use of the Web and other electronic aids to accessing information. These electronic sources of information can be thought of as Computer Tutors designed to help us learn and to accomplish tasks we want to accomplish. Thus, readers of this book are routinely involved in being tutored by both people and computers.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. Think back over your personal experiences of tutoring (including helping your friends, fellow students, siblings), being tutored, being helped by peers, receiving homework help from adults, and so on. Name a few key tutoring-related ideas you learned from these experiences.
3. Have you made use of computer-assisted learning and/or computer-based games as an aid to learning or teaching math? If so, comment on the pros and cons of your experiences. What are your thoughts on a computer-as-tutor versus a human tutor?

Chapter 3

Tutoring Teams, Goals, and Contracts

"There is no I in TEAMWORK." (Author unknown.)

"No matter what accomplishments you make, somebody helped you." (Althea Gibson; African-American tennis star; 1927–2003.)

A tutor and a tutee work together as a team. The *tutor* part of a team may include a human and a computer system. The *tutee* part of the team may be just one student, but sometimes it consists of a small group of students who are learning together.

In all cases, the tutor(s) and tutee(s) have goals. It is desirable that these goals be explicit but quite flexible. The goals need to be agreed upon by the human tutor(s) and tutee(s). It should be possible to measure progress toward achieving the goals. This chapter discusses these issues.

Tutoring Scenario

Kim was a fourth-grade student who did not like math. Alas, early in the school year, her math grade was a D. Kim did better in other subjects. Kim's mother Jodi was sure that Kim could do much better with a little help, so she hired a tutor who would come to their home once a week, help Kim do her math homework, and hopefully help Kim to like math better, or at least dislike it less. Jodi knew that Kim did well in subjects she liked.

Jodi and the tutor talked. "Aha" thought the tutor, who loved math games. "This is a splendid opportunity to use games to make math fun for Kim." The tutor suggested to Jodi that each tutoring gig spend some time playing games as well as doing the homework. Jodi readily agreed.

Tutoring began. Each tutoring session, Kim and the tutor spent 30 to 40 minutes doing homework and then played math games. Kim loved the math games. After a few tutoring sessions, she became more at ease doing the homework because she knew that she would soon play a game. Better yet, she began trying to do more homework before the tutor arrived in order to have more time to play games.

Kim became very good at playing games, including games at a higher math maturity level than usual for a fourth grader. It became clear to the tutor that Kim was very smart in math.

Kim and the tutor played many games. Her favorite game was **Number Race 0 to 12**, a game in which you try to move racers from 0 to 12 on five tracks. (See Chapter 5 for a detailed description of this game.) To move your racer, you roll three 6-faced dice (3D6) and use the numbers on the dice to create numerical expressions to move the racers on their tracks.

As the weeks rolled by, Kim became better and better at creating numerical expressions. After a few weeks, she became as good as the tutor in rolling 3D6 and using addition, subtraction, multiplication, and parentheses to create numbers to move her five racers on their five tracks.

Spring rolled around and Science Fair beckoned. Kim and her mother asked the tutor to suggest science fair topics. He did. Among the topics was one of his favorites, making homemade batteries from fruit, vegetables, and metal electrodes. Kim liked this idea and chose it as her science fair project.

Kim, with great support from her mother, made batteries using apples, bananas, lemons, oranges, potatoes, and other electrolytes. She experimented with pairs of electrodes selected from iron, aluminum, carbon, zinc, and copper. Jodi bought a good quality multimeter (about $40) for Kim to use in order to measure the voltages produced by various combinations of fruit, vegetables, and metals. Kim found that copper and zinc electrodes produced the highest voltage using several fruits and vegetables as electrolytes. Figure 3.1 shows her final project.

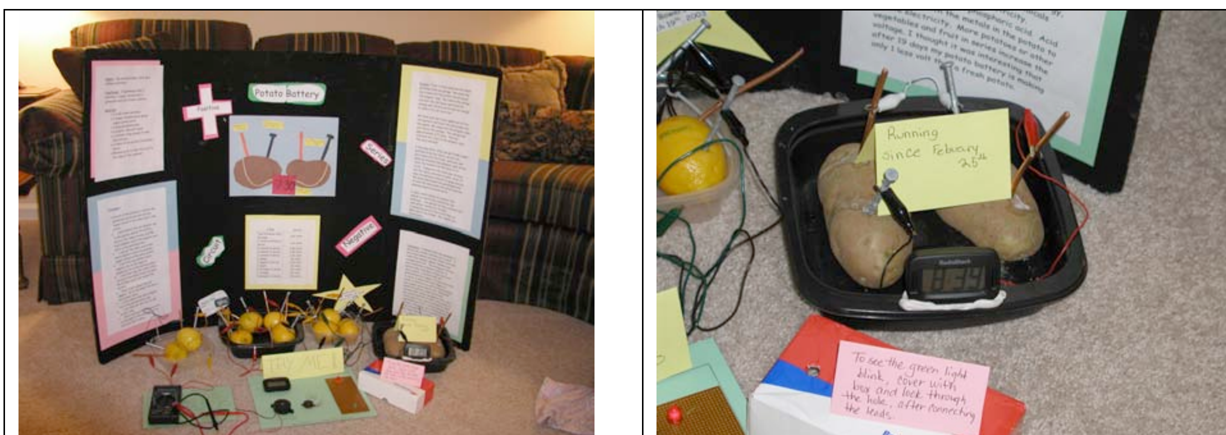

Figure 3.1. Science fair project done by tutee with her mother's help.

This story has a very happy ending. Kim's Science Fair project was outstanding! And, Kim became a very good math student! In retrospect, we can conjecture that Kim's previous home and school environments had not appropriately fostered and engaged Kim's abilities in math and science. The combination of two tutors (mother and paid tutor) helped Kim to develop her interests and talents in both math and science.

The active engagement of Kim's mother was a very important part of this success story. Jodi was an excellent role model of a woman quite interested in and engaged in learning and doing science. This story also illustrates the power of a team engaged in the tutor/tutee process. The active engagement of all three members of this tutor/tutee team was outstanding.

This story also illustrates another important point. The tutor had a very broad range of knowledge, skills, and approaches to getting a tutee engaged. The real breakthrough came via games and the Science Fair project rather than through the original "contract" on homework tutoring.

> With the help of the paid tutor and her mother, the tutee became a very good math and science student.

Contracts

A parent might use a paid tutor without a formal written contract—the "contract" is an oral agreement or implied by the situation.

A Scenario from Bob Albrecht's Tutoring

> The mother of a 5th-grade student that I tutored at home for an entire school year said, "I want my son to have fun." Wow! (I thought). We can do homework for part of the hour and play games or do experiments for the rest of the hour.
>
> One day we went outside with the goal of measuring the height of tall objects in the neighborhood such as utility poles, the top of the tutee's home, trees, et cetera. From each tall object, we walked and counted a number of steps, and then used an inclinometer to measure the angle to the top of the object. We drew all this stuff to scale and used our scale drawings to estimate the heights of the tall objects in units of the tutee's step length and my step length—thus getting different values for the heights. We discussed the desirability of having a standard unit of measurement, and then did it again using a metric trundle wheel.

This is an excellent example of "play together, learn together." It shows the value of a flexible contract and a highly qualified and versatile tutor.

Tutoring is often a component of an Individual Education Program (IEP). The IEP itself is a contract. However, this does not mean that a tutor helping to implement an IEP is required to have a written or informal contract or agreement with the tutee. A similar statement holds when a tutoring company, a paid tutor, or a volunteer tutor works with a tutee outside of the school building.

Many schools routinely provide tutoring in environments that fall between these two extremes. The school provides a "Learning Resource Center" that is staffed by paid professionals (perhaps both certified teachers and classified staff), a variety of adult volunteers, and perhaps peer tutors who may be receiving academic credit or "service credit" for their work.

A student (a tutee) making use of the services of a school's Learning Resource Center or Help Room may have an assigned tutor to engage with on a regularly scheduled basis, or may seek help from whoever is available. By and large there are some written or perhaps unwritten rules such as:

1. Tutors and tutees will be respectful of each other and interact in a professional manner. This professionalism includes both the tutor and the tutee respecting

the privacy of their communications. This holds true both for the tutoring and the mentoring aspects of the tutor-tutee communications and other interactions.

2. In a school setting (such as in a Learning Resource Center or a Help Room) each of the tutors (whether paid or a volunteer) is under the supervision of the professional in change of the Center. The tutor is expected to take advantage of the knowledge and skills of the Center's director and so seek help when needed.
3. The tutee has academic learning goals and agrees to use the tutoring environment to help move toward achieving these goals. Some of these academic goals may be quite specific and short term and others much broader and longer term. Some are math content specific and some are learning to be a responsible student who is making progress toward becoming a responsible adult. Here are a few examples:
   - I need help in getting today's homework assignment done.
   - I want to pass my math course.
   - I want to move my C in math up to a B.
   - I need to pass the state test that we all have to take next month.
   - I need to learn to take responsibility for doing my math homework and turning in it in on time.
   - I want to become a (name a profession). I need to do well in math to get into college and to get a degree in that area.
   - I want to understand the math we are covering in the math class. Right now I get by through memorization, but I don't think that is a good approach.
4. The tutor has the academic knowledge, skills, and experience to help the tutee move toward achieving the tutee's academic goals. The desirable qualifications of a tutor are discussed later in this chapter.

Notice the main emphasis in the above list is on academics. But—what about non-academic goals? A student may be doing poorly academically due to a bad home environment, due to being bullied, due to poor health, due to identified or not-identified learning disabilities, and for many other reasons.

> Individual paid or volunteer academic tutors should use great care in—and indeed, are often restricted from—moving outside the realm of the academic components of tutoring. They are tutors, not counselors.

A school or school district's counseling and other professional services may well have the capacity to deal with such problems. However, individual paid volunteer academic tutoring should use great care in—and indeed, are often restricted from—moving outside the realm of academic tutoring. An academic tutor who senses the need for non-academic counseling, tutoring, or other help should communicate this need to their tutoring supervisor or employer.

A Lesson Plan

A tutor/tutee team has instructional and learning goals. Before a tutoring session begins, the tutor creates some sort of a plan for the session. If there are to be multiple sessions, the tutor creates some sort of unit plan or multiple unit plans.

These types of plans can be quite detailed or quite sketchy, such as a few quickly scribbled notes. Good tutoring often requires extreme flexibility in adjusting to situations that arise and in being able to "seize the moment."

Here is a very rough outline for an individual session lesson plan:

1. **Begin.** Establish social contact with the tutee. Typically this includes friendly, non-threatening and non-academic conversation relevant to the tutee. Students can find tutoring sessions to be stressful. If a tutee seems overly tense and stressed out, work to reduce the tension and stress levels. Some tutors find that a little light humor helps. Others find it helps to talk about non-academic topics of mutual interest.
2. **Phase into academics.** This might begin with a question such as, "How has school been going for you since our last meeting?" The question can be more specific. For example, if the previous tutoring session focused on getting ready for a math test, the question might be, "Last time we helped you prepare for a math test. How did the test go for you?" If getting better at doing and turning in homework is one of the major tutoring goals, the tutor might ask for specifics on how the tutee did on this since the previous session. The goal is to move the conversation into academics and gives the tutor a chance to pick up on possible pressing problems.
3. **Session goals.** Remind the tutee of the very general goal or goals of the tutoring sessions. Ask if there are specific other topics the tutee would like to address during the session. In 1-2, both tutor and tutee get an opportunity to practice active listening and focusing their attention on the tasks at hand. This component of the tutoring session can end with a brief summary of the session's specific goals and tasks. Notice that the tutor may need to make major adjustments in the predetermined lesson plan.
4. **Content-specific tutoring.** This might be broken into several relatively self-contained activities of length consistent both with good teaching/learning practices and with the attention span of the tutee. A 30-minute block of time might be broken into two or three pieces of intense effort, with a "breather" between pieces. (A breather might be quite short, such as 30 seconds or a minute. It can be a short pause to make a small change in direction. It might be asking the question, "How are we doing so far in this session.") Part of the breather time might be spent on talking about the value and/or uses of the content being explored, with an emphasis on transfer of learning.

5. **Wrap up (debrief) and closure.** This might include asking the tutee "How do you think this session went?" Get the tutee actively involved in self-assessment and tutoring session assessment. The tutor provides a summary of what has been done during the session, makes suggestions of what the tutee might do before the next tutoring session, and suggests a possible plan for the next session.
6. **Tutor's personal debrief.** Soon after the session ends, make some case notes about what was covered, what went well, what could have gone better, and suggestions to oneself for the next tutorial session.

Qualifications of Tutor/Tutee Team Members

Suppose that a tutor/tutee team consists of a human tutor, a computer, and a tutee. There are expectations or qualifications that one might expect for each of these team members. A later chapter will discuss computerized tutoring systems. This section discusses the human members of a tutor/tutee team.

This section mainly applies to tutoring being done by adults. More detail about peer tutoring is given in the chapter on that topic.

Qualifications of a Tutee

A tutee is a person. A tutee has physical and mental strengths, weaknesses, interests, and disinterests. A tutee has a steadily growing collection of life experiences and learning experiences.

A tutee knows a great deal about him or her self. This self-knowledge and insight covers areas such as: friends and social life; interests and disinterests; academic and non-academic capabilities and limitations; current knowledge and skills; current and longer-range goals; home, school, and community life; and so on.

> A tutee is a human being who is facing and attempting to deal with a host of life's problems—both in school and outside of school—and including having learning problems.

Generally speaking, a tutee is in a math tutoring situation in order to facilitate more, better, and faster learning of math. Think about a typical third grade class. The math knowledge and skills of students in the class will likely range from 1st grade (or below) to 5th grade (or above). Students at the lower end of this scale may be learning math at one-half the rate of average math students. Students at the other end of the scale may be learning math at twice the rate of average math students.

Students at the lower end of the scale may receive math tutoring that is designed to help them move toward catching up with the mid-range students, or at least to not fall still further behind. Students at the upper end of the scale may receive math tutoring designed to help them continue to rapidly develop their math knowledge and skills —and to keep them from being "bored" in the math components of their education.

Schools throughout the country vary widely in the special services they make available to talented and gifted students. In situations where schools do little, parents may well provide special instruction to their TAG children and/or hire others to do so. As a personal example, Dave (one of your authors) is deeply involved in helping teachers learn to make use of calculators and computers in math education. His older daughter showed interest in learning about computers when she was quite young. Through Dave's help, she became a skilled computer programmer and computer gamer well before she finished elementary school. She has gone on to a very successful career as a computer programmer and gamer. Bob (your other author) can tell similar stories about his son who showed an early interest in computers.

However, the typical student a math tutor encounters tends to be struggling in our math education system. An in-school tutoring arrangement might begin with an intake interview conducted by a professional in the school's Learning Resource Center. In this interview a potential tutee might make statements and/or ask questions such as the following:

- I just can't do math.
- I hate math.
- Math scares me.
- The stuff we do in math class is not relevant to my life. Why do we have to learn this stuff?
- The math teacher makes me feel dumb and picks on me.
- Math is boring.
- I've got better things to do in life than to waste time doing homework.
- My parents get along fine in life, and they don't know how to do this stuff.

After tutoring sessions begin, the tutee may express similar sentiments to the tutor. Experienced math tutors have had considerable practice in dealing with such situations.

Qualifications of a Tutor

Tutors range from beginners, such as students learning to do peer tutoring and parents learning to help their children with homework, to paid professionals with many years of experience and a high level of education. Thus, it is important that the expectations placed on a tutor should be consistent with the tutors knowledge, skills, and experience.

> Tutor qualification areas: math content knowledge, math pedagogical knowledge, math standards knowledge, communication skills, empathy, and learning in areas relevant to math education.

Becoming a Better Math Tutor

This section is targeted mainly to desirable qualifications of professional-level math tutors, whether they be paid or volunteers. (A parent, volunteer, or peer tutor can be very successful without having this full set of qualifications.)

Here are nine qualification areas:

1. **Math content knowledge.** Be competent over a wide range of math content below, at, and higher than the content being tutored. Have good math problem solving knowledge and skills over the range of his or her math content knowledge.
2. **Math maturity.** Have considerably greater math understanding and math maturity than the tutee.
3. **Math pedagogical knowledge.** Know the theory and practice of teaching and learning math below, at, and somewhat above the level at which one is tutoring. This includes an understanding of cognitive development and various learning theories, especially some that are quite relevant to teaching and learning math.
4. **Standards.** Know the school, district, and state math standards below, at, and somewhat above the level at which one is tutoring.
5. **Communication.** This includes areas such as: a) being able to "reach out and make appropriate contact with" a tutee, and b) being able to develop a personal, mutually trusting, human-to-human relationship with a tutee.
6. **Empathy.** Knowledge of "the human condition" of being a human student with life in and outside of school, facing the trials and tribulations of living in his or her culture, the school and community cultures, and in our society.
7. **Learning.** A math tutor needs to be a learner in a variety of areas relevant to math education. Information and Communication Technology (ICT) is such an area. An introductory knowledge of brain science (cognitive neuroscience) and the effects of stress on learning are both important to being a well-qualified tutor (Moursund and Sylwester, October 2010; Moursund and Sylwester, April-June 2011).
8. **Diversity.** A math tutor needs to be comfortable in working with students of different backgrounds, cultures, race, creed, and so on. In addition, a math tutor needs to be able to work with students with dual or multiple learning-related exceptionalities, such as ADHD students who are cognitively gifted.
9. **Uniqueness (Signature Traits).** A math tutor is a unique human being with tutoring-related characteristics that distinguish him or her from other math tutors. As an example, Bob Albrecht (one of the authors of this book) is known for his wide interest in games, use of math manipulatives, use of calculators, and broad range of life experiences. He integrates all of these into his work with a student.

Tutoring Tips, Ideas, and Suggestions: Fun with Numbers

Math contains a large number of "fun" but challenging activities and challenges for students. A math tutor can have a repertoire of such activities and draw an appropriate one out of the bag when time and the situation seem right. Here is an example.

Positive Integers Divisible by 3

We know that some positive integers are exactly divisible by the number 3 and others are not. The number 7,341 is an example of 4-digit number divisible by 3:

7341/3 = 2447

Now, Let's form other 4-digit numbers from the four digits 7, 3, 4, and 1. Examples include 3741, 1437, 4137, and so on. It turns out that each of these is exactly divisible by 3.

3741/3 = 1247   1347/3 = 449   4137/3 = 1379

Interesting. Perhaps we have found a pattern. Try some other 4-digit numbers formed from the digits 7, 3, 4, and 1. It turns out that each of the 4-digit numbers you form will be evenly divisible by 3. [It also works for 2-digit numbers, 3-digit numbers, et cetera.]

Here are some "junior mathematician" questions:

1. How many different 4-digit numbers can one make from the digits 7, 3, 4, 1? This question is relevant because we may want to test every one of them to see if it is divisible by 3.

   Note to tutors: Use a 3-digit version of this question for tutees you feel will be overwhelmed by the 4-digit version. Your goal is to introduce the idea of careful counting and a situation in which your tutee can experience success.

2. Are there other 4-digit numbers that are divisible by 3 and such that any number formed from these four digits is divisible by 3? This question is relevant as we work to find then the divisibility conjecture might be true. Some exploration will lead you to a conjecture that this "divisible by 3" pattern works on the variety of 4-digit numbers that you try. Of course, that does not prove that it works for all 4-digit numbers that are divisive by 3. How many different 4-digit numbers are there that are divisible by 3? Is it feasible for a person to list all of these and then test for each one all of the 4-digit numbers that can be made from the digits? (A computer could complete this task in a small fraction of a second.)

3. Does the divisible by 3 property we have explored for 4-digit numbers also hold for 2-digit numbers, 3-digit numbers, 5-digit numbers, and so on? Some trials might well lead you to conjecture that the answer is "yes." But now, we have a situation in which an exhaustive test of all possible numbers is not possible. What is needed next is a "mathematical proof" that the conjecture is correct, or finding an example for which the conjecture is not correct.

4. Explore the following conjectures:

   4a. If the sum of the digits in a positive integer is divisible by 3, then the integer is divisible by 3.

   4b. If a positive integer is divisible by 3, then the sum of its digits is divisible by 3.

Final Remarks

Being a tutor or a tutee is being a member of a teaching and learning team. A team is guided (indeed, driven) by goals that are mutually acceptable to the team members. Success depends on the various team members being committed and actively involved. It also depends of the team members being qualified to effectively participate in achieving the goals.

Through education, training, and practice, all team members can get better in fulfilling their particular roles. Effective tutoring over an extended period of time needs to include a strong focus on the human and humane aspects of the process—on the humans communicating with each other and working together to accomplish the agreed upon goals.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. Read through the list of nine tutor-qualification areas. If you like, make additions to the list. In the original or expanded list what are your greatest strengths? What are your relative weaknesses? What are you doing to improve yourself in your areas of relative weakness? One of the ideas that David Perkins stresses in his book about Whole Games (Perkins, 2010) is identification of one's weaknesses and spending much of one's study and practice time on these weaknesses.
3. In your initial conversation with a new math tutee, the tutee says: "I am not good at math and I hate math." How would you deal with this situation?

Chapter 4
Some Learning Theories

> "Give a man a fish and you feed him for a day. Teach a man to fish and you feed him for a lifetime." (Chinese Proverb.)
>
> "They know enough who know how to learn." (Henry B. Adams; American novelist, journalist, and historian; 1838–1918.)

A human brain is naturally curious. It is designed to be good at learning making effective use of what it learns.

People vary considerably in terms of what they are interested in learning, how rapidly they learn, how deeply they learn, and how well they can make use of what they learn. There has been substantial research on similarities and differences among learners. A variety of learning theories have been developed. These help to guide teaching and learning processes and the development of more effective schools and other learning environments.

This chapter provides a brief introduction to a few learning theories. As an example, constructivism is a learning theory based on the idea that a brain develops new knowledge and skills by building on its current knowledge and skills. This theory is particularly important in a vertically designed curriculum such as math. Weaknesses in a student's prerequisite knowledge and skills can make it quite difficult and sometimes impossible for a student to succeed in learning a new math topic.

Tutoring Scenario

> One of my first tutoring gigs was tutoring two 8th-grade girls in algebra. The three of us met twice a week for the entire school year in the home of one of the girls.
>
> For the first few weeks, we spent our hour doing the assigned homework. The tutees did not do the assignment prior to my visit, but waited until I arrived. Then we slogged through the assignment together.
>
> One day we finished early, so I asked, "Want to play a game?" They said, "OK."
>
> We played Pig [described in Chapter 5] for the rest of the hour, and I stayed on for a while afterwards because they were having so much fun.
>
> Before I left, I said, "Hey, if you do your homework before I arrive, we can go over it, and then play games. I have lots of games."
>
> From that day on, they did their homework before I arrived and we went over it. Because we were not pressed for time, we could delve more deeply into what the girls were learning and/or could be learning in doing the homework assignment problems. We always finished with ample time to play a fun math game.

This example illustrates using a potential reward to shape behavior. Behavirorist learning theory is discussed later in this chapter. The example also illustrates that the tutees were quite capable of doing their homework without the aid of a tutor. The tutoring environment provided a type of structured social and educational learning situation that made the homework more fun.

The Essence of Teaching and Learning

Education researchers and practitioners have accumulated a great deal of knowledge about the theory and practice of teaching and learning. We know, for example, that an intact human brain is naturally curious and has a great capacity to learn. You know that both nature (inborn potentials) and nurture (all informal and formal learning-related life experiences) are important to a child's development.

> The human brain is naturally curious and has a great capacity to learn. Good teachers and a "rich" learning environment improve the speed and quality of learning.

Learning requires access to what is to be learned, focused attention, and feedback.

Sensory disabilities and/or problems in focusing and maintaining attention are major challenges to learning. High quality tutoring can be tremendously beneficial to students with these learning challenges.

Here is an example of a theory about how infants learn. Infants receive input from their five senses. The input is processed and "understood" in terms of what the infant has already learned. That is, new knowledge is built on (constructed on) what has already been learned. Feedback plays a key role in this cyclic process.

Consider an infant experiencing a situation of some form of discomfort or distress. Perhaps the situation is a feeling that we would describe as hunger, dirty diapers, being too hot or being too cold. The infant tries out a particular type of crying. If the type of crying leads to an improvement in the situation, this behavior is quickly learned.

As a child babbles, feedback from listeners helps to shape language development. A healthy human brain has a tremendous capacity to learn spoken language, but feedback is essential. In a bilingual or trilingual home environment, a child can readily become bilingual or trilingual through the informal instruction and feedback provided by parents and other caregivers.

This learning occurs because of a combination of innate capability of the learner and the feedback provided by the parents and other caregivers. There are other very important factors, such as the loving care and routine one-to-one "tutoring" provided by the caregivers. Being a good parent is a very challenging task!

Probably you have heard about the idea of a "rich" learning environment. It is an environment that includes many varied opportunities for learning and for using one's learning. Substantial research with a wide range of learners show that a rich physical and mental environment leads to long-lasting improvement in brain capabilities.

As children phase in to more formal teaching/learning environments, the quality of the environment—especially the quality of the caregivers and teachers—makes a huge difference. The teachers are both sources of information to be learned and feedback during the learning process. A good learning environment is rich in learning opportunities such as materials, activities, and explorations. It also provides good opportunities for children to learn from each other, often quickly switching roles from being a teacher (tutor) to a learner (tutee).

Cognitive Development (Brain Development)

The term cognitive development may suggest the name Jean Piaget to you. Piaget's 4-stage theory of cognitive development has been somewhat modified over the years, but the underlying ideas have stood the test of time. They are a useful theory in the development of teaching and learning materials and in understanding human learning.

A person's brain slowly matures as it grows in size, learns to make effective use of its neurons, and develops more connectivity among the neurons. Physical maturity of a human brain is usually complete by age 25 or so. However, the ability to learn and to adjust to new situations continues for many years after that. Remember the adage, "Use it or lose it."

Piaget's 4-state theory of cognitive development consists of:

Stage #	Name	Approximate Age Range
1	Sensorimotor	Birth to 2
2	Preoperational	Roughly ages 2 to 7
3	Concrete Operations	Ages 7 to 11 or 12, or higher
4	Formal Operations	Most people achieve their full brain development by about age 25 or so.

Figure 4.1. Piaget's 4-stages of cognitive development.

Here are important points.

1. The rate of cognitive development varies considerably with students.
2. Many people never fully achieve Formal Operations. Throughout their lives, they function at a Concrete Operations level, perhaps moving into the lower level of Formal Operations in some areas.
3. The level that a person reaches can vary from discipline to discipline and is dependent on cognitively challenging informal and formal education in the various disciplines.

Piaget's theory of cognitive development is an example of a Development Stage Theory. One of the modern versions of Developmental Stage Theory has 16 stages. In it, various Piagetian stages are divided into a sequence of levels, and some levels are added at the top end.

One of your authors—David Moursund—has developed a 6-stage math model for use in his teaching, writing, and workshops. It is not a research-based model, but it helps to explain why so many students get in "over their heads" in our math education system.

Stage Number, Name, and Age	Math Cognitive Developments

Range	
Level 1. Piagetian and Math sensorimotor. Birth to approximately age 2.	Birth to age 2. Infants use sensory and motor capabilities to explore and gain increasing understanding of their environments. Research on very young infants suggests some innate ability to deal with small quantities such as 1, 2, and 3. As infants gain crawling or walking mobility, they can display innate spatial sense. For example, they can move to a target along a path requiring moving around obstacles, and can find their way back to a parent after having taken a turn into a room where they can no longer see the parent.
Level 2. Piagetian and Math preoperational. Approximately ages 2 to 7.	During the preoperational stage, children begin to use symbols, such as speech. They respond to objects and events according to how they appear to be. The children are making rapid progress in receptive and generative oral language. They accommodate to the language environments (including math as a language) they spend a lot of time in, so can easily become bilingual or trilingual in such environments. In math-rich environments, they learn quite a bit about the number line, the language of math, and solving simple math problems.
Level 3. Piagetian and math concrete operations Approximately ages 7 to 12.	During the concrete operations stage, children begin to think logically. This is characterized by 7 types of conservation: number, length, liquid, mass, weight, area, and volume. Intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking develops (mental actions that are reversible). While concrete objects are an important aspect of learning during this stage, children also begin to learn from words, language, non-electronic and electronic games, pictures/video. They learn about and via objects that are not concretely available to them. For the average child, the time span of concrete operations is approximately the time span of elementary school (grades 1-5 or 1-6). During this time, learning new math topics is linked to having previously developed some knowledge of math words (such as counting numbers) and concepts. However, the level of abstraction in the written and oral math language quickly surpasses a student's previous math experience. That is, math learning tends to proceed in an environment in which the new content materials and ideas are not strongly rooted in verbal, concrete, mental images and understanding of somewhat similar ideas that have already been acquired. There is a substantial difference between developing general ideas and understanding of conservation of number, length, liquid, mass, weight, area, and volume, and learning the mathematics that corresponds to this. These tend to be relatively deep and abstract topics, although they can be taught in very concrete manners.
Level 4. Piagetian and math	Starting at age 11 or 12, or so, thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use

formal operations. Age 11 or 12, up to about age 25.	of symbols related to abstract concepts, problem solving, and gaining and using higher-order knowledge and skills. Math maturity supports the understanding of and proficiency in math at the level of a high school math curriculum. Beginnings of understanding of math types arguments and proof. Piagetian and Math formal operations includes being able to recognize math aspects of problem situations in both math and non-math disciplines, convert these aspects into math problems (math modeling), and solve the resulting math problems if they are within the range of the math that one has studied. Such transfer of learning is a core aspect of Level 4. Level 4 cognitive development can continue well into college, and most students never fully achieve Level 4 math cognitive development. (This is because of some combination of innate math ability and not pursuing cognitively demanding higher-level math courses or equivalent levels on their own.)
Level 5. Abstract mathematical operations. Moving far beyond math formal operations.	Mathematical content proficiency and maturity at the level of contemporary math texts used at the upper division undergraduate level in strong programs, or first year graduate level in less strong programs. Good ability to learn math through some combination of reading required texts and other math literature, listening to lectures, participating in class discussions, studying on your own, studying in groups, and so on. Solve relatively high level math problems posed by others (such as in the text books and course assignments). Pose and solve problems at the level of one's math reading skills and knowledge. Follow the logic and arguments in mathematical proofs. Fill in details of proofs when steps are left out in textbooks and other representations of such proofs.
Level 6. Research mathematician.	A very high level of mathematical proficiency and maturity. This includes accuracy and understanding in reading the research literature, writing research literature, and in oral communication (speak, listen) of research-level mathematics. Pose and solve original math problems at the level of contemporary research frontiers. This level is reached through having considerable natural ability in math and through many years of hard work.

Figure 4.2. A Piagetian-like 6-state math cognitive development scale.

Think about what happens when math instruction in a course is at a cognitive developmental level that is significantly higher than that of a student. Many such students fail the course. Many others pass by using a memorize, regurgitate, and forget strategy. Such students are not ready for the next math course. However, our precollege math education system pushes them forward, and many pass the course by continuing their use of the memorize, regurgitate, and forget strategy. Such a strategy causes them to fail (fail miserably) when they reach a course (such as a rigorous course in College Algebra or beginning Calculus) that requires math knowledge and skills based on understanding math, solving challenging problems, and having a level of math maturity that they have not yet achieved.

Alas, use of the *memorize, regurgitate, and forget* strategy is doomed to ultimately fail when a student reaches a course that requires the student to **understand math**.

Math tutors frequently encounter the challenges described in the previous paragraph. Some push their students along, helping them in use of the memorize and regurgitate strategy. Others help their tutees to build (rebuild) an understanding of math content and math maturity foundational knowledge and skills to a level needed for success in the math course they are currently taking.

Four Important Learning Theories

There are many learning theories. Accompanying them are many different theories and practices of teaching. Teaching and learning are not "one size fits all" situations. The task of converting a learning theory into effective teaching practice is particularly challenging because each individual student is unique.

This uniqueness provides one way to look at learning theories especially applicable in tutoring. Can a theory readily be translated into effective practice that helps fit the individual, unique needs of each student? One of the strengths of one-on-one tutoring is that both instructional processes and content can be adjusted to the individual needs of the tutee. Increasingly, computer-assisted learning is proving useful in this endeavor.

Behaviorism

Quoting Kendra Cherry (n.d.).

> The term behaviorism refers to the school of psychology founded by John B. Watson based on the belief that behaviors can be measured, trained, and changed. Behaviorism was established with the publication of Watson's classic paper Psychology as the Behaviorist Views It (1913).
>
> Behaviorism holds that only observable behaviors should be studied, as cognition and mood are too subjective. According to behaviorist theory, our responses to environmental stimuli shape our behaviors. Important concepts such as classical conditioning, operant conditioning, and reinforcement have arisen from behaviorism.

Nowadays, the name B. F. Skinner is most often associated with behaviorism learning theory. Likely you are familiar with many different forms of extrinsic motivation based on a system of rewards and/or punishments. Many parents use this approach from time to time in working with their children:

> "Eat your vegetables and then you can have dessert." (Reward.)
>
> "If you don't eat your vegetables, you don't get dessert." (Punishment.)
>
> "After you put your toys away, we will go play in the park." (Reward—if the child wants to play in the park. Neither reward nor punishment if the child does

> not want to play in the park. Perhaps the child would rather continue playing with the toys.)

Simple rewards and punishments are common in drill & practice and in tutorial types of computer-assisted instruction learning materials. These stimulus/response types of training have proven to be quite effective. Quoting from http://wik.ed.uiuc.edu/index.php/Behaviorism:

> Behaviorism is in practice throughout our schools. If one employs behaviorism in the classroom, it is imperative that it be used correctly. Skinnerian teachers would avoid the use of punishment. Research indicates that reinforcing appropriate classroom behaviors, such as paying attention and treating classmates well, decreases misbehavior and behaviorist classroom management techniques are often effective when others are not. Behavioral teaching and learning tends to focus on skills that will be used later.

> Research suggests that positive reinforcements are much more effective than negative reinforcements in education of children.

Consider a math tutor helping a tutee with a set of homework problems. The assigned problems are done one at a time with immediate feedback for each problem when it is completed. If the problem is solved correctly, positive reinforcements might include praise from the tutor or acknowledgement of the good thinking done in dealing with a difficult part of the problem. The tutor may have promised that after all of the homework is done, the remaining time in the tutoring session can be used to play a fun math game. Thus, the tutee works in an environment of positive reinforcement.

Research suggests that positive reinforcements are much more effective than negative reinforcements in education of children. Such research also suggests that teachers and parents tend to use too much negative reinforcement.

Constructivism

We mentioned constructivist-learning theory in the introduction to this chapter. One way to approach the subject of constructivism is to study how a human brain takes in, processes, and stores information at a neuron level. For the neurons that are affected, some dendrites are strengthened and perhaps some new dendrites are grown. This can be thought of as constructing better and/or new components of neurons.

To understand constructivism from a teaching and learning point of view, think about prerequisite knowledge and skills that are assumed or required as one approaches a new topic in math. The new math knowledge and skills are “constructed” on the prerequisite knowledge and skills. This helps to explain why math is a challenging topic. Students have covered (and, perhaps learned) the math that is assumed as a prerequisite to the next math topics they are studying. Many students will have forgotten (or, never learned) some of the essential prerequisites knowledge and skills.

To complete the required assignments and pass the tests, many students are forced into a memorize (with little understanding) and regurgitate approach. However, this leads to the students not having the understanding that is needed as prerequisite for the next math topic to be covered.

Here is an example. When studying division of arithmetic fractions, a student may be encouraged to memorize:

> "Ours is not to reason why, just invert and multiply."

Think about how this bit of "memorized wisdom" transfers to working with a complex algebraic expression. Many students don't know what to invert and become still further lost in manipulating algebraic expressions were understanding is required.

Many students seeking tutoring help in math have fallen into the memorize with little understanding, use memorized knowledge and skills to do the assignments and pass the tests, and then quickly forget the material. Good tutoring addresses this problem and helps to change a tutee's approach to taking math courses. Readers interested in learning more about constructivism in math education may want to explore the work of Catherine Fosnot (http://condor.admin.ccny.cuny.edu/~cfosnot/fullvita.doc).

Situated Learning

Situated learning is a theory stating that what you learn is highly dependent on the situation in which you learn it. Situated cognition and the culture of learning (Brown et al., 1989 is a seminal article on situated learning. Quoting from the article:

> Recent investigations of learning, however, challenge this separating of what is learned from how it is learned and used. The activity in which knowledge is developed and deployed, it is now argued, is not separable from or ancillary to learning and cognition. Nor is it neutral. Rather, it is an integral part of what is learned. Situations might be said to co-produce knowledge through activity. Learning and cognition, it is now possible to argue, are fundamentally situated.

Suppose, for example, that you grow up using the US Common System of measurements, (foot, yard, mile, ounce, pound, pint, quart, gallon, et cetera) and learn about the metric system in a math or science class. You then travel to a country where everybody uses the metric system. The chances are you will have considerable difficulty transferring your math and science classroom knowledge of the metric system into dealing with its everyday use during life in another country.

The Brown et al. (1989) article provides a number of other math-related examples that illustrate weaknesses in our math education system.

Situated learning theory provides some insight into the David Perkin's "Whole Game" ideas discussed in Chapter 1. Consider the whole game of living in a foreign country. In preparation for this trip, one learns about the country and some elements relevant to life in the country. Elements such as the language and the metric system can be taught as junior games where the games create simulated learning environments that are much like will be encountered when one encounters the whole game. If instead, they are taught in a manner relatively far removed from the whole game, there will be less transfer of learning and the traveler will face major challenges.

Situated learning theory helps to explain the value of apprenticeship types of education and training. In apprenticeship situations, the learner is engaged in hands-on activities that are closely related to the desired learning outcomes. For example, an apprentice carpenter gets to carry wood, measure wood, saw wood, plane wood, sand wood, and so on. The apprentice gets to help put pieces of wood together to help form objects such as cabinets and shelving.

In summary, apprenticeships provide good illustrations of effective application of situated learning theory. An apprentice is provided with small-group or one-on-one instruction that is quite specific to the desired learning outcomes. This instruction occurs in a situation where the new learning is immediately used to do productive work. The instruction and the assessment are authentic. In many apprenticeship settings, the apprentice does sufficient work to cover or more than cover the cost of providing the individualized help.

> An apprentice learns in a situation where new learning is immediately put to work by the apprentice and becomes an important component of the apprentices' repertoire. Contrast this with the long-delayed or relatively little use—once a topic has been "covered"—that is common in math education.

Part of a good math tutor's repertoire is to engage the tutee in junior games of math that are relevant to the tutee's life and interests. In this context, effectively using math in the tutee's "real work" is the whole game, and the tutor provided junior games (situated learning games) that are simulations of important elements of the whole game.

Transfer of learning

Transfer of learning is one of the most important ideas in education. It involves learning in a manner that facilitates retaining and using one's learning in the future, as well as building future learning upon it. There are various theories about how to teach and how to learn in a manner that facilitates such transfer of learning.

For many years, the prevailing theory of transfer of learning was quite simple. The actual transfer was called either near transfer or far transfer. In near transfer, one applied his or her learning to contexts and situations that were closely related to (near) the context and situation of the learning. In far transfer, the application was to contexts and situations that were rather different (far from) the learning context and situation. It was also common to first define near transfer and then define any learning that did not readily transfer as far transfer.

Perkins and Solomon (1992) describe this process:

> Near transfer refers to transfer between very similar contexts, as for instance when students taking an exam face a mix of problems of the same kinds that they have practiced separately in their homework, or when a garage mechanic repairs an engine in a new model of car, but with a design much the same as in prior models. Far transfer refers to transfer between contexts that, on appearance, seem remote and alien to one another. For instance, a chess player might apply basic strategic principles such as "take control of the center'" to investment practices, politics, or military campaigns. It should be noted that "near" and "far" are intuitive notions that resist precise codification. They are useful in broadly

characterizing some aspects of transfer but do not imply any strictly defined metric of "closeness."

> Near transfer refers to transfer between very similar contexts. Far transfer refers to transfer between contexts that, on appearance, seem to be different from one another ("far" from one another).

The low-road/high-road theory of transfer of learning developed by Perkins and Solomon (1992) has proven quite useful in designing curriculum and instruction. In low-road transfer, one learns some facts and procedures to automaticity, somewhat in a stimulus-response manner. When a particular stimulus (a particular situation) is presented, the prior learning is evoked and used.

> **Standards:** First-grade students are expected to learn the addition table for numbers 1 to 9 so well that given two one-digit numbers (the stimulus) they can quickly rattle off the sum (the response). This is an example of low-road transfer.

Low-road transfer is encouraged by helping students to identify problems that share many characteristics in common. This is a "hugging" strategy. For example, what types of rope or string tying problem situations can be solved by use of a bowknot? Examples include shoe tying, package tying, bowtie tying and other situations in which one wants to easily untie the knot. The human brain is very good at this type of learning.

High-road transfer is based on learning some general-purpose strategies and applying them in a reflective manner. It focuses on critical thinking and understanding. Here is an example. When faced by a complex problem, try the strategy of breaking the complex problem into a number of smaller, less complex problems. This is called the divide-and-conquer strategy. If the resulting problems are simple enough, you may well be able to solve each of them by drawing upon your current knowledge and skills.

In math, consider the problem of adding two 4-digit numbers. You learned an algorithm that breaks this into four separate problems—adding the 1's digits, adding the 10's digits, adding the 100's digits, and adding the 1,000's digits. Also, there is the task of what to do with each of the separate sums. Many students find it a challenge to learn to deal with the sums that are 10 or larger.

One can do a similar analysis for the "standard" paper and pencil multiplication algorithm that you learned many years ago. In both the addition and the multiplication examples, notice the value of memorizing the one-digit facts and being able to recall them both rapidly and accurately.

For a non-math example, consider the task of writing a relatively long article. By careful thinking, you can produce an outline. In essence, each item in the outline is a smaller, more manageable writing task. This careful thinking may include talking with others about your topic, doing online research, and a lot of mental trial and error. Divide and conquer is a very useful strategy in many different problem solving situations, but it does not take the place of careful thinking.

In addition, after you deal with each of the smaller individual problems, you have to put all of the smaller written pieces together and then check to make sure the original problem has been solved.

Here is a strategy (a bridging strategy) for learning for high-road transfer of learning. When you encounter a new strategy within a course:

1. Identify the generalizable strategy that is being illustrated and used in a particular problem solving or higher-order thinking situation.
2. Give the strategy a name that is both descriptive and easily remembered. (The divide and conquer strategy listed above is a good example.)
3. Analyze and reflect on the strategy. Identify a number of different examples in other disciplines and situations in which this named strategy is applicable. Practice using the strategy in these various situations.

There are many different strategies that are useful in a variety of problem-solving areas. Each can be learned and practiced both in the situated learning environment of a traditional academic course and in a manner that promotes transfer of learning. David Moursund's book, *Introduction to using games in education,* contains a large number of problem-solving strategies that are applicable over a wide range of problems (Moursund, 2006). The book illustrates high-road transfer of many of these strategies in the context of games and game playing.

A Mentoring Example from Bob Albrecht

> When I was the math/science mentor in Richard Zimmer's Mars Society (fall semester) and Mars Habitat (spring semester) courses at Sonoma State University, I posed the task of designing learning environments for the children in the three communities on Mars that our students were designing. At first, each group designed "schools" like the schools they were used to. I encouraged them to jump out of that box and indulge their most outrageous fantasies about learning environments that they would like to grow in from age 0 and up, on Mars or on Earth. Beautiful! Loverly! They responded with learning environments that used the entire community as the learning environment and included learning much stuff from the computer-based artificial intelligence network, low-road stuff, and also high-road stuff using AI simulations and AI-human dialogs. And, tra la, tra la, apprenticeships. Every person in the community who had knowledge and skills became a possible mentor for anyone who wanted to acquire that knowledge or those skills at any age. On Mars, no schools. Instead, everyone helps everyone to learn. Serendipity!

Tutoring Tips: Learning the Metric System of Measurement Units

Measurement of length, mass, weight, volume, and so on is an important aspect of math and is used throughout our daily lives. The United States, Burma (Myanmar) and Liberia are the only countries that do not use the International System of Units (SI), commonly called the metric system, as the official standard. The United States uses the US Common System (USCS) of measurements.

- International System of Units (SI) http://physics.nist.gov/cuu/Units/index.html
- United States Customary Units http://en.wikipedia.org/wiki/United_States_Customary_System

In the United States, students in math classes learn about the metric system and how to move between it and USCS units. Students get practice in using the metric system in their secondary school science classes. However, few students get a chance to routinely practice using the metric system. Most students do not develop fluency in moving between the two systems. A tutee can gain in "metric maturity" by hands-on measurements of the lengths and masses of objects.

> The best way to get an intuitive feeling for metric measurement is to do many "hands-on" measurements of the lengths, areas, volumes, and masses ("weights") of real objects. Another way: Learn the metric measures of objects that you see every day. Every time you see them, think about their metric measure.

Measure short lengths in millimeters (mm) and centimeters (cm). 10 millimeters = 1 centimeter. Measure "people-size" lengths in centimeters and meters (m). 100 centimeters = 1 meter. Metric rulers are available in 15-centimeter, 30-centimeter, and 100-centimenter (1 meter) lengths with centimeters and millimeters on one side of the ruler and inches on the other side.

- 1 inch = 2.54 centimeters.
- A short ruler displays 15 centimeters on one side and 6 inches on the other side. 15 centimeters = 5.91 inches. About 6 inches.
- 1 foot = 30.48 centimeters = 0.3048 meter.
- A longer ruler displays 30 centimeters on one side and 12 inches on the other side. 30 centimeters = 11.81 inches. About 12 inches.
- 1 yard = 91.44 centimeters = 0.9144 meter
- A meter stick displays 100 centimeters on one side and 39.37 inches on the other side. 1 yard = 91.44 centimeters = 0.9144 meter. 1 yard = 36 inches. 1 meter is approximately equal to 1.1 yard (10% longer than 1 yard).

Ready? Set? Go! With metric ruler in hand, you and your tutee can measure these objects and verify (we hope) our measurements of objects that we think have the same measurements here, there, everywhere.

- Width of one's little fingernail: about 1 centimeter. [Perhaps between 0.9 cm and 1.1 cm.]
- Diameter of a penny: 1.905 centimeter. [1.9 cm is very close and 2 cm is only about 5% off.]
- Credit card: 8.5 cm (85 mm) long by 5.4 cm (54 mm) wide.
- Length and width of a dollar bill: 15.6 centimeters by 6.6 centimeters.

- Sheet of 8.5 in by 11 in paper: 21.6 cm by 28 cm.
- Height of a standard door: approximately 200 centimeters = 2 meters.

Calculate areas of small objects in cubic centimeters.

- Area of your little fingernail. Bob's is about 1.0 cm long by 0.9 wide. Area = (1.0 cm)(0.9 cm) = 0.9 square centimeters.
- Area of a penny: $\pi r^2 = \pi$ (diameter / 2)$^2 = \pi$ (1.905 cm / 2)^2 = 11.4 square centimeters.
- Area of a dollar bill = (15.6 cm)(6.6 cm) = 103 square centimeters.
- Area of a 8.5 in by 11 in sheet of paper = (21.6 cm)(28 cm) = 605 square centimeters.

Remember these volumes:

- Volume of soda or water in a 1-liter bottle: 1 liter = 1000 cubic centimeters.
- Volume of soda or water in a 2-liter bottle: 2 liters = 2000 cubic centimeters.

The mass of an object is the quantity of *matter* in the object, the amount of *stuff* in the object. If you look up mass, matter, and stuff in a dictionary, you may find that they are defined in terms of one another, thus completing a circular definition. To measure mass in metric units, you need a metric scale. If you don't have one handy, the next best thing is to remember the masses of common objects.

- Mass of a dollar bill: 1 gram.
- Mass of a penny dated 1981 or earlier: 3.1 gram.
- Mass of a penny dated 1983 or later: 2.5 gram.
- Mass of a standard roll of 50 pennies dated 1983 or later: 50 x 2.5 grams = 125 grams.
- Mass of water or soda in a 1-liter bottle: about 1000 grams = 1 kilogram.
- Mass of water or soda in a 2-liter bottle: about 2000 grams = 2 kilograms.
- Mass of 1000 dollar bills: about 1000 grams = 1 kilogram.

The gram and kilogram are units of mass. The mass of an object is the same on Earth, Moon, Mars, and elsewhere in the universe. The USCS pound is not a unit of mass; it is a unit of force or weight. It depends on gravity. Your weight is different on Earth, Moon, and Mars. On Earth, you can use these conversions to go between USCS units (ounce, pound) and metric units (gram, kilogram).

- 1 ounce = 28.35 grams (1 ounce = 30 grams is close).
- 1 pound = 0.454 kilogram.
- 1 kilogram = 2.205 pounds. (1 kilogram = 2.2 pounds is very close.)

Final Remarks

A tutor is a teacher. The whole game of teaching and the whole game of tutoring overlap. Much of the underlying theory and practice is the same. Thus, a tutor benefits from knowing the learning theories and other general pedagogical knowledge of classroom teachers. A tutor learns to make use of these types of teacher knowledge and skills in a tutoring environment.

In addition, because a tutor is working with an individual tutee or a very small group of tutees, the tutor can create situated learning environments that are relevant to the "real worlds" of the tutees. This aids in tutee engagement and interests, and it helps promote transfer of learning.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. Study the 6-Level Math Cognitive Development Scale given in Figure 4.2. Which level best describes you, and why? Imagine in your mind a math tutee that you feel well qualified to work with. How might you go about making a good estimate of this tutee's math cognitive development level?
3. Briefly summarize the "conglomerate" of math learning theories that best describe your personal learning theory. How does "your" learning theory affect how you tutor/teach math?

Chapter 5

Uses of Games, Puzzles, and Other Fun Activities

> "It's not whether you win or lose, it's how you play the game." (Grantland Rice; American sportswriter; 1880–1945.)
>
> "Play is the work of the child." (Multiple sources, including Friedrich Froebel 1782–1852 and Jean Piaget 1896–1980.)
>
> Maybe we were borne to play,
> Maybe we were borne to amuse,
> When I hear children singing in play,
> I feel myself merrily shaking.
> (Goshirakawa; Emperor of Japan; 1127–1192.)

A fundamental component of successful tutoring is establishing an environment in which the tutee is actively and willingly engaged in learning and in building attitudes and skills that contribute to being a successful learner.

A good tutor develops a "bag of tricks" to help in this endeavor. One of your authors (Bob Albrecht) likes to use math-oriented games and intellectual puzzles in engaging tutees. Your two authors have written extensively on the use of games in education.

The general idea covered in this chapter is that a tutor's repertoire of math related, games, puzzles, brainteasers, and other activities can be integrated into math tutoring. The examples presented have been chosen for their success in improving a tutees level of math maturity and interest in math-related things.

Tutoring Scenario (Bob Albrecht Working with a Student)

In this scenario, Bob Albrecht working with a student identified by the initials SD. The scenario is written in the present tense.

> SD is diagnosed Asperger's Syndrome. He spends much time in the high school's Special Services Center. I tutor him 1st period Tuesday and Thursday, and frequently see him when I am tutoring other students. He rarely smiles, and he frequently seems tired and/or distracted.
>
> One Thursday, SD's Geometry homework was light and we finished it in 25 minutes. I was about to ask, "What shall we talk about the rest of the period?" SD preempted me and asked, "What shall we talk about the rest of the period?" I said, "You choose." He laughed, smiled, and said, "Dungeons & Dragons." SD's body language went from neutral to gleeful anticipation.

> For the rest of the period, SD was animated, smiling, laughing, and clearly having a good time. During the next 30 minutes, SD was highly interactive, engaging in a dialog that we shared equally.
>
> I shared some of my personal experience with *D&D* and other Role-Playing Games and described how I had used them with elementary school students. We discussed the six characteristics of a *D&D* character: Strength (STR), Constitution (CON), Dexterity (DEX), Intelligence (INT), Wisdom (WIS), and Charisma (CHA). SD knows about rolling 3D6 to create a beginning character so that each characteristic value is 3 to 18 inclusive.
>
> Game Masters are likely to let you roll 4D6 and choose the "best" 3D6 in order to avoid a very low score for a characteristic. We calculated the probability of rolling 3D6 and getting three ones and the probability of rolling 4D6 and getting four ones. We agreed that, rolling 3D6, an average characteristic value is (3 + 18)/2 = 10.5. But, no roll of three dice can produce 10.5. So, we talked about the idea that the average of a set of numbers can be different than any number in the set.
>
> We continued our investigation of the math involved in using dice to create *D&D* characters until the bell rang.

It is exciting and rewarding to participate in such a tutoring session. The tutee chose a topic and the tutor was able to bring both personal experience and math into the topic. Bob (the tutor) has a huge repertoire of games to draw on, as gaming has been one of his life passions.

Characteristics for a Good Game Useful in a Tutoring Environment

Here are several possible characteristics:

1. Low threshold and possibly a relatively high ceiling. This makes it easy to get started in a short period of time, and makes the game useful over a wide range of tutees.
2. The game involves "serious" use of math, math thinking, and math-related strategies.
3. There is a good chance that the tutee will find the game to be fun (intrinsically motivating) and that the tutee may well play it outside of the scheduled tutoring times.
4. Some of problem-solving activities used in playing the game provide an opportunity to practice transfer of learning.

If you have played some of the modern *Dungeons and Dragons* type games, you realize that they do not have a low initial threshold. People interested in getting started typically learn in an apprenticeship-like environment in which they work closely with a more skilled player.

A complete *D&D* game has a number of elements. In some cases, an element may be considered to be a junior level game in its own right. Rolling a character provides a good example. Many game players enjoy rolling characters and making decisions as to which aspects of a character should be given the greater strengths. The mathematical probability aspects are fun and mathematically challenging.

Competitive, Independent, and Cooperative

You are probably aware that boys are more "into" computer games that girls. Over the years, game developers have worked to develop games that are appealing to girls. This has narrowed the gap.

The puzzle video game Tetris created in 1984, along with many variations of this game, is particularly attractive to girls and women. (See http://en.wikipedia.org/wiki/Tetris.) It typically is played as a solitaire, non-competitive game. It is playable at many different levels of difficulty. It is a game in which one can improve their skill through practice. It involves quick recognition of certain geometric shapes and good hand-eye coordination.

Tetris is certainly a challenging game. A key to its success with girls and women is that the competition is not with another player. Rather, it is competition with oneself. The pleasure comes from doing well relative to oneself. You don't have to defeat anyone in playing the game.

Think about non-competitive, competitive, and collaborative situations in education. Some teachers like to create competitive situations in which there are clear winners and losers. Others like to create collaborative learning environments in which students work together to help each other learn and to accomplish a mutual goal.

> Some teachers like to create competitive situations in which there are clear winners and losers. Others like to create collaborative learning environments in which students work together to help each other learn and to accomplish a mutual goal. In today's world, most students need to become skilled in both of these environments.

A tutor may be influenced by his or her personal insights into the trio competitive, collaborative, and non-competitive. The tutor may also strive to fit the personal beliefs and feelings of the tutee.

Number Race Games

This is the first of several sections that provide details of math-oriented games that might be used as a "reward" in a tutoring session or as a specific goal in a tutoring session. These games are designed to help tutees gain in both math content knowledge and in math maturity. There is considerable emphasis on problem solving—in particular, in developing and testing strategies.

In this section we give brief introductions to two racetrack games that are played with two 6-faced dice (2D6) or three 6-faced dice (3D6) and a five-track racetrack such as the one illustrated in Figure 5.1. The game can be played in a solitaire mode, or two or more players may play it in a competitive mode. Each player has his or her own copy of the racetracks.

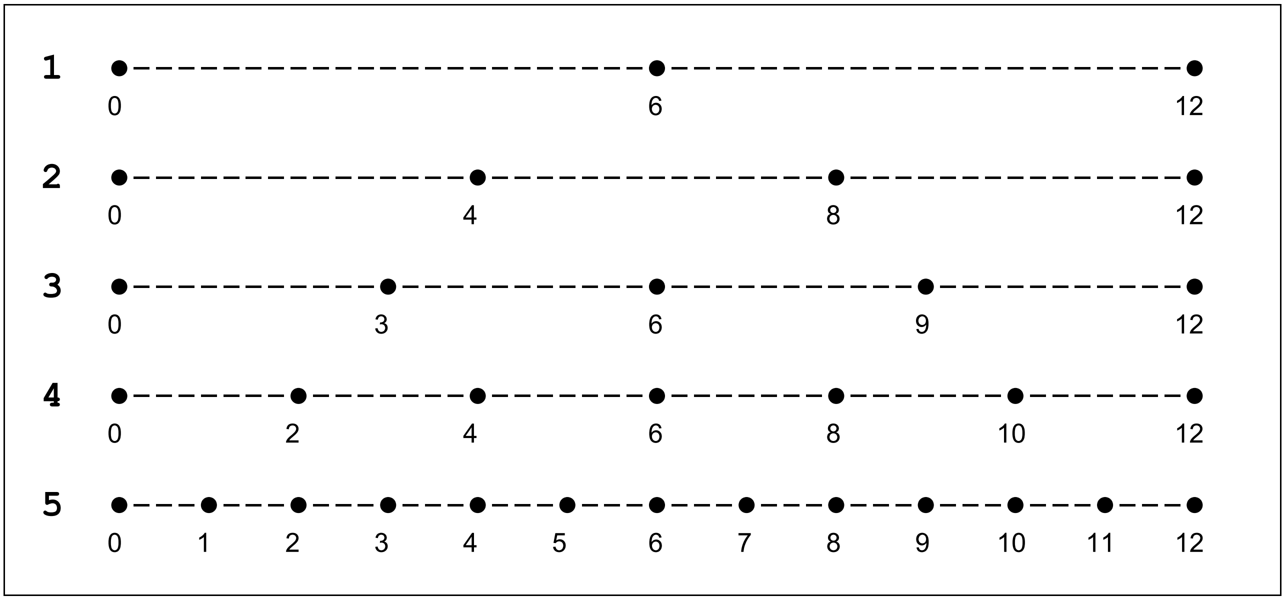

Figure 5.1. Five-track racetrack for Number Race 0 to 12.

To play Number Race 0 to 12, you first put a "racer" at the starting line (0) on each of the five tracks. The racer may be a game token such as the ones used in Monopoly or other board games. We frequently use base-10 block unit cubes as racers. Lima beans work well.

The object of the game is to move all of the racers from the starting line (0) to the finish line (12). In one round, every player gets a turn. The winner is the player who gets all five racers to 12 at the end of a round, so it is possible for two or more players to tie for a win.

To play a game, you roll 2D6 (Version 1, lower-level game) or 3D6 (Version 2, higher-level game) and use the outcome of the dice roll to calculate moves for your racers. Possible moves for a racer are from a numbered dot (●) on its track to another numbered dot on the same track. On Track 1, possible moves are from 0 to 6, 0 to 12, and 6 to 12. On Track 2, possible moves are from 0 to 4, 0 to 8, 0 to 12, 4 to 8, 4 to 12, and 8 to 12.

Number Race Game 1: A Beginner's Game

In Number Race 0 to 12 Game 1, you roll 2D6 and use the outcome of the dice roll to move racers from 0 to 12 along the five racetracks shown in Figure 5.1. To begin the game, put a racer at 0 on each racetrack. The object of the game is to move all five racers to the finish line (12).

To move your racers towards the finish line (12), roll 2D6 and use the outcome to move one or two racers from a numbered dot (●) to another numbered dot on the same track in any of these ways:

- Use the numbers on the **two dice** to move **two racers** along their tracks from one dot to another dot. If you roll **3** and **4**, you may use 3 move a racer on Track 3 or Track 5 a distance of 3 and also use 4 to move a racer on Track 2, 4, or 5 a distance of 4. If you have a racer on 6 on Track 1, 3, 4, or 5 and roll a **6**, you may move that racer from 6 to 12 because 6 is the distance from 6 to 12. If a racer is on 8 on Track 2, you must roll a **4** to move it to 12. You may not use a roll of **5** or **6** to move a distance of 4.

- Add the numbers on the two dice and use the sum to move **one racer** that distance on its track. If you roll **6** and **6**, you may add them and use the sum (12) to move one racer all the way from 0 to 12.
- Subtract the numbers on the two dice and use the difference to move **one racer** that distance on its track. This can be handy near the end of a game. Suppose all of your racers have finished except the racer on track 5, which is at 10. You need a **2** to move that racer to 12. You roll 3 and 5, calculate 5 – 3 = 2 and move your racer to the finish line.

As the game proceeds, you may find that a particular roll does not provide any legitimate move. For example, suppose all of your racers have finished except the racer on track 5, which is at 10. You roll 3 and 6. Alas, you are unable to calculate the 2 that you need, so you must pass and hope for a better roll next turn.

Number Race Game 2, the Higher-level Game

In Number Race 0 to 12 Game 2, you roll 3D6 and use the outcome to calculate a move for one, two, or three racers along their tracks from one numbered dot (●) to another numbered dot in any of these ways:

- Use the individual numbers on the three dice to move three racers along their tracks.
- Use one die to move one racer. Add or subtract the other two dice and use the sum or difference to move another racer. Or multiple them and use the product to move one racer.
- Add and/or subtract the three dice and use the result to move one racer. For example, if you roll 2, 5, and 5, you can calculate 2 + 5 + 5 = 12 and move one racer from to 0 to 12.
- Let a, b, and c represent the numbers on the three dice in any order. You may use the following algebraic alakazams to calculate moves:

  Two-dice calculations using addition, subtraction, and multiplication: Use $a + b$, $a - b$, or $a \times b$ to move one racer. Use the 3rd die to move another racer, if possible.

  Three-dice calculations using addition, subtraction, and multiplication: Use $a + b + c$, $a + b - c$, $a - b - c$, $a \times b + c$, $a \times b - c$, or $a \times b \times c$ to move one racer.

  Three-dice calculations using addition, subtraction, multiplication, and parentheses: Use $a(b + c)$ or $a(b - c)$ to calculate a move for one racer. For example, if you roll 1, 2, and 5, you can calculate 2(1 + 5) = 12 and move one racer from 0 to 12.

Number Race Math Notes

Each Number Race 0 to 12 game has a five-track racetrack with the individual tracks labeled Track 1 through Track 5. In each racetrack, the **minimum distance** between numbered dots (●) is a **factor of 12**. A legal move on any track is a **multiple** of the minimum distance on that track. To begin a game, put a racer at **0** on each track. To move a racer, roll 2D6 (Game version 1) or 3D6 (Game version 2).

- **Track 1**. Minimum distance: 12 / 2 = 6. Possible moves: 6 and 12.
- **Track 2**. Minimum distance: 12 / 3 = 4. Possible moves: 4, 8, and 12.
- **Track 3**. Minimum distance: 12 / 4 = 3. Possible moves: 3, 6, 9, and 12.

- **Track 4**. Minimum distance: 12 / 6 = 2. Possible moves: 2, 4, 6, 8, 10, and 12
- **Track 5**. Minimum distance: 12 / 12 = 1. Possible moves: 1, 2, 3, 4, 5, …, 11, 12. This is the only track in which you can move your racer a distance of 1. If your racer is stuck on 11 on Track 5 and you roll 5 and 4, calculate 5 – 4 = 1 and zoom that racer to the finish line.

In our experience, elementary school students who play Number Race games become quite adept at creating **algebraic expressions** that will move their racers toward the finish line.

In our experience, elementary school students who play Number Race games become quite adept at creating **algebraic expressions** that will move their racers toward the finish line.

WordsWorth

WordsWorth is a game that intertwingles math and words. It is designed to help children increase their knowledge of both language and math. The game comes with many variations that make it useful over a very wide range of grade levels and student abilities, from grade 1, "up, up, and away" to high levels of learning.

To play WordsWorth, assign a **letter value** to each letter in the alphabet, **a** through **z**, as shown in Figure 5.2. Uppercase letters have the same letter values as their lowercase counterparts.

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9
j = 10	k = 11	l = 12	m = 13	n = 14	o = 15	p = 16	q = 17	r = 18
s = 19	t = 20	u = 21	v = 22	w = 23	x = 24	y = 25	z = 26	

Figure 5.2. Letter values for the letters a through z.

The WordsWorth of a word is the sum of the letter values of the word's letters.

- The WordsWorth of **fun** is 6 + 21 + 14 = 41
- The WordsWorth of **game** is 7 + 1 + 13 + 5 = 26
- The WordsWorth of **learn** is 12 + 5 + 1 + 18 + 14 = 50

To make sure that a "word" is a word, we require that it can be found in one of the dictionaries listed below. You may choose a different dictionary or perhaps use a list of words that you want your students to learn.

- The *Official Scrabble Players Dictionary*
- Internet: Dictionary.com (http://dictionary.reference.com/)

A simple version of WordsWorth—and a good starting point for beginners—is to give them a list of words and have them calculate the WordsWorth of each word. This idea is illustrated in the examples given above. A teacher or tutor may want to give students a list of words and tasks such as:

- Find the WordsWorth of each word in the list of words.
- Arrange the words in alphabetical order.
- Arrange the words in WordsWorth (numerical value) order.
- Find the word or words in the list that have the least WordsWorth.
- Find the word or words in the list that have the greatest WordsWorth.

Using a dictionary of your choice, you might ask students to do these tasks:

- In the designated dictionary, what 2-letter word has the greatest WordsWorth? Is there more than one 2-letter word that has this value? What 2-letter word has the least WordsWorth? Is there more than one 2-letter word that has this value? One can ask the same set of questions for 3-letter words, 4-letter words, and so on.
- Is it possible to find 26 words, with the value of the first being 1, the value of the second being 2, the value of the third being 3, and so on up to 26? Think about this for a minute. The only letter that has a value of 1 is the letter *a*. Fortunately, *a* is a word. The only possible letter strings having a value of 2 are *b* and *aa*. Is *b* a word? No. Is *aa* a word? Fortunately it is—it is a type of lava. Otherwise, the problem would have no solution. Now you are off to a good start.
- Can you think of a word having a value of 3? It is easy to make a list of all possible letter strings that might be a solution. They are *c*, *ba*, *ab*, and *aaa*. Then all you have to do is decide whether one or more of these possible solutions is actually a word. Look them up in the dictionary that you are using.
- What is the longest word (most letters) that has a given WordsWorth? For example, what is the longest word that has a WordsWorth equal to 18? Is it *acacia*, which has six letters? Can you find a word with more than six letters that has a WordsWorth equal to 18? What is the longest word that has a WordsWorth equal to the number of weeks in a year (52)? We know a word with 11 letters that has a WordsWorth equal to 52. (Hint: It is a word sometimes spoken by magicians.)

WordsWorth has a number of characteristics of a good educational game. Much of the value of a good educational game rests in the hands of the people who help children learn to play the game, and provide guidance to the learners that explicitly focuses on the important educational aspects of the game.

Games such as WordsWorth provide an opportunity for students to develop strategies that will help them play the game more efficiently and effectively. Of course, someone can tell a student a particular strategy and explicitly teach the strategy. However, one of the goals in the use of games in education is to help students learn to develop strategies on their own, and then to explore possible uses of the strategies in problem solving outside of the realm or context in which the strategy was discovered and learned. A good teacher or tutor can be of great help in this learning task.

Bob Albrecht (one of your authors) is writing Play Together, Learn

Together books published as Amazon Kindle books. To learn about them, go to http://www.amazon.com. Select Kindle Store, and search for **albrecht firedrake**.

Place Value Games

The decimal number system is built on the concept of place value. Using only the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, we can construct any whole number. A one-digit number has only a **ones place**, a two-digit number has a **tens place** and a **ones place**, a three digit number has a **hundreds place**, a **tens place**, and a **ones place**. More-digit numbers have more places.

Students begin learning about place value in 1st grade and learn more in grades 2, 3 and up. You the tutor can use place value games to make learning place value an enjoyable activity. If you don't understand everything you know about place value, see http://www.aaamath.com/plc.html. This Web site presents interactive place value tutorials using several methods of presentation. It begins with two-digit numbers, then three-digit numbers, gently moves on to more-digit whole numbers, and culminates in decimal place values (tenths, hundredths, et cetera, et cetera).

We will briefly describe a three-digit place-value game using one 6-faced die (1D6). In this Try for High game, the objective is to get the highest (greatest) three-digit number. Using 1D6, the greatest possible three-digit number is 666.

The tutee can play a solitaire game with the tutor observing and responding to questions from the tutee, or the tutee and tutor can play a two-person game, taking turns rolling the die and placing its outcome in one of the three place values.

To play a solitaire game, you roll 1D6. The outcome is a number 1 to 6. Place the result in the hundreds place, the tens place, or the ones place to begin forming a three-digit number. Roll 1D6 again and place the result in one of the two place-value positions not selected in the first roll. Roll 1D6 a third time and place the result in the one-and-only remaining place value. Here are three games played a tutee whose initials are MJ.

Try for High Game 1	**Place values**		
	100	**10**	**1**
MJ rolled 1D6 and got 5. She placed it in the hundreds place.	5		
MJ rolled 1D6 and got 2. She placed it in the ones place.	5		2
MJ rolled 1D6 and got 6. She placed it in the remaining place, the tens place. Her three-digit try for high number is 562.	5	6	2

Try for High Game 2	**Place values**		
	100	**10**	**1**
MJ rolled 1D6 and got 3. She placed it in the ones place.			3

MJ rolled 1D6 and got 1. She placed it in the tens place.		1	3
MJ rolled 1D6 and got 2. She placed it in the remaining place, the tens place. Her three-digit try for high number is 213.	2	1	3

Try for High Game 3	**Place values**		
	100	**10**	**1**
MJ rolled 1D6 and got 2. She placed it in the ones place.			2
MJ rolled 1D6 and got 4. She placed it in the hundreds place.	4		2
MJ rolled 1D6 and got 5. She placed it in the remaining place, the tens place. Her three-digit try for high number is 452.	4	5	2

Figure 5.3 Three Try for High three-digit place-value games played by MJ

MJ explained her Try for High strategy:

- 1st roll. If the outcome is 4, 5, or 6, I place it in the hundreds place. If the outcome is 1, 2, or 3, I place it in the ones place.
- 2nd roll. If the outcome is 4, 5, or 6 and the hundreds place is not filled, I place it there. If the hundreds place is already filled, I put 4, 5, or 6 in the tens place. If the outcome is 1, 2, or 3 and the ones place is not filled, I place it there. If the ones place is already filled, I put 1, 2, or 3 in the tens place.
- 3rd roll. I place the outcome in the remaining place, the place not chosen in the first two rolls.

MJ's strategy worked well in Game 1, not so well in Game 2, and OK in Game 3. Her three place-value numbers were 562, 213, and 452. Aha! A splendid opportunity to do a little statistics by calculating the mean of the three scores.

The mean of MJ's three numbers = (562 + 213 + 452) / 3 = 1227 / 3 = 409

MJ created the data, so she owns the data: her scores for three games. So, tra la, tra la, we think that calculating the mean of the three scores that she owns is meaningful and relevant to her. She has just reinforced an important element of math, calculating the mean of a set of numbers. She did this calculation in a context that is meaningful and relevant to her, playing the place value game, a game that she likes to play.

Some Tutorial Ideas

- Don't show MJ's strategy to your tutee.
- Have the tutee play five three-digit Try for High games and record each roll of each game as MJ did in Figure 5.3.
- Calculate the mean of the five numbers in the five games.
- Ask the tutee to design and write a strategy showing what to do on each of the three rolls.

- Have the tutee play five games using the strategy that she or he designed. You are the referee who makes sure that the tutee follows the written strategy exactly.
- Calculate the mean of the five numbers using the tutee's strategy.
- Discuss the strategy and the results of applying the strategy.
- Challenge the tutee to play more games, record the moves, and calculate the means for each set of games before your next tutoring gig.
- Suggest an investigation. Suppose you played 1000 games using your strategy, what might be the average three-digit number? You can answer these questions by playing 1000 games and calculating the mean of the 1000 three-digit numbers. That would be a wonderful experiment, but very time consuming. A mathematical alakazam called **expected value** predicts an average score for many games equal to 499.875. We played 100 games and got an average score equal to 498.78, very close to 499.875.

You can continue this tutorial scenario by discussing the tutee's strategy and then having him or her test it again, or by having the tutee design and test another Try for High strategy.

Variation

Instead of Try for High, Go for Low. The object of the three-digit Go for Low place-value game is to get the lowest (least) three-digit number. Using 1D6 for each roll, the lowest possible number is 111.

> We like to use Try for High and Go for Low variations of games so that players have to think in two different ways and develop different strategies for each variation.

Factor Monster

Factor Monster is our name for a classic game about natural numbers, factors, proper factors, prime numbers, and composite numbers. Your tutee can play Factor Monster as a solitaire game while you observe and perhaps answer the tutee's questions as she or he learns how to play.

Factor Monster is also called Taxman or the Factor Game. See

- Taxman Game: http://www.maa.org/mathhorizons/pdfs/feb_2007_Moniot.pdf.
- NCTM Illuminations: Factor Game http://illuminations.nctm.org/ActivityDetail.aspx?ID=12.

Start with a list of **natural numbers** 1 to n. A small value of n makes the game easy, and a larger value of n makes the game more challenging. Let's use a small value, $n = 6$, to illustrate the game.

In playing the game with the natural numbers 1, 2, 3, 4, 5, and 6, your tutee can select a natural number that has one or more **proper factors** in the list. A proper factor of a number is a factor that is less than the number. See Figure 5.4.

Number	Factor(s)	Proper factor(s)	Comments
1	1	none	1 is neither a prime number nor a composite number.
2	1, 2	1	2 is a prime number.
3	1, 3	1	3 is a prime number.
4	1, 2, 4	1, 2	4 is a composite number.
5	1, 5	1	5 is a prime number.
6	1, 2, 3, 6	1, 2, 3	6 is a composite number.

Figure 5.4. Factors and proper factors of the numbers 1 to 6.

In your first move, you cannot take 1 because 1 does not have a proper factor. The only factor of 1 is itself. In the list of natural numbers 1 to 6, for your first move you may take 2 or 3 or 4 or 5 or 6.

After you select your number, alas, alack, and oh heck, greedy Factor Monster gets **all of the proper factors of the number you took**.

- If you take 2, Factor Monster gets 1, the only proper factor of 2.
- If you take 3, Factor Monster gets 1, the only proper factor of 3.
- If you take 4, Factor Monster gets 1 and 2, the two proper factors of 4.
- If you take 5, Factor Monster gets 1, the only proper factor of 5.
- If you take 6, Factor Monster gets 1, 2, and 3, the three proper factors of 6.

The number that you took and the proper factors that Factor Monster gobbles up are removed from the list. Then it is your turn again. You may take another number only if it has a proper factor in the new list.

More bad news! If the list contains only numbers that have **no** proper factors remaining in the list, then Factor Monster gets those numbers and the game is over. The player and Factor Monster add the numbers they have collected. The winner is the one who has the larger total.

Sample Game #1. Played by tutee JD with the list of natural numbers 1 to 6.			
	The List	**JD**	**Factor Monster**
Starting list of natural numbers 1 to 6:	1, 2, 3, 4, 5, 6		
JD takes 6 and removes it from the list.	1, 2, 3, 4, 5	6	
Factor Monster gleefully takes all of the proper factors of 6 and removes them from the list.	4, 5		1, 2, 3
JD's turn. Oops! There are no proper factors of 4 or 5, the two remaining numbers in the list, so JD cannot take a number and Factor Monster gets 4 and 5.			4, 5
JD and Factor Monster add their numbers.		6	15

Figure 5.5. A sample Factor Monster (FM) game in which the FM wins big.

Suggestion: When playing with paper and pencil, circle the numbers that the player takes, and then cross out the numbers that FM gets.

JD has played one game and lost. He starts thinking about a strategy that might help him do better next time. Starting with the 6 proved to be a poor strategy. JD did not think ahead about the consequences of this choice.

Here goes JD with his second game using the list 1 to 6. JD has learned from experience to not take 6 as his first number, and so he tries a different starting number. See Figure 5.6.

Sample Game #2	The List	JD	Factor Monster
Starting list of natural numbers 1 to 6:	1 2 3 4 5 6		
JD takes 4 and removes it from the list.	1 2 3 5 6	4	
Factor Monster chomps the proper factors of 4.	3 5 6		1, 2
JD takes 6, the only number that has a proper factor in the list.	3 5	6	
Factor Monster slurps up the proper factor of 6.	5		3
JD's turn. But there is no proper factor of 5 (JD's only choice), so FM gleefully snarfs 5.			5
JD and Factor Monster add their numbers.		10	11

Figure 5.6. Sample Factor Monster game #2. JD loses by a tad.

JD did better in Game 2, losing by a score of 10 to 11. After playing more games with the list of numbers 1 to 6, JD figured out how to win with the highest possible score: JD 15, FM 6, as shown in Figure 5.7.

Sample Game #3	The List	JD	Factor Monster
Starting list of natural numbers 1 to 6:	1 2 3 4 5 6		
JD takes 5 and removes it from the list.	1 2 3 4 6	5	
Factor Monster gets the only proper factor of 5.	2 3 4 6		1
JD takes 4.	2 3 6	4	
Factor Monster gets 2, the proper factor of 4.	3 6		2
JD takes 6.	3	6	
FM gnashes his teeth and takes 3.			3
JD and Factor Monster add their numbers.		15	6

Figure 5.7. Sample Factor Monster game #3. JD wins big.

Note to tutors: For you to know and your tutee to discover, the best first move is to take the greatest prime number in the list of numbers. JD figured this out by

> playing and recording his moves. He learned that if you take the greatest prime number, FM gets only 1.
>
> For any first turn choice by the player, Factor Monster will get 1 because 1 is a proper factor of any number greater than 1. The only way for the player to get a prime number is to take it in the first turn, so it is good strategy for the player to take the greatest prime number in his or her first turn.

After playing several Factor Monster games, JD now has a good handle on important math alakazams such as factors, proper factors, composite numbers, and prime numbers. Better yet, by playing and recording his games, he learned how to design a strategy that enables him to win by the best possible score. He has increased his **math maturity level**. His tutor has aided and abetted this activity by being there while JD played and providing support as JD taught himself how to win big playing Factor Monster.

JD was confident that he could play Factor Monster with longer lists of numbers such as 1 to 7, 1 to 8, 1 to 9, and so on. He did and for each list was able to attain the highest possible winning score. Factor Monster gnashed his teeth each time JD played with a new list of numbers and won by the highest possible score.

The table in Figure 5.8 shows the maximum possible score that the player can make for Factor Monster games with lists of numbers (value of *n*) from 1 to 6 to 1 to 9. You can give your tutees the task of playing the game for these values of *n*, and trying to achieve the maximum score given in the figure.

You can also give your tutees the task of determining the maximum scores that can be attained for values of *n* greater than 9. This is a more difficult task.

List of natural numbers	**Player's maximum**	**In this case, Factor Monster then gets**	**Sum of scores**
1 2 3 4 5 6	15	6	21
1 2 3 4 5 6 7	17	11	28
1 2 3 4 5 6 7 8	21	15	36
1 2 3 4 5 6 7 8 9	30	15	45

Figure 5.8. Maximum score the player can make.

Gift Giver

The game Gift Giver is the flip side of Factor Monster. Using the same rules as Factor Monster, you choose your numbers so that **Gift Getter** gets the higher score. For a given list of numbers, try to minimize your score and maximize Gift Getter's score.

Factor Monster and Gift Getter are variations that require the player to think in two different ways and develop different strategies for each variation.

Tutor Tips: Mathemagical Black Holes

A black hole is a region of space having a gravitational field so intense that no radiation or matter can escape from its gravitational pull.

If you are a science fiction fan, you have encountered black holes in a variety of books and videos. A story might involve space ships being sucked into a black hole, and the intrepid hero narrowly escaping being sucked in and killed.

Black holes are important astronomical objects created by collapsing stars. In a star, pressure caused by nuclear reaction of hydrogen and other fuels balances out the star's gravity. However, eventually a star runs out of fuel. When the star runs out of fuel, its own gravity causes it to collapse.

A small star, such as our sun, becomes a white dwarf. For stars with a greater mass, one of two things happens. If the star is of "medium" size, it collapses into a neutron star. If it is still larger, it collapses into a black hole. For more scientific detail, read about the Chandrasekhar limit at http://en.wikipedia.org/wiki/Chandrasekhar_Limit.

Mathematicians have discovered various numbers that have "black hole" or "suction-like" properties. The number 123 is an example.

Here is a simple procedure that can be applied to any natural number (that is, to any integer greater than 0). Let's name our original number N, and follow our procedure for $N = 123$.

1. Count the number of even digits in N. The number 123 has 1 even digit.
2. Count the number of odd digits in N. The number 123 has 2 odd digits.
3. Count the total number of digits in N. The number 123 has 3 digits.
4. Now, form a new natural number by concatenating (putting together) the three results in the order **even, odd, total**. (Notice that in this example, the result is 123.) The procedure, when applied to 123, produces 123. Black hole!

Now, here is the "magic," Start with any natural number. Apply the procedure (steps 1-4). If the result is 123, stop. If the result is not 123, call the result N and start over at step 1. **It turns out that no matter what the original natural number is, this procedure will eventually "suck it in" to the number 123.** The number 123 in a mathemagical black hole!

For example, start with $N = 1223334444$.

1. Count the number of even digits. 1223334444 has 6 even digits.
2. Count the number of odd digits. 1223334444 has 4 odd digits.
3. Count the total number of digits. 1223334444 has 10 total digits.

4. Form a new natural number by concatenating (putting together) the three results in the order even, odd, total. The new number is 6410. Apply the procedure to 6410.

1. Count the number of even digits. 6410 has 3 even digits.
2. Count the number of odd digits. 6410 has 1 odd digit.
3. Count the total number of digits. 6410 has 4 total digits.
4. Form a new number by concatenating (putting together) the three results in the order even, odd, total. The new number is 314. Apply the procedure to 314.

1. Count the number of even digits. 314 has 1 even digit.
2. Count the number of odd digits. 314 has 2 odd digits.
3. Count the total number of digits. 314 has 3 total digits.
4. Form a new number by concatenating (putting together) the three results in the order even, odd, total. The new number is 123. Black hole!

This mathemagical black hole may amuse or amaze your tutee and your tutee's friends. Trying the procedure out with a variety of starting values gives practice in following a mathematical set of instructions.

The underlying math can be explored. How might one go about proving that no matter what natural number is chosen for a starting value, the procedure eventually produces the number 123?

For more examples of mathematic, mathemagical black holes, and related topics see:

- http://www.amazon.com/s/ref=nb_sb_noss?url=node%3D154606011&field-keywords=Bob+Albrecht&x=0&y=0
- http://www.pleacher.com/handley/puzzles/blackhol.html

Tutoring Scenario (Bob Albrecht Working with a Student)

> In a recent tutoring session, we finished the required work before the period was over. So, I showed my tutee the **Square Root Black Hole**.
>
> The process: Start with any real number. Take the square root of that number. Then take the square root of the answer. Keep on truckin'—take the square root of each answer.
>
> This process is easy to carry out on any calculator that has a square root key. We were using a TI-84. I showed my tutee how to use [2nd] ANS on the TI-84, and then he began the task, starting with 100. Below I use SQRT to indicate the TI-84's square root function.
>
> SQRT(100) [Enter] = 10

SQRT(ANS) [Enter] = 3.16227766

SQRT(ANS) [Enter] = 1.77827941

SQRT(ANS) [Enter] = 1.333521432

Then we discovered that he could press [Enter] without SQRT and get the next answer.

[Enter] = 1.154781985

[Enter] = 1.074607828

[Enter] = 1.036632928

[Enter] = 1.018151722

[Enter] = 1.0090035045

[Enter] = 1.004507364

[Enter] = 1.002251148

[Enter] = 1.001124941

[Enter] = 1.000562313

I asked my tutee to make conjectures about what is happening. His conjectures:

- The numbers are decreasing.
- The numbers are all greater than 1.
- The numbers are getting closer to 1.
- The decimal to the right of the decimal point goes down by about a factor of about two each time.

Abracadabra! Alakazam! I had never noticed that last conjecture. Score math maturity points for my tutee.

The tutoring period ended before we had time to explore using the same process for positive numbers less than 1. In addition, we did not have time to discuss rounding errors in calculator arithmetic. All in all, this is a quite interesting topic! My tutee agreed that it was amazing, fun, et cetera, et cetera.

Final Remarks

There are lots of math-oriented games that can be used to help students gain in math content knowledge, math maturity, and math confidence. When you make use of a math-oriented game in tutoring, you might measure success by some combination of the following:

- Tutee is actively engaged.
- Tutee is enjoying the activity.
- Tutee "sees" that the game helps both in learning math content and in developing math maturity.

- Tutee is so interested in the game that he or she takes actions to make more of the tutoring time available for playing the game. For example, the tutee might get all math homework done before the session. During the session the tutor and tutee work together to check the homework for correctness and understanding—and then have time left over to play a game.
- Tutee is so enamored by the game that he or she plays it outside of the tutoring session.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. Select one of the games discussed in this chapter. Play it enough times so that you can play it with fluency and confidence. As you do this, make notes on your progress in gaining expertise. Make notes on what you are learning about yourself as a learner.
3. Select another game from the chapter or from some other source. Repeat (2) above. When you have done this activity for two games, compare and contrast the learning value of the two games for the types of students you tutor or feel qualified to tutor.

Chapter 6

Human + Computer Team to Help Build Expertise

> "Any teacher that can be replaced by technology—should be." (Arthur C. Clarke; British science fiction author; 1917–2008.)
>
> "Computers are incredibly fast, accurate, and stupid. Human beings are incredibly slow, inaccurate, and brilliant. Together they are powerful beyond imagination." (Albert Einstein; German-born theoretical physicist and 1921 Nobel Prize winner; 1879–1955.)

Chapter 3 mentioned the idea of a tutor/tutee team consisting of both humans and computers. It provided a brief introduction to computer-assisted instruction (computer-assisted learning).

This chapter delves more deeply into capabilities and limitations of human and computer tutors. It also explores two major forms of tutee/computer interactions. In one form, the computer is programmed to be a tutor and to provide various forms of computer-assisted learning. In the other form a computer is a general-purpose aid to retrieving information and helping to solve various types of problems. Math tutors and tutees can both learn to make effective use of this human and machine learning environment.

In this chapter we are particularly interested in what a human math tutor can do better than a computer math tutor, and vice versa.

Tutoring Scenario

DQ was a high school student when I (Bob) tutored him for two years. He has Asperger syndrome. Quoting from the Wikipedia:

> The lack of demonstrated empathy is possibly the most dysfunctional aspect of Asperger syndrome. Individuals with AS experience difficulties in basic elements of social interaction, which may include a failure to develop friendships or to seek shared enjoyments or achievements with others (for example, showing others objects of interest), a lack of social or emotional reciprocity, and impaired nonverbal behaviors in areas such as eye contact, facial expression, posture, and gesture.

Your authors know people in the computer field who have AS and have been very successful in their careers. We emphasize with their social problems and admire their intellectual achievements.

The relationship I developed with DQ was a combination of being a math tutor and mentor. When I first met DQ, I soon learned that he is a Star Trek fan and identifies with Spock and Data. He likes logic and frequently talks about logic. One day I asked him, "You like logic and you are a Trekkie, so is it OK if I call you SpockData?" Big smile and an enthusiastic "Yes!"

SD (formerly DQ) has well defined aspirations for his future. He wants to go to college, get an MBA, quickly make a lot of money, and then spend the rest of his life going to college and learning about everything. A major goal in tutoring SD was helping him to make progress toward achieving his goals. He has strong intrinsic motivation and our tutoring built on this intrinsic motivation.

Since these goals include a college education, some tutoring time is spent helping him to understand what is required to get into a college and what it takes to be successful in college. Many high school students have little insight into the nature of the academic challenges and requirements of higher education. SD is doing well in moving toward his future.

Standing on the Shoulders of Giants

Isaac Newton was a genius. He made major contributions to his world in many different disciplines. Moreover, he recognized the contributions of those who contributed to his successes.

> "If I have seen further it is by standing on the shoulders of giants." (Isaac Newton, English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian; letter to Robert Hooke, February 5, 1675; 1642–1727.)

Perhaps the single most important idea in problem solving is learning to build on the previous work of one's self and others. We learn with the help of our parents, teachers, peers, and others. Humans have accumulated a huge amount of data, information, and knowledge. Newton was a prolific researcher who built upon the research frontiers of his time.

So it is with today's students. For a moment, think of yourself as a young child. Your world is full of adults who are physical giants relative to your size. These adults are also mental giants compared to your current stage of cognitive development. Your world is full of the products that have been produced in the past. Buildings of all sizes and shapes, electric lights, vehicles for transportation, telecommunication devices, computer technology, multimedia … all of these things are in some sense "giants" that were developed by others.

Our education system is another of these giants. Students grow on the backs of teachers, tutors, and others who help them learn. Who knows—perhaps the next Marie Curie or Isaac Newton is one of your students!

Feedback is Essential in Gaining Increased Expertise

Learning by your brain requires:

1. **Content to be learned.** This content may come from outside your body, from inside your brain and body, or a combination of the two.
2. **Acquiring and processing content.** For example, content represented as light striking your retinas, is coded as neural impulses, goes into your brain, and is processed by your brain. Learning in your brain occurs through making new neural links or strengthening existing neural links among content that has already been stored.
3 **Feedback.** Feedback from internal and/or external sources helps the brain mechanism decide on which neural connections to further strengthen. For example, suppose you are trying to memorize that 3 x 4 = 12. You carry out some learning activities on this number fact. Then someone or some thing

(such as a computer) asks you "What is 3 x4?" You respond 12, and you receive feedback from the question asker that you have given a correct answer.

> One of the benefits of one-on-one tutoring is the immediate feedback that the tutor gives to the tutee.

In summary, think of learning as building and strengthening of neural connections, while forgetting is a weakening of neural connections. Remember the adage, "Use it or lose it."

Some General Categories of Expertise

Learning in any area helps to build your level of expertise in that area. Here are four types or measures of expertise:

1. A level of expertise that meets your current personal needs in a particular area and provides a foundation for achieving a level of expertise that will meet your future needs and wants. A person may have many such pockets of expertise (individual areas of expertise). Over time, new ones can be developed and one's level of expertise in some current areas may decline through disuse.
2. A level of expertise that meets the levels being specified by other people such as parents, relatives, siblings, friends, and other people you routinely interact with. For example, parents want their children to have some expertise in being respectful and responsible children who "pick up" after themselves and do their assigned chores.
3. A level of expertise established by various systems and institutions such as one's culture and community, schools, religious institution, our legal, moral, and ethical systems, and so on. Our schools want students to gain a level of expertise that meets "contemporary standards" in a variety of areas. Schools specify these areas and levels somewhat independently of student levels of intelligence and interest in these various areas.
4. A level of expertise needed to be successful in a variety of adult tasks such as being a spouse, parenting, employment, being a responsible adult citizen, and so on. An important aspect of "growing up" is gaining wisdom and foresight that leads to becoming a responsible and self-sufficient adult.

Figure 6.1 shows a scale of increasing levels of expertise.

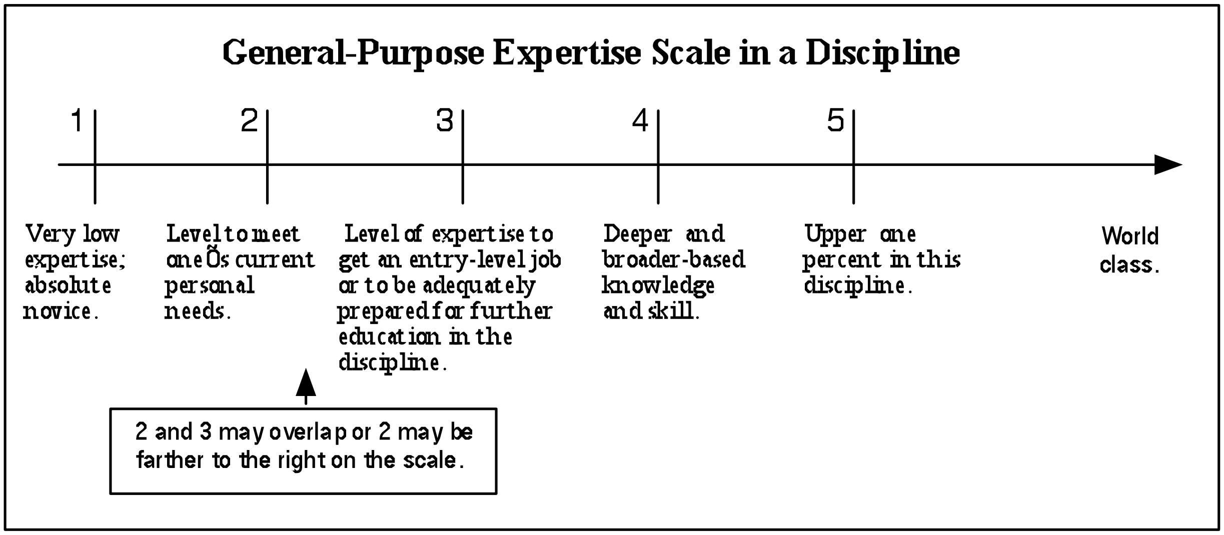

Figure 6.1. An Expertise Scale

Figure 6.1 can be interpreted in terms of having various pockets of expertise that are combined into a general area of expertise. Suppose, for example, that you are considering your overall qualifications to get a job as a clerk in a store. Certainly reading, writing, and arithmetic are pockets of expertise that are needed. However, honesty, punctuality, good grooming, good interpersonal skills and communication skills, teamwork skills, being a responsibly person, learning on the job, and many other pockets of expertise are important.

Figure 6.2 gives a rough estimate of the learning effort required to gain varying levels of expertise. How rapidly you gain expertise in a particular area depends on both your nature and your nurture.

- Nature: You may have innate ability in an area that allows you both to learn the area rapidly and to progress to a very high level.
- Nurture: A new area you are studying may be closely related to areas you have previously studied. A substantial amount of transfer of learning from your previous learning may occur.

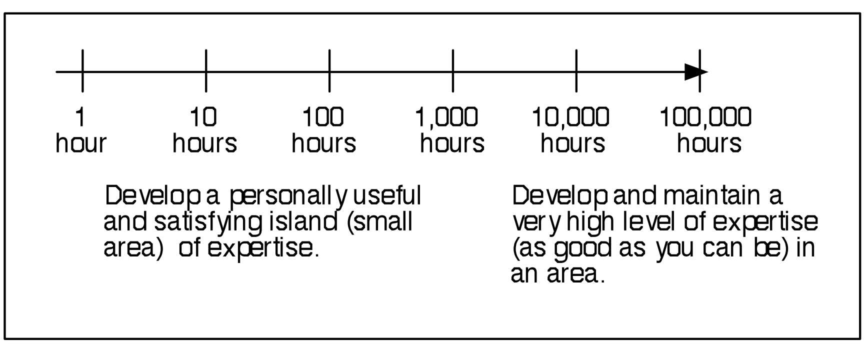

Figure 6.2. Learning time required in gaining varying levels of expertise

In many physical and mental skills areas it takes 10,000 hours or more of study and practice to get about as good as you can be. A combination of natural gifts, good teachers/coaches, and hard work brings a person to a national-class or world-class level of expertise. Although there are prodigies in many areas, the 10,000 hours is a low estimate for most people in most areas. Successful research professors in strong academic institutions put in about 10,000 hours by the time they finish a doctorate. In post doctorates and then "regular" employment, they may well spend over 2,000 hours a year building and maintaining their discipline-oriented expertise.

Computer-Assisted Learning

The early history of computers includes an emphasis on two different types of using a computer to provide instruction:

1. The automation of drill and practice types of instruction that makes use of multiple choice and True/False types of questions. In this setting a question is presented and the user selects an answer from the choices that are available. The computer system processes the answers, provides feedback, and keeps records of the number of correct and incorrect responses. The computer system then provides the next question.
2. Computer simulations that engage or immerse a learner in a simulated learning environment. Typically the goal is to help the learner learn to deal with relatively complex learning challenges. There are simulators to help train and educate car drivers, repair technicians in many areas, medical personnel, and so on.

As curriculum developers gained in knowledge and skill, and as computers became much more capable, both types of computer-assisted learning (CAL) became much more sophisticated. For example, computer simulations developed for helping to prepare airplane pilots now provide simulated environments that are nearly indistinguishable from those involved in actually flying a plane. Even experienced pilots can use such simulators to refresh and improve their skills.

While drill and practice types of software are still widely used, such software has gradually become much more versatile and "intelligent." As a simple example, think of CAL software that can provide instruction using a variety of media, ask questions and process a variety of types of responses, keep track of all of a student's correct and incorrect answers, detect patterns of types of errors, and "on the fly" adjust the instruction being presented to specifically help the student get better in areas that he or she is having the most difficulty. Nowadays, such a CAL system might be called a computer tutor.

A still better computer tutor system will "know" a variety of ways of presenting instruction and will adjust its instruction to better fit the learning needs of an individual student. A good human teacher can do this quite well. Highly Interactive Intelligent Computer-assisted Learning (HIICAL) systems are getting better and better at this task.

A variety of CAL companies are making use of the term "computer tutor" to stress the idea that their software can interact with an individual student in a "tutor-like" manner. Such software may be developed to fit needs of a student with specific learning challenges, or it might be developed for the broad ranges of students in a typical classroom. The quality of such computer tutor systems can be evaluated by comparing learning results of students using such systems versus the learning results of students using other modes of instruction.

Distance Learning

A steadily increasing number of precollege and higher education courses are available on the Web. The course on the Web might be completely self-contained, not requiring an instructor. Alternatively, a student will interact both with the computer instructional materials and with a teacher. This interaction will typically be done using email or text "chat," but also might be via audio or video conferencing. If a number of students are taking the same course at the same time, the students may be encouraged or required to interact with each other using similar communications technology.

Research suggests that students who successfully complete distance-learning courses learn as well as students who successfully complete "traditional" courses. However, the drop out rate in much higher in distance learning courses. Many students find that they are unable or unwilling to take the personal responsibility required in a distance-learning environment.

The content being covered in a distance learning course may consist of print materials to be read (perhaps from hard copy books and/or the Web), live or recorded lectures from a course instructor, multimedia and other support material developed specifically for the course, and so on. There is no fine dividing line between Computer-Assisted Learning and Distance Learning.

A mixed mode of instruction—a combination of HIICAL and traditional classroom instruction—is called hybrid instructional system. The term hybrid instruction is also used to in talking about a combination of distance education (distance learning) and traditional classroom instruction. You need to be aware that just because a program of instruction is being delivered by a computer system, this does not mean it is HIICAL. It might well just be videoed lectures or a combination of videoed lectures and drill and practice types of CAL.

Computer as Tutor

We have discussed computer-assisted learning. Many students have access to CAL systems purchased or leased by their schools. In addition there is a steadily growing amount of such educational software available free on the Web. Here are a few examples:

AAAMath (n.d.). AAAMath sorted by grade level. Retrieved 7/14/2011 from http://www.aaamath.com/.

Kahn Academy (n.d.) Watch. Practice. Learn almost anything—for free. Retrieved 5/19/2011 from http://www.khanacademy.org/.

Math Diagnostic Test (n.d.). Learning materials and resource centre. Retrieved 7/14/2011 from http://www.pedagonet.com/maths/diagnostic.htm.

Math Forum (n.d.). The Math Form @ Drexel. Retrieved 7/14/2011 from http://mathforum.org/.

ThinkQuest (n.d.). Think Quest library. Retrieved 7/14/2011 from http://www.thinkquest.org/pls/html/think.library. The 3DVinci Math Forum Page provides an excellent example of their materials. See http://mathforum.org/sketchup/.

Here is a different approach to using a computer as a tutor. Each time you as a tutor or a student who is your tutee look up something on the Web, you and/or your tutee are making use of the Web as a tutor.

Nowadays, essentially all students learn to make use of the Web. However, relatively few students learn to make use of the Web to access math-related materials. They learn to access CAL sites specified by their teachers. Few learn to use the Web to explore math-related materials beyond what the teacher requires. Few learn to make use of the software designed to help represent and solve math problems. For example, are you and your tutees familiar with Wolfram Alpha (http://www.wolframalpha.com/)? (Also see http://iae-pedia.org/Free_Math_Software.)

Your authors believe this weakness in information retrieval and making use of Web-based resources represents a major flaw in our math education system. The combination of not developing appropriate levels of skill in reading math and not learning how to access math-related materials that they need or want to know about represents a major impediment to being a life long learner of math.

A Combination of Human Tutor and Other Instruction

A tutoring team learning environment—combining human tutor(s), computer tutor(s), and tutees(s)—is increasingly common. This environment helps a tutee learn to learn from a tutor and a computer, and to make progress in becoming a self-responsible learner.

As a tutor, you need to keep in mind that typically your tutee is also receiving traditional classroom instruction, help from peers, and perhaps help from parents and others. Your tutee may find it quite a challenge to integrate all of these aids to learning. You can take it upon yourself to help your tutee deal effectively with this challenge.

Tutoring Tips, Ideas, and Suggestions: Pockets of Expertise

> "First say to yourself what you would be; and then do what you have to do." (Epictetus; Greek philosopher; 55 AD –135 AD.)

The activity suggested here is designed for a situation in which you are doing long term tutoring of a precollege student. It is adapted from the work of Joseph Renzulli, a world leader in gifted and talented education. See Chapter 3 in the free book *Computers in education for talented and gifted students* (Moursund (2007).

What math-related pockets of expertise does your tutee need and/or want? To answer this question, you need good insight into the current and long term math-related needs and goals of your tutee. This comes through getting to know your tutee and by specific enquiries on your part.

Here is a brief summary of the 4-item list under the Some General Categories of Expertise heading given earlier in this chapter.

1. Current personally specified personal needs.
2. Specified by people (such as parents; your culture) in your everyday life.
3. Specified by "systems" such as school, religious institution, and other organizations.
4. Needed in order to become a self-sufficient responsible adult.

The basic idea in this ongoing activity is that you and your tutee work together to specify one or more pockets of expertise or areas of expertise under each of the four headings. You and your tutee talk about current levels of expertise in these areas, progress (or, lack thereof) that is occurring in increasing expertise in these areas and goals in these areas. Help your tutee learn to self-assess in each of the areas.

Once a list is established, you and your tutee each keep a copy. In the ongoing tutoring, you and your tutee can occasionally revisit the list. This might be done a couple of times a month. In a short revisit to the list do two things:

1. Talk about progress (or, the lack thereof) that is occurring and what is needed to start, maintain, or increase the rate of progress. Include an emphasis on the tutee taking increased personal responsibility for progress. Point out the huge number of resources that are available to help the tutee.
2. Some possible items to be added to the list, and possibly some items to be deleted from the list.

Final Remarks

Each of us faces a lifelong challenge of building and maintaining levels of expertise in a wide variety of areas. There are many different resources to help us learn. With practice we can learn to use these resources and we can become more efficient (that is, gain increased expertise) in using these resources.

Here are two areas that a tutor should emphasize:

1. Learning to self-assess.
2. Taking an increasing level of responsibility for one's own education.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. Create a personal "areas of expertise" that you feel relate to you being a good math tutor. Use the list to analyze your strengths, relative weaknesses, and what you are doing to improve your tutoring knowledge and skills. (Your authors hope that you included computer technology in your list.)
3. Do (2) above, but for the topic of you being a self-sufficient lifelong learner in areas related to your life and to being a responsible adult citizen.

Chapter 7

Tutoring for Increased Math Maturity

> "Education is what remains after one has forgotten what one has learned in school." (Albert Einstein; German-born theoretical physicist and 1921 Nobel Prize winner; 1879–1955.)
>
> "One looks back with appreciation to the brilliant teachers, but with gratitude to those who touched our human feelings. The curriculum is so much necessary raw material, but warmth is the vital element for the growing plant and for the soul of the child." (Carl Jung; Swiss psychiatrist; 1875–1961.)

Earlier parts of this book have mentioned both math content and math maturity. Math content and math maturity strongly overlap. In a good math education system, students make thoroughly integrated progress toward:

1. **Math content goals:** Increased math content knowledge and skills.
2. **Math maturity goals:** Increased ability to think mathematically and make effective use of math knowledge and skills.
3. **Math Habits of Mind:** Developing increasingly strong math "Habits of Mind." (The next chapter focuses on this topic.)

As indicated in Chapter 4, an average person reaches their full level of cognitive development by approximately 25 years of age. With proper math education and encouragement, a person's level of math cognitive development will increase substantially over these years.

In Chapter 1 we gave the following very brief definition of math maturity:

> Math maturity is being able to make effective use of the math that one has learned through informal and formal experiences and schooling. It is the ability to recognize, represent, clarify, and solve math-related problems using the math one has studied. Thus, a fifth grade student can have a high or low level of math maturity relative to the math content that one expects a typical fifth grader to have learned.

Good math curriculum, in the hands of good math teachers, lays a foundation for learning to think mathematically. It prepares students for their future math studies and their future uses of math.

In this chapter we give a much more extensive definition of math maturity. We also provide some ideas that math tutors can use to help improve the math maturity of their tutees.

Tutoring Scenario

Becoming a Better Math Tutor

Most of this book focuses on formal tutoring situations in which there is a clearly defined tutor and a clearly defined tutee. However, your authors are also quite interested in informal tutoring situations in which a pair of people or a small group of people are talking about education-related topics.

The topic of math maturity provides an excellent example. Many students, parents, and math teachers have never heard about the idea of math maturity. We find it fun to bring up this topic in a conversation and do some informal exploration of the topic. This is easy to do when talking with a math teacher and exploring the topic of whether and/or how students are changing over the years.

For example, many algebra teachers find their students highly resistant to doing word problems. They believe that something has changed over the years, and that many of their students lack the math cognitive development and math habits of mind that are necessary to deal with this type of problem solving. So, immediately the conversation can be become one of talking about whether students are developing a level of math maturity that is suited to the course content and methods of teaching. What can be done to help increases a student's level of math maturity that related to dealing with math word problems?

In this type of very brief "tutoring" session, a seed can be planted. In addition, your authors like to take advantage of such situations to tell teachers about the free and/or very inexpensive materials that we have developed. (The Math Maturity Website http://iae-pedia.org/Math_Maturity has received more than17,000 hits as of 8/6/2011.)

Math Education is More Than Just Learning Math Content

We want students to learn to solve math problems they have previously studied. However, we also want students to learn to effectively deal with math-related problems they have not previously encountered—especially those that can readily be solved using the math that they have studied. We also want students to transfer their math knowledge and skills to other math-related problems both in school and outside of school, both as they are learning the math and in the future. In summary, we want students to build an increasing level of expertise in the discipline of mathematics.

The following characterization of a discipline is copied from Chapter 1.

> An academic discipline can be defined by a combination of general things such as:
>
> 1. The types of problems, tasks, and activities it addresses. In math education, we want students to gain a level of understanding that is well beyond rote memory.
> 2. Its tools, methodologies, habits of mind, and types of evidence and arguments used in solving problems, accomplishing tasks, and recording and sharing accumulated results.
> 3. Its accumulated accomplishments such as results, achievements, products, performances, scope, power, uses, impact on the societies of the world, and so on. Note that uses can be within their own disciplines and/or within other disciplines. For example, reading, writing, and math are considered to be

"core" disciplines because they are important disciplines in their own rights and also very important components of many other disciplines.

4. Its methods and language of communication, teaching, learning, and assessment; its lower-order and higher-order knowledge and skills; its critical thinking and understanding; and what it does to preserve and sustain its work and pass it on to future generations.
5. The knowledge and skills that separate and distinguish among: a) a novice; b) a person who has a personally useful level of competence; c) a reasonably competent person, employable in the discipline; d) a state or national expert; and e) a world-class expert.

The K-12 math curriculum touches on all of these areas. Some are given much more emphasis than others. Math curricula vary from country to country, from state to state within a country, from school district to school district, and so on. No matter how much a country or state attempts to standardize their math curriculum, there are major variations from teacher to teacher and in students' outside-of-school informal and formal math education.

In each country, considerable thought has gone into specifying the math content that is to be taught in the K-12 curriculum. In the United States, for example, second-degree polynomial functions with real coefficients are deemed to be a very important topic in algebra. Students learn to graph such functions and to find points where the function has a value of zero. Adults who have taken Algebra II sometime in the past are apt to remember the terms *quadratic equation* and *quadratic formula.* Over time, however, most forget the underlying mathematics. They may have had to solve a quadratic equation in a high stakes state or national test, or a college entrance exam. However, quadratic equations are not an important aspect of their everyday lives.

Now, think about what the average adult knows about general aspects of math functions and equations. Why are polynomial and trigonometric functions especially important? Why are functions such a fundamental component of math? How are functions used both in the STEM disciplines and in other disciplines? A good indicator of increasing math maturity is growing insight into these "deeper" types of math questions.

In the next few sections we give further examples of math content and math maturity.

Word Problems in Math and Across the Curriculum

In this section, we explore math word problems from a combined math content and math maturity point of view.

You know that math is a useful aid to dealing with various problems in all academic disciplines. We want students to recognize problem situations in which math may be a useful aid to understanding and solving the problem. In essence, we want a student's math knowledge and skills to transfer into other disciplines the student studies and/or encounters outside of the math classroom.

> We want students to recognize problem situations in which math may be a useful tool for understanding and solving the problem. We want a student's math knowledge and skills to transfer into other disciplines such a biology, business, chemistry, physics, and social studies.

In math education, one way we address this transfer of learning goal is to help students learn to solve word problems. Word problems are part of every academic discipline. In a discipline outside of math, a problem or problem situation is often stated using a combination of natural language, the special language of the discipline, and the language of math.

That is, there are math word problems that essentially deal strictly with math, and there are math-related word problems in other disciplines. (Physics provides an excellent example. Most of the problems presented in a physics text are word problems.) In math education, we want students to gain skill in solving math-related word problems "across the curriculum" This is akin to goals in reading instruction, in which we want students to learn to read across the curriculum.

Mathematizing is a word that you might not have heard before. It is a process of analyzing a problem that is not clearly stated in the language of math, and extracting from it carefully stated math problems that may be relevant to solving the original problem. The task is also called math modeling—developing a math model or a math representation of a problem.

We want students to become skillful in mathematizing word problems across the curriculum and then solving the math problems that result from this task. We want students to be able to interpret the results of solving the math problems in light of the original word problem to see if the results make sense and are an aid to solving the original problem.

Solving math-related word problems requires a combination of math content knowledge, math maturity, knowledge of the discipline from which the problem comes, and general cognitive maturity. Increasing ability to effectively deal with math-related word problems across the disciplines is an indicator of increasing math maturity.

Math is a Human Endeavor

> "God created the natural numbers. All the rest [of mathematics] is the work of mankind." (Leopold Kronecker; German mathematician; 1823–1891.)

Math is a human endeavor. Part of an increasing level of math maturity is a growing understanding of math as a human endeavor. Notice the title of the following book.

> Jacobs. Harold (3rd edition, 1994). Mathematics, a Human Endeavor. H.W. Freeman.

Quoting from a review available at http://dimacs.rutgers.edu/Volumes/schools/paper82/node3.html:

> *Mathematics, A Human Endeavor* was clearly a textbook ahead of its time. After going through its third revision in 1994, it is as popular today as a classroom text and a teacher resource as it was back in the late 60s. … **Much of the mathematics is connected to real world applications, and there are many**

> **science and mathematics connections. The book is jammed with problem-solving activities and ``What if ...?" questions, and emphasizes mathematical thinking.** It is also full of photos, drawings (e.g., from Escher), and mathematical cartoons (e.g., from *Peanuts*, *BC*, and *The New Yorker*). [Bold added for emphasis.]

The book emphasizes **mathematical thinking**. Mathematical thinking is both dependent on the specific math content being considered and cuts across the math curriculum. You have probably heard the expression, "Looking at the world through rose-colored glasses." Think about what it might mean to "Look at the world through math-colored glasses."

How does a mathematician view the world? The view is built on a combination of math content, looking for math-types of patterns in the world, representing problems in terms of math, careful and precise communication about problems, careful representations of how to solve certain types of problems, using mathematical logic in one's arguments, and so on.

Quoting from (Stacey, 2008):

> Mathematical thinking is a highly complex activity, and a great deal has been written and studied about it. Within this paper, I will give several examples of mathematical thinking, and to demonstrate two pairs of processes through which mathematical thinking very often proceeds:
>
> • Specializing and Generalizing
>
> • Conjecturing and Convincing.

…

> In my own work, I have found it helpful for teachers to consider that solving problems with mathematics requires a wide range of skills and abilities, including:
>
> • Deep mathematical knowledge
>
> • General reasoning abilities
>
> • Knowledge of heuristic strategies
>
> • Helpful beliefs and attitudes (e.g. an expectation that maths will be useful)
>
> • Personal attributes such as confidence, persistence and organization
>
> • Skills for communicating a solution.
>
> Of these, the first three are most closely part of mathematical thinking.

An increasing ability to do mathematical thinking is an indicator of an increasing level of math maturity. The value of mathematical thinking is most evident as one works to solve problems they have not previously encountered.

Math maturity includes of a number of aspects of math that are often not given much emphasis in the traditional content-oriented school math curriculum and in assessment. As an example, consider the following quotes from famous mathematicians. They all relate to mathematics as a human endeavor.

> "Mathematics is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency." (René Descartes, French philosopher, mathematician, scientist, and writer; 1596–1650.)
>
> “Mathematics is the queen of the sciences.” (Carl Friedrich Gauss, German mathematician, physicist, and prodigy; 1777–1855.)
>
> "Mathematics, rightly viewed, possesses not only truth but supreme beauty—a beauty cold and austere, like that of sculpture." (Bertrand Russell; English philosopher, historian, logician, and mathematician; 1872–1970.)

What do we want students to know about math as a human endeavor? For example, do we want our students to have an understanding of the types of ideas represented by the three quotations given in this section?

Each of the quotations given above represents a profound set of ideas. They are all related to math maturity. What understanding of these ideas do we want students to gain? Here is one answer.

> **We do not want to add to the math curriculum a requirement that students must memorize the names and dates of each of the famous mathematicians quoted above.** What we want is for students to gain some understanding of math as a human endeavor and of how math content and math thinking have affected our world in the past and will continue to do so in the future. We want them to become more skilled at math thinking.

While it is quite appropriate to integrate a history of math into the math curriculum, it is also appropriate to integrate a history of math and a history of many other disciplines into the history curriculum. Math is an important human endeavor, and we want students to have insight into the history of this endeavor and how it relates to other components of human history`.

Communication in the Language of Mathematics

The development of reading, writing, and a language of math facilitated the large accumulation of math knowledge and skills that we have today. As children learn oral communication in a natural language, they also learn some math vocabulary and ideas. As they begin to learn symbols for the written representation of a natural language, they also learn some math symbols. Thus, for example, young children learn both oral words and (abstract) written symbols for natural numbers 1, 2, 3, 4, 5, ….

Moreover, they learn to make a one-to-one correspondence between natural numbers and the items in a small collection. They learn an association between number words and quantity. This is a major step forward in learning math.

As children begin their formal math education in Kindergarten or the first grade, they are taught math through a combination of oral instruction and use of written math notation. They develop some fluency in moving between written and oral math communication.

At the same time, they are beginning to learn to read and write in a natural language. For most students, this is a long and challenging process. By the end of the third grade, average students begin to transition from learning to read to reading to learn. By middle school, a significant part of the instructional process involves learning by reading.

In math, however, something different happens. To a large extent, elementary school math is taught using oral methods, and students make relatively slow progress in learning to read math. The level of math that the content standards want students to learn remains quite a bit above an average student's ability to learn the math by reading math. For a great many students, this situation persists until they encounter a "rigorous" college-level calculus course.

This does not mean that average students cannot learn to read and write math at a level of the math they are studying in middle school and high school. Rather, it means that our current methods of math instruction do not foster this skill.

An indicator of increasing math maturity is a growing ability to read, write, speak, and listen in the language of math. This can include learning to read math at the level of the math one is working to learn, so that reading can be a viable aid to learning math.

> An indicator of increasing math maturity is a growing ability to read, write, speak, and listen in the language of math, and to learn math by reading math.

You can engage your tutee in math-related communication. A major component of each tutoring session can be an interactive give and take math communication involving reading, writing, speaking, and listening designed to help your tutee develop an increased level of skill in math-related communication.

Learning to Learn Math

The development of reading and writing a little over 5,000 years ago produced a need to have formal schools in which students learned to read, write, and do arithmetic. Thus, we now have a 5,000-year history of how to teach and learn the three Rs. You might think that in 5,000 years of study and practice in this aspect of schooling, we humans would have perfected the teaching and learning processes for the three Rs. Such is not the case.

One of the difficulties is that human brains are not all alike. Individual tutoring is effective partly because the instructional methodology and the specific content being taught can be adjusted to an individual tutee. Another difficulty in math education is that the content that one might learn has grown immensely, and the possible uses of one's math learning have changed immensely.

There are many other math education challenges. In the early history of math education, there were no books. It is only in the last few hundred years that our publishing technology has made it possible for a student to have a math book.

A Personal Story

Dave Moursund, one of your authors, likes to tell a story told to him by his mother. In the late 1920s, his mother was a graduate student in mathematics at Pembroke, the women's component of Brown University. One of her math

> professors brought his “old, yellowed, handwritten notes” to each class meeting and read them out loud, paragraph by paragraph. He read each paragraph twice, to facilitate students making hand written copies.
>
> The material being taught was relatively advanced and was unique to the professor. Thus, no textbook existed. Methods of inexpensively making good quality copies of “old, yellowed, handwritten” notes did not exist. Thus, the teaching/learning process consisted of coming to class to copy the notes, and then going off and learning from the notes.
>
> Of course, the “spirit duplicator” had existed for many years. Dave’s Mother suggested that the professor was “of the old school” and taught the way he was taught …

Our math education system is somewhat divided between learning by rote memorization and learning for understanding. A great many elementary school teachers of math teach math the way they were taught when the emphasis was rote memory with only a modest level of understanding.

Consider the challenge of learning to solve math problems. One approach is to study important categories of math problems and memorize procedures for solving them. Do a lot of drill and practice to solidify the memorization and to develop speed and accuracy in carrying out the procedures.

Many leading math educators believe this is a poor approach. It does not prepare students to deal with unfamiliar problems. It does not serve students well as they forget memorized facts and procedures over time.

In addition, it is not in tune with the steadily growing capabilities of calculators and computers both to “know” the procedures and can carry them out rapidly and accurately. In summary, your authors recommend less emphasis on rote memory oriented drill and practice, more emphasis on learning for understanding, and more learning to effectively use calculators and computers.

Proofs and Problem Solving

A math proof is a careful communication attesting to the correctness of a math assertion. It is a communication that is understandable by and convincing to others who have an appropriate level of math knowledge and skills. It stands the test of time, as mathematicians over the years scrutinize the detail of the proof looking for flaws in the arguments.

Students first encounter the concept of proof when they are asked to show details of their math work. This is a start of learning to give careful arguments about why a problem-solving process they have developed/used is correct.

George Polya was a world-class mathematician and math educator. The following Polya quote was given in the first chapter.

> To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. **For the higher aims about which I am now talking are some general tactics of problems—to have the right attitude for problems**

> **and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school.** But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems. [Bold added for emphasis.]

An indicator of increasing math maturity is a growing ability to solve math problems and make math proofs. This is quite different than developing an increasingly large repertoire of memorized proofs and memorized strategies to solve specific categories of problems.

> An indicator of increasing math maturity is a growing ability to solve math problems, understand math proofs and make original math proofs.

Polya also mentions having the "right attitude for problems and to be able to attack all kinds of problems. This growing attitude and ability are indications of increasing math maturity.

Mathematical Intuition

As one's knowledge of and experience in using math grows, one's math intuition grows. Herbert Simon, a Nobel Prize winning polymath, defined intuition as "frozen analysis." He noted that in any disciplines where one studies and practices extensively, a subconscious type of intuition is developed. One's math intuition may well be able to quickly detect an error that one has made in math thinking and math problem solving, very quickly decide a way to attack a particular type of problem, or provide a "feeling" for the possible correctness of a conjecture.

Suppose that you are grading a math test and you see a student has written 5 + 8 = 40. At a subconscious level your brain might say, "something is wrong." Your subconscious might next tell you, "The number 40 is way too large." Your experience and math teaching intuition might tell you, "perhaps the student multiplied instead of added." Through grading lots of student papers, you have developed some math intuition that makes you into a faster paper grader.

For a deeper view of math intuition, read Henri Poincaré 1905 paper, Intuition and Logic in Mathematics, available at http://www-history.mcs.st-and.ac.uk/Extras/Poincare_Intuition.html.

A growing level of math intuition is an indicator of an increasing level of math maturity.

Information and Communication Technology

All of the above ideas about math maturity need to take into consideration the various tools that have been developed to aid in representing and solving math problems.

Earlier in this chapter we discussed communication in the language of mathematics. Historically, this type of communication included reading and writing print materials. Any action to be taken based on such print materials was carried out by the person dong the reading.

Now, we have computerized information and storage systems. The systems are powerful aids to locating information a person wants to retrieve. In addition, the computer system may well be able to:

1. Provide interactive instruction to help you learn the material. One way to learn math is to make use of high quality computer-assisted learning materials.
2. Carry out math procedures that are discussed in the documents.

Calculators and computers are useful both in representing and solving math problems and also in the task of learning math. Moreover, computational mathematics is now one of the major subdivisions of the overall field of mathematics. Thus, increasing levels of math maturity are evidenced by increasing knowledge and skills in making effective use of Information and Communication Technology both as an aid to representing and solving math problems and as an aid to learning math. See http://iae-pedia.org/Two_Brains_Are_Better_Than_One.

Tutoring Tips, Ideas, and Suggestions: Fun with Domino Fences

Dominoes are used in a variety of domino games. They can also be used as building blocks. The activities given below uses them as building blocks. A domino has a length exactly twice its width. The height does not matter in these examples. In these examples, a domino has a length of 2 (units), a width of 1 (unit), and an area of 2 (square units).

> If you don't have a set of dominoes, use quadrille paper and draw "domino" fence pieces. Or print dominoes on stiff paper at http://www.first-school.ws/theme/printables/dominoes-math.htm.

Here are five examples of fencing in enclosed areas using dominoes.

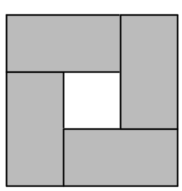

Figure 7.1

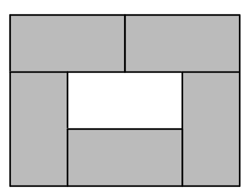

Figure 7.2

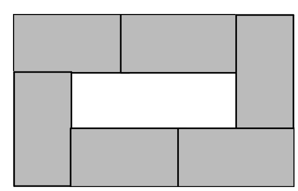

Figure 7.3

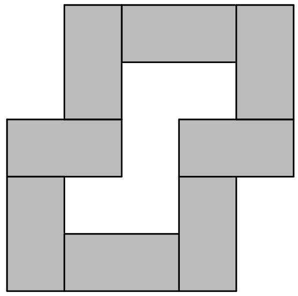

Figure 7.4

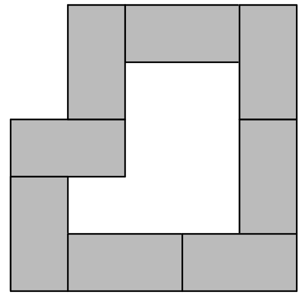

Figure 7.5

Notice that the 4-domino fence of Figure 7.1 encloses an area of 1, the 5-domino fence of figure 7.2 encloses an area of 2, and the 6-domino fence of figure 7.3 encloses an area of 3. Do you see a pattern linking the enclosed area with the domino length of the fence?

A young student might notice that in Figures 7.1, 7.2, and 7.3, we add one domino to increase the enclosed area by 1. Repeating this "add one domino" approach, if we add 4 dominoes in a correct manner to Figure 7.1, the enclosed area is increased by 4. If we add 6 dominoes, the area is increased by 6. Such a student may illustrate a systematic way to add one domino at a time in this process. The student will be satisfied with this type of demonstration/argument. Notice that the way the "pattern" in which the dominoes are put together is different for even and odd numbers of dominoes.

Is this a proof? What would make it into a proof? A student with some knowledge of algebra may express this pattern using algebraic notation. Would this be helpful in making a proof? What level of Piagetian cognitive development, math content knowledge, and math maturity are needed to deal with an infinite sequence of positive integers 1, 2, 3, 4, 5, and so on? The point is, with a simple set of manipulatives we can span a wide range of math content and math maturity.

> Using algebraic notation:
>
> Area = number of dominoes - 3.
>
> $A = n - 3$.
>
> But how do you prove this? Proof must take into account that the dominoes are put together differently for an even number of dominoes and an odd number of dominoes.

Following the rectangular pattern example illustrated by Figures 7.1 to 7.3, 8 dominoes enclose an area of 5. Is it possible to use 8 dominos that do not form a rectangle but that enclose an area of 5? Is it possible to use 8 dominoes and enclose a larger area? These are examples of posing a math problem. Problem posing is a very important component of math. An increasing level of skill in posing challenging and interesting math problems is a good indicator of an increasing level of math maturity.

To explore this new question, we might try to make a rectangle that is wider and not so long. Or. We might not restrict ourselves to a rectangular-shaped figure. Take a look at Figures 7.4 and 7.5. Each is made from 8 dominoes. The first has an enclosed area of 5 and the second has an enclosed area of 6. We now have three different fences made from 8 dominoes, with enclosed areas of 4, 5, and 6 respectively. Is it possible to enclose a still larger area by an 8-domino fence?

Let's examine Figure 7.1 in more detail. In terms of domino geometry, what we have is a fence that is four dominoes in length. The outer edge of the domino fence and the inner edge of the domino fence are both squares. What is the next larger square (outer edge of the fence is a square and inner edge of the fence is a square) that one can make from dominoes? As one makes still longer square fences using dominoes, do you see a pattern in the number of dominoes needed for each successive square fence? Can you prove your conjectured pattern is correct for all sizes of square domino fences?

Simple manipulatives can create an environment in which a tutee can deal with challenging math problems. In the examples given above, the problems can be attacked in a quite concrete manner, and they can also be attacked using ideas from algebra. The environment is also a good one for posing problems (making conjectures). As a more general statement, math manipulatives are a powerful aid to teaching and learning math for students of widely varying math content knowledge and math maturity.

Final Remarks

Math maturity does not have a precise definition, and a person's level of math maturity cannot be precisely measured. However, we know that with proper math education and related educational experience, a student can grow in both math content knowledge and math maturity.

Math tutoring provides an excellent environment for a tutor to gain increased insight into math maturity. A session's tutoring plan can be (should be, we think) designed both to help a tutee gain increase math content knowledge and skills, and also to gain in math maturity.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. Consider a tutee you are well qualified to tutor. How would you explain to this tutee why the teacher requires students to "show your work"? Answer the same question for a tutee's parents.
3. How would you explain to a tutee what a math proof is? Answer the same question for a tutee's parents.
4. Assess your level of mathematical intuition. If possible, give examples in which you had a "feeing" or intuition that something you were doing in math was correct or incorrect, and whether your intuition proved to be correct or incorrect.

Chapter 8

Math Habits of Mind

> When the mind is thinking it is talking to itself. (Plato; Classical Greek philosopher, mathematician, writer of philosophical dialogues, and founder of the Academy in Athens, the first institution of higher learning in the western world; 428/427 BC– 348/347 BC.)

> The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill. To raise new questions, new possibilities, to regard old problems from a new angle, requires creative imagination and marks real advances..... (Albert Einstein; German-born theoretical physicist and 1921 Nobel Prize winner; 1879–1955.)

The previous chapter included Math Habits of Mind as a component of math maturity. The history of education is replete with discussions about learning content versus learning to think—that is, to develop and make routine use of good Habits of Mind. During the past few decades, Arthur Costa and Bena Kallick have emerged as world-class experts and leaders in a movement to incorporate Habits of Mind into the everyday school curriculum. Their Habits of Mind Website (Costa and Kallick, n.d.) lists 16 transdisciplinary Habits of Mind.

This chapter is about the general topic of Habits of Mind. The bulk of the chapter examines a list of 16 Habits of Mind identified by Costa and Kallick. Each is transdisciplinary—that is, applicable in many different disciplines. For each, we examine transfer of learning to math and math education.

An alternative way to study Habits of Mind is to select a specific discipline (such as math) and explore its Habits of Mind. There are many such lists that can be found via Web search. One can then do transfer of learning from the discipline-specific Habits of Mind to other disciplines or as part of the effort to develop a comprehensive transdisciplinary list of Habits of Mind.

Near the end of this chapter we also suggest some transdisciplinary habits of mind that are not in the Costa-Kallick list. Computational Thinking—which is very important to math—provides a good example. Computational Math is now a well-established component of the discipline of math.

Tutoring Scenario

I (Dave Moursund) was once a reasonably successful math researcher. After completing my math doctorate at the University of Wisconsin-Madison, I was hired as an assistant professor in Mathematics and the Computing Center at Michigan State University. During my four years there, three of my students competed doctorates in math and I published enough math research papers to be promoted to the rank of Associate Professor. I also co-authored a Numerical Analysis book that was published by McGraw-Hill.

As my children grew, I helped to create a math and computer-oriented home environment. Gradually I moved out of my "being a mathematician" career into mainly being a math educator and computer educator. I had no trouble helping my children with their math homework at the precollege level.

However, eventually my older daughter reached college and began to take the calculus and more advanced math courses required of computer science majors. Increasingly, as she came to me for help, I found that I could not solve her math homework problems "off the top of my head." While I had considerable math maturity and good math Habits of Mind, I had forgotten key details of the math content. Thus, these homework help sessions became situations in which we worked together and learned together. They allowed be to illustrate math Habits of Mind and that one forgets details of math that they are not using.

My message to math tutors, teachers, and students: "Use it or lost it!" A general message to math tutors: "Many of your tutees don't use the math they have covered in their coursework enough to avoid losing it."

Costa and Kallick's List of 16 Habits of Mind

The table given below contains the Arthur Costa and Bena Kallick (n.d.) collection of 16 habits of mind. These are general-purpose, interdisciplinary habits of mind they define as follows:

> A "Habit of Mind" means having a disposition toward behaving intelligently when confronted with problems, the answers to which are not immediately known. When humans experience dichotomies, are confused by dilemmas, or come face to face with uncertainties, our most effective actions require drawing forth certain patterns of intellectual behavior. When we draw upon these intellectual resources, the results that are produced through [their use] are more powerful, of higher quality and greater significance than if we fail to employ those patterns of intellectual behaviors.

The left side of the table given below includes a number of quotes from Costa and Kallick (n.d.). That reference is abbreviated (C/K). The right side of the table provides brief analyses and comments from a tutoring in math education point of view.

Habit of Mind	Applications in Math Tutoring
1. Persisting. Stick to it through task completion. Remain focused—keep your eye on the ball. Try alternative approaches when you are stuck. Don't give up easily.	This is one of the key ideas in math problem solving. ADD and ADHD students have special difficulties in this area. A great many other math students have not learned the need for persistence in dealing with challenging math problems. However, be aware that not all math problems are solvable, and that others are beyond a student's current capabilities. One aspect of learning math problem solving

	is to develop insight into when to temporarily or permanently give up. Of course, if a math researcher gives up too early, then important discoveries are not made. Examples: The equation $2x - 3 = 0$ is unsolvable in the domain of integers, but is solvable in the domain of rational numbers. The equation $x^2 - 2 = 0$ is unsolvable in the domain of rational numbers, but is solvable in the domain of real numbers.
2. Managing impulsivity. Think before you act, and consider the consequences of your actions before taking the actions. Remain calm, thoughtful, and deliberate. Don't be driven by a need for instant gratification; with practice, one can learn to control this impulse.	This habit of mind is applicable both in interacting with other people and in carrying out tasks such as problem solving. In math problem solving, one has a goal in mind. Learn to mentally consider various approaches to achieving the goal. Learn to analyze whether the steps one is taking or considering taking will actually contribute toward achieving the goal. Students who are driven by the need for instant gratification seem to have trouble in their math studies when they reach algebra. See http://i-a-e.org/newsletters/IAE-Newsletter-2009-24.html.)
3. Listening with understanding and empathy. It is difficult to listen with empathy and with a goal of achieving understanding. "Some psychologists believe that the ability to listen to another person, to empathize with, and to understand their point of view is one of the highest forms of intelligent behavior." (C/K)	A math tutor needs to learn to promote two-way conversations with tutees, and needs to learn to listen with understanding and empathy. A tutee's math learning problems often extend far beyond the math classroom and the math content being taught in the classroom. It is common to start a tutoring session with the question, "How's it going with you?" You are seeking "deeper" insight into the trials and tribulations in your tutee's life that might be affecting math-learning performance.
4. Thinking flexibly. Don't let your thinking and approaches to dealing with the world get stuck in a rut. Entertain new ideas; learn to change perspectives, generate alternatives, consider a broad range of options, and understand other's points of view.	It is interesting to see students who are studying algebra struggle with problems that are readily solved without the use of algebra. They get stuck in an algebra rut. Descarte's development of analytic geometry through combining ideas from Euclidean geometry and algebra represents a major change in perspective. Folk math (Maier, n.d.) is math developed by people to solve the math problems and accomplish the math-related tasks they encounter in day-to-day life. Their approaches are often far different than use of school math.
5. Thinking about your thinking (metacognition). You have the ability to think about what you know and what you don't	In any discipline, metacognition is a powerful aid to cognitive development. Math education stresses the idea of "showing your work." What they are really trying to get at is having students understand and explain their thinking

know. You have the ability to think about your thought processes. You can develop increased skill in being aware of your own thoughts, strategies, feelings and actions, and their effects on others.	as they solve math-related problems and accomplish-related tasks. This is a useful activity. However, be aware that students (especially TAG students) can make math intuitive leaps that are beyond their power of explanation. Much research consists of working to explain and understand a "hunch" or "intuitive leap."
6. Striving for accuracy and precision. Routinely check your work and thought processes. "To be craftsman like means knowing that one can continually perfect one's craft by working to attain the highest possible standards, and pursue ongoing learning in order to bring a laser like focus of energies to task accomplishment." (C/K.)	One of the great strengths of the discipline of math is the careful statement of problems and tasks, and providing details of solving the problems and accomplishing the tasks. This is done in a manner that allows the steady accumulation of math knowledge and skills so that others can build on them with confidence. In math education, we stress: "check your answers." What we are really trying to teach is the concept that math can be done with a level of correctness that both the doer and others can act on and build on the results with confidence.
7. Questioning and problem posing. "Effective problem solvers know how to ask questions to fill in the gaps between what they know and what they don't know." (C/K) Effective questioners are inclined to ask a range of questions such as: "How do you know that?" What evidence do you have?" Problem posing (asking relevant, interesting, challenging questions) is an important habit of mind.	Problem posing and problem solving lie at the heart of math and many other disciplines. Math is a powerful aid to representing and helping to solve problems in many disciplines. Thus, one aspect of good math instruction is helping students do the transfer of learning required to make effective use of their math learning in new settings. A tutoring environment can provide the time and individualization needed to help make math useful and relevant to a tutee's life outside of the math classroom. Math tutoring also provides an environment in which a tutee can learn to ask meaningful math related questions far beyond the traditional one of: "Is my answer right?"
8. Applying past knowledge to new situations. (C/K) stress learning to make effective use of what one has learned. Know what you know, and learn to apply it in new, different, challenging problem situations. From your author's point of view, Costa and Kallick fail to emphasize applying the accumulated knowledge and skills of others, such as one can readily do making use of the Web. They do not emphasize Computational Thinking (IAE-pedia, n.d.) as a desirable habit of mind. See the next main	One of the most important ideas in math problem solving is building on the previous work of oneself and others. This "building on" is a type of transfer of learning that has been done by one's self and by others. One gets better at it by long and concerted effort to solve challenging math problems. Our current math education system is weak in helping students learn to read and understand math so they can make effective use of the accumulated math that can be assessed on the Web and through other library resources. A tutor with network access can work on this aspect of math education as "targets of opportunity" arise in tutoring sessions. Our math education system also is weak in helping students to understand that the use of tools such as calculators and computers is a way of storing, using, and accessing accumulated math knowledge and skills.

section of this current chapter.	Remember, a computer is both a way to access math knowledge and a way to automate a wide range of math procedures.
9. Thinking and communicating with clarity and precision. "Language and thinking are closely entwined. Like either side of a coin, they are inseparable. When you hear fuzzy language, it is a reflection of fuzzy thinking." (C/K) Be clear! Strive for accurate oral and written communication.	The written and oral language of mathematics allows of precise communication in the discipline of mathematics. However, it takes a great deal of study and practice to learn to precisely communicate in this language. A tutoring environment allows an ongoing oral and written math communication between a "native math speaker"—in this case, a well-qualified math tutor—and a novice in the field. See "Communicating in the language of mathematics" available at (IAE-pedia, n.d.). Writing using the language of math should begin early elementary and be given a high priority every year. (See http://www.pbs.org/teacherline/courses/rdla230/docs/session_1_burns.pdf.)
10. Gather data through all senses. Pay attention to—be alert to, be actively involved in—the world around you and the steady stream of input provided through your senses.	Math word problems are a standard part of the math curriculum. Many good math word problems are embedded in the world outside of mathematics. They challenge a student to draw on his or her world knowledge and understanding both in representing the problem mathematically and in checking to see if the results of the math problem-solving work make sense. See "Word problems in math" (IAE-pedia, n.d.).
11. Creating, imagining, and innovating. "All human beings have the capacity to generate novel, original, clever or ingenious products, solutions, and techniques—if that capacity is developed." (C/K) The challenging problems and tasks in each discipline "call for" creativity. Creative people are often uncomfortable with the status quo.	Robert Sternberg's three-component of definition of multiple intelligences includes creativity as one of the three components. (See http://en.wikipedia.org/wiki/Triarchic_theory_of_intelligence.) Humans have considerable innate creativity, and this creativity can be fostered through instruction, encouragement, and use. In math, there are many different ways to solve a problem and/or make a proof. Posing math problems, conjecturing math theorems, and solving math problems, and proving math theorems provide outlets for mathematical creativity. And, of course, tutoring and teaching provide great opportunities for creativity.
12. Responding with wonderment and awe. Quoting Albert Einstein, "The most beautiful experience in the world is the experience of the mysterious." A dedicated lifelong learner follows a pathway strewn with new experiences, excitement, beauty, wonder, awe, and enjoyment.	Some people who study math come to appreciate the excitement, beauty, wonder, and awe of the discipline. Alas, most don't! One might point to issues of nature versus nurture, and argue that those who are not endowed by nature with great natural math ability cannot come to appreciate math as a human endeavor that is full of fun, creativity, excitement, beauty, and so on. Others argue that our informal and formal math education system (the "nurture component of

	the puzzle") is responsible for so many people growing up claiming they hate math and could never learn to do math. A good math tutor directly addresses these issues and helps tutees personally experience some of the wonder and awe of math.
13. Taking responsible risks. Be flexible and venturesome. "Flexible people seem to have an almost uncontrollable urge to go beyond established limits. They are uneasy about comfort; they 'live on the edge of their competence'. They seem compelled to place themselves in situations where they do not know what the outcome will be." (C/K) Quoting Bobby Jindal from the (C/K) document: "The only way to succeed is to be brave enough to risk failure." Quoting (C/K): "Some students hold back [participating in] games, new learning, and new friendships because their fear of failure is far greater than their experience of venture or adventure. They are reinforced by the mental voice that says, 'if you don't try it, you won't be wrong' or "'f you try it and you are wrong, you will look stupid'".	Our math education system teaches students to believe that math problems have "right or wrong" (correct or incorrect) answers and solution processes. A student of math frequently fails in attempts to solve a specific problem or accomplish a specific task. Such mistakes and/or failures are an essential component of a good learning environment. They provide a basis for self-analysis, more learning, and persistence. However, they can also "beat down" a student. Rather than failure and mistakes being a useful learning experience, they become something to be avoided. One way to avoid failure and mistakes is to withdraw—don't participate. A slightly different result is a decision to participate at a minimal level in a manner that reduces one's risks. In a tutoring environment, a tutee is expected to show and explain his or her work. A good tutor uses this situation to help a tutee learn to learn from errors in logic/analysis and errors in computation/symbol manipulation. The tutor can help the tutee fill in gaps in math knowledge and skill that come from forgetting or from not learning needed prerequisites. The following quotes capture important ideas: "I have not failed. I've just found 10,000 ways that won't work." (Thomas Edison) "In the book of life, the answers aren't in the back." (Charles Schulz; American cartoonist best known worldwide for his Peanuts comic strip; the quoted statement is from the comic strip character Charlie Brown.) "What is your radius of exploration? How far from your comfort center are you willing to wander? As you meander and encounter wonders, how many possibilities do you squander because of baggage you carry? Travel light – carry a few essentials. Make room in your backpack for new experiences, new tools, new toys." (Laran Stardrake)
14. Finding humor.	There are, of course, many mathematical jokes. A recent

"[Laughter] has been found to liberate creativity and provoke such higher level thinking skills as anticipation, finding novel relationships, visual imagery, and making analogies. People who engage in the mystery of humor have the ability to perceive situations from an original and often interesting vantage point." (C/K) Lighten up. Look for and enjoy humor. Be able to laugh at one's self.	Google search of *math jokes* produced about 1.7 million hits. Many require some insight into math in order to be humorous. For example, what makes the following math joke funny? Teacher: "Who can tell me what 8 times 7 is?" Student: "It's 56!" Teacher: "Very good! And who can tell me what 7 times 8 is?" Same student: "It's 65!" Humor is an important component of education. (See "Using humor to maximize learning" at http://iae-pedia.org/Using_Humor_to_Maximize_Learning. A math tutor might want to make a collection of math jokes suitable to the math level and interests of his or her tutees. These could be "doled out" one or two per session to help lighten up the tutoring sessions.
15. Thinking interdependently. Think in terms of collaboration in work and learning. "Human beings are social beings. We congregate in groups, find it therapeutic to be listened to, draw energy from one another, and seek reciprocity." (C/K) In work, play, and learning, engage with others, depend on others, and be dependent on others.	Many people view learning and doing math as solitary activities. They view math-oriented people as somehow a different (and somewhat peculiar) breed. That is a misconception. The (C/K) quote to the left certainly applies to mathematicians. Like specialists in other academic disciplines, they like to interact with their peers, and they often work together to learn and to do their research. Math project-based learning and team-based problem-based learning are useful approaches to engage students as social beings who are learning and doing math together (PBL, n.d.). In math tutoring, a 3-component team (human tutor, computer tutor, and tutee) brings together and draws on each member's unique capabilities in addressing a math teaching/learning problem.
16. Learning continuously. It is the nature of a human brain that is it continually engaged in learning and in processing what it has learned. We are all lifelong learners. Some of us develop a habit of mind that focuses our learning capabilities toward gaining an increasing level of knowledge, skill, and expertise in areas that serve our individual interests and needs, and help us to better serve needs of others. In our lifelong quest for knowledge and wisdom, we each in our own way "boldly go where no	Math is a huge and vibrant discipline, with great breadth and depth. The world we live in is alive with math-related problems, tasks, and challenges. Thus, in our everyday lives we routinely encounter opportunities to see math in use, to use math, and to learn math. Successful math education helps students learn to view the world through "math-colored" glasses and to effectively deal with the math-related problems and tasks that they encounter in their everyday lives. It helps them to develop math-related habits of mind. It starts them on a livelong pathway sets them on a path of using, maintaining, and strengthening these habits of mind. A good math tutor role models such behavior and helps his or her tutees to develop such behaviors. A math tutor might want to make such transfer of learning from the

one has gone before."	math classroom to the rest of the world be an integral component of tutoring sessions.

Computational Thinking as a Habit of Mind

> "Computers are incredibly fast, accurate, and stupid. Human beings are incredibly slow, inaccurate, and brilliant. Together they are powerful beyond imagination." (Albert Einstein; German and American theoretical physicist and Nobel Prize winner; 1879–1955.)

This section is your authors' recommendation for an addition to the Costa and Kallick list of 16 Habits of Mind. Moursund's 7/5/2011 email exchange with Arthur Costa indicates his support in making such an addition.

Integrating use of Human and Computer Brains

Integrating use of human and computer brains. A human brain and a computer "brain" have overlapping capabilities. Human brains are much better than computer brains in some areas, while computer brains are much better than human brains in other areas. The quote from Einstein captures this key idea. More recently, the term Computational Thinking (IAE-pedia, n.d.) has come into common use as a transdisciplinary important approach to problem solving. Quoting Jeannette Wing from the Computational Thinking document:

> Computational thinking builds on the power and limits of computing processes, whether they are executed by a human or by a machine. Computational methods and models give us the courage to solve problems and design systems that no one of us would be capable of tackling alone. Computational thinking confronts the riddle of machine intelligence: What can humans do better than computers, and what can computers do better than humans?

Genetically, human brains are not changing very rapidly. However, our cognitive capabilities have been greatly increased by the development of aids such as reading and writing, libraries, and the STEM disciplines of science, technology, engineering, and mathematics. The combined artificial intelligence (machine intelligence) and "brute force" power of computer brains continues to grow very rapidly.

Here are three key aspects of computer capabilities to keep in mind:

1. The size and capabilities of electronic digital libraries such as the Web are growing very rapidly. Such libraries help support a "look it up" approach to education, solving problems, and accomplishing tasks.
2. Computer systems can solve or greatly help in solving many of the problems that students learn about in school. (Consider a parallel between machines used to automate physical tasks and machines used to automate mental tasks.)
3. Telecommunications systems facilitate widely dispersed teams of people and computers to work together.

Computational Thinking and the growing capabilities of computer systems present a major challenge to our educational system. If a computer can solve or greatly help in solving the problems addressed in a component of our school curriculum, how should this affect the content being taught, the teaching methods being used, and the assessment?

> Computational Thinking and the growing capabilities of computer systems present a major challenge to our educational system. How do we teach **tool selection**? How do we teach students how and when to use a computer, and when not to use a computer?

A related part of this question is what role computers should play in teaching. Marshall McLuhan's statement that "The medium is the message" is especially poignant here. If we are using computers to help teach the curriculum, and if computers are a powerful aid to representing and helping to solve the problems in that discipline, then it makes sense to thoroughly integrate the medium (computer) and the message (solve the problems).

Applications in Math Tutoring: We have previously emphasized the idea of a human tutor, computer tutor, and tutee constituting a team working together in a one-on-one or small-group tutoring environment. In this environment, a tutee can gain experience in working with both a human and a computer in learning and in demonstrating learning.

Math education struggles with the challenge of preparing students for:

- The rest of their current math course.
- Their next math course.
- Knowing and understanding the math they will need in their future non-math courses.
- Relearning needed math they have forgotten and to deal with new math-learning challenges they will face throughout their lives.
- Using math in their current and future work, play, and everyday lives.

Computer technology can play an important role in all of these. All of these uses of computer technology can be integrated into the activities of a tutoring team.

Think Globally, Act Locally as a Habit of Mind

You are familiar with the statement, "Think globally, and act locally." This is a way to talk about sustainability—maintaining Earth as a good pace for humans and others to live.

Here is a definition of sustainability quoted from http://www.arch.wsu.edu/09%20publications/sustain/defnsust.htm:

> Sustainability embodies "stewardship" and "design with nature," well established goals of the design professions and "carrying capacity," a highly developed modeling technique used by scientists and planners.
>
> The most popular definition of sustainability can be traced to a 1987 United Nations conference. It defined sustainable developments as those that "**meet present needs without compromising the ability of future generations to meet their needs**"(WECD, 1987). Robert Gillman, editor of *In Context* magazine,

> extends this goal oriented definition by stating "sustainability refers to a very old and simple concept (The Golden Rule)...**do onto future generations as you would have them do onto you**." [Bold added for emphasis.]

The discipline of sustainability draws on the work in many different disciplines. From math it draws on mathematical modeling and projecting into the future. Such modeling and projections routinely makes use of Computational Thinking. This requires careful definitions and careful measurements.

Children can learn about recycling and incorporate this idea into their everyday lives. They can learn about the "carbon footprint" of themselves and their families. A recent Web search of *mathematics of carbon footprint* returned more than a million hits.

Tutoring Tips, Ideas, and Suggestions

The Website http://math4teaching.com/2010/01/27/developing-mathematical-habits-of-mind/ provides the definition:

> "A habit is any activity that is so well established that it occurs without thought on the part of the individual."

The Website also lists a number of math Habits of Mind. The key point made by the short article is:

> Learning mathematics is not just about knowing, understanding, and applying its concepts, principles and all the associated mathematical procedures and algorithms. It's not just even about acquiring the capacity to solve problem, to reason, and to communicate. It is about making these capacities part of students' thinking habits. It is only then that one can be said to be mathematically literate.

Here is a five-step plan.

1. Begin by talking with your tutee about Habits of Mind. Facilitate this conversation so that your tutee recognizes and talks about two or more of his or her general habits of mind that are applicable in a variety of situations. If your tutee can't think of any, make a suggestion that you think is appropriate. For example, in reading, if one comes to a word that they do not understand, a good habit of mind consists of first trying to figure out the meaning from the context of the document, and if that fails, looking up the word in a dictionary.
2. Select one of the Habits of Mind suggested by your tutee that seems broadly applicable. Explore with your tutee how it serves his or her needs in dealing with problems and tasks that require thinking. Make sure that there is considerable focus on non-math examples.
3. Then you and your tutee work together to do a transfer of learning from this general Habit of Mind to studying math. Help your tutee find examples in which he or she has used this math Habit of Mind and/or would have benefited by using it.

4. Suggest to your tutee that he or she look for more examples before the next tutoring session.
5. At the next tutoring session, check for results. (Do not spend more than five minutes on this activity.) Whether or not your tutee has anything to report, using the previously selected math Habit of Mind, go back to the second sentence in (2) above and continue down through the steps. Do this at each tutoring session until your tutee has considerably strengthened this math Habit of Mind and can recognize when he or she is using it in learning and doing math.

   When engaged in step 5, be open to the idea of your tutee selecting a different math Habit of Mind to investigate. Consider making a change if little or no progress is occurring or if the current math Habit of Mind being explored seems to have been mastered.

Final Remarks

Math content and math maturity are thoroughly intertwined. Math Habits of Mind are an important component of math maturity.

However, Math Habits of Mind can be thought of as math interpretations of general transdisciplinary Habits of Mind. Thus, in teaching and learning Habits of Mind, we want students to fluently move from general Habits of Mind to Math Habits of Mind, and vice versa. A math-tutoring environment is great for practicing such transfers of learning.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. From the (C/R) list, name two or three of your best math Habits of Mind. Name two or three of your relatively weak math Habits of Mind. Think back over your math informal and formal education up to the current time. What helped you to gain your strongest math Habits of Mind, and what helps explain those you listed as your relatively weak math Habits of Mind?
3. Analyze your current strengths and weakness in the math Habit of Mind named "Integrating use of Human and Computer Brains."

Chapter 9

Tutoring "to the Test"

"Chance favors only the prepared mind." (Louis Pasteur; French chemist and microbiologist; 1822–1895.) "Math is a game, and assessment in math education is a game. In today's math education system, students need to learn to play both games." (Dave Moursund and Bob Albrecht.)

Here are two different views of goals of math education:

1. **Top-down point of view.** We (the country's leadership) want students to learn math in a manner that helps serve their current and future needs as well as the needs of employers and the nation. We want to define and enforce high standards across the nation. One way to measure student math knowledge and skills is by use of widely used "high stakes" assessment instruments that allow comparisons with other schools and school districts across a state and across the nation.
2. **Some students' points of view.** Math education has some game-like characteristics. Playing the game well means getting a good grade on an up-coming in-class exam, passing a course with a good grade, and scoring well on high stakes tests such as state tests and college entrance exams. Many students want to play the game well enough to meet requirements that are placed on them by parents, teachers, schools, higher education, potential employers, and so on. Some students find the game so difficult and/or discouraging that they essentially give up.

Math tests are a well-engrained component of math education. Many schools provide tutoring especially designed to help improve student scores on high stakes tests. From an individual student's point of view, the goal might be to score a little bit higher in order to pass a course or meet a graduation requirement. To a school, the goal might be to have enough students pass a state test so that the school is able to show annual progress and/or "look better" than some of the other schools in the district, state, or nation.

Thus, math tutors often face the challenge of tutoring in a manner that will help tutees score better on math tests. This chapter discusses this aspect of tutoring.

This chapter discusses three topics:

1. Teaching to the test.
2. Testing to the teaching, using tests created by the teacher or textbook publisher.
3. High stakes external testing. (District, state, national, and college entrance tests.)

Tutoring Scenario of Tutoring to the Test

In the school where Bob Albrecht tutors, some students receive special tutoring designed to help them score higher on the state test. In these tutoring sessions, Bob works with a small group of one or two tutees. In the situation described below, Bob worked with a total of 17 tutees, with no more than two at one time. Most were high school sophomores and juniors. Sample questions from previous tests were used in these tutoring sessions.

Bob makes use of the following scale to help describe student responses to various math questions.

1. Abracadabra! Alakazam! Solved with no help or almost no help. Oh happy day!
2. Solved with here a nudge, there a hint.
3. Solved with much ado and heaps of help.
4. Alas, alack, and oh heck. Led through every step of the solution and still boggled.

Here are three sample test questions. Student average scores (using the above scale) are included. When the average score is less than 1.0, it is shown in **red**.

01. Average score = 1.8 (17 students) No student noticed that $(1/2)^4$ is less than 1 and therefore can be ignored. I had to show almost everyone how to use a TI-84 to calculate $(1/2)^4$. Some students asked where the fraction key was. How sad.	Which number has the greatest value? A. $\left(\frac{1}{2}\right)^4$ B. $\sqrt{5}$ C. $\frac{7}{3}$ D. 2.324

02. Average score = **0.6** (17 students) Most did not know how to calculate percent. Students who knew how to calculate percent calculated 60% of 800 and selected A. 480. I had to point out "60% greater than …" and explain what it meant.	Given a choice of beverage, the number of students who prefer cola is 60% greater than the number of students who prefer milk. If 800 students prefer milk, how many students prefer cola? A. 480 B. 1120 C. 1280 D. 2000

03. Average score = 0.8 (17 students) Most selected **A. two times as great**. No one drew diagrams. I drew diagrams of a 4 by 2 rectangle and an 8 by 4 rectangle with dashed lines showing four 4 by 2 rectangles inside the 8 by 4 rectangle. Some still looked puzzled.	If the length and width of a rectangle are doubled, what is the effect on the area? It becomes A. two times as great. B. three times as great. C. four times as great. D. eight times as great.

Comments About The Tutoring Scenario

Over time, students forget most of the math that they cover in the courses they take. Teachers and curriculum developers know this, so curriculum includes a great deal of review. Nowadays, it is common for math classroom teachers to also spend considerable time teaching to upcoming high stakes tests. Thus, the tutoring scenario could just as well have occurred in a regular math classroom.

Teaching or tutoring using sample test questions can have several different purposes. For example:

1. Acclimate students to the types of questions and the testing environment that they will encounter. This helps them to get used to the time pressure, the tools they will have available (pencil, paper, graph paper, calculator, formulas) and how they will indicate their answers (such as a mark sense form or on a computer), and how they can go back and make corrections.
2. Help students relearn content and problem-solving strategies that they have covered in the past.
3. Help students gain increased skill in transferring their math knowledge and skills into a good performance in a high stakes testing situation.
4. Help students learn to play the game of taking multiple choice math tests. This includes learning how to deal with questions that may be poorly stated.

Teaching to the Test

We all have our own opinions about teaching to the test (Johnson, 6/2/2011). In situations in which educational or training goals are very clear, it is possible to develop assessment instruments that are very good measures of how well students will do when they need to use their newly acquired knowledge and skills in "real world" settings. In essence, the content being taught, the methods of teaching, and the assessment can be "authentic" in that they are all closely aligned with performance outside of the unit or course of instruction. Here, "outside" may mean performance in future courses, performance in competitive events, performance on the job or in future jobs, performance as a responsible adult citizen, performance as a responsible parent, and so on.

Under such circumstances, teaching to the test means teaching to increase performance on authentic measures of well defined and agreed upon assessments. For example, suppose that we are training airplane pilots or astronauts. Computerized simulations have been developed that are so authentic that performance in such simulators is indeed a good measure of future performance in flying an airplane or spaceship. There are many training and education situations in which simulators or simulated situations can be created that are sufficiently authentic that assessment in the simulated situations is a very good predictor of performance in the corresponding "real world" situation.

> Simulation is the imitation of some real thing, state of affairs, or process (Wikipedia). In a good simulation, the student has a sense of "being there," interacting, doing things, solving problems, et cetera. The student is engaged and continually interacting with the simulated environment.

Education and training for athletic performance provides many fine examples. For example, four sprinters train individually and as a group for the 400-meter relay. Each time they do a practice team run, they are doing a simulation of being in an actual track meet. If the track team has enough sprinters, it can have two relay teams compete against each other. This is closer to actually competing in a track meet.

However, good coaching is much more complex than the above example suggests. Suppose that I am a basketball coach. I want individual players to become better players and I want the team to become a better team. I also want individual players and the team to have a high level of sportsmanship and to conduct their daily lives in a manner that makes their peers, school, town, and so on proud.

I can assess how well a player shoots foul shots in practice. How do I create a final exam on free throw shooting that captures the pressure of performance in a game setting? Is my assessment of how high a player can jump a good measure of how high and well a player will jump when trying to capture rebounds or block a shot?

In essence these are transfer of learning types of questions. I want the teaching (coaching, tutoring) to be authentic to the performance measures.

There is a large and growing "business" of teaching to the test. This business goes on in many different countries. Very roughly speaking it consists of:

1. Helping acclimate students to the test-taking environment. If a computer is administering a test, it is helpful to have practiced taking tests administered by a computer.
2. Helping students review content they have previously studied and/or that the test developers assume has been in the student curriculum.
3. Teaching students techniques that lead to higher test scores and that have very little focus on learning and understanding the content being tested. (This has come to be called, "gaming the test.") A number of companies include this type of instruction in their test preparation course.

Teaching Estimation and Answer Checking

Math educators agree that it is important for students to learn to check their answers. One way to check an answer is via estimation. Another way is determine if the result "makes sense."

Lets look at the first example from the test questions given earlier.

Which number has the greatest value?

A. $\left(\frac{1}{2}\right)^4$

B. $\sqrt{5}$

C. $\frac{7}{3}$

D. 2.324

Listen to me as I (one of your authors) imagine "playing the game" of answering the question.

> The statement of the question suggests that there is only one correct answer. I glance at the four answers and notice that the first one is quite small relative to the others. So, the correct answer is B, C, or D. I carry around in my head that the square root of 5 is approximately 2.236. I quickly compare this with 7/3, which I know is 2.333…That eliminates answer B, and a tiny bit more thinking allows me so see it also eliminates answer D. In a modest number of seconds I conclude the answer is C, using only my brain as a calculator and information retrieval device.

If I did not remember an approximate value for the square root of 5, I could have used my calculator. A calculator provides an alternative to memory, mental calculation, and paper and pencil calculation.

Your authors are well aware that this type of thinking is way beyond most of the students that receive tutoring. For the most part, it is a type of math maturity that is not taught in their classes. The quickest way for them to find the answer is to use a calculator. Alas, some students did not know how to use their calculator to calculate $(1/2)^4$.

Now, listen to me as I (one of your authors) analyze the second test question.

Given a choice of beverage, the number of students who prefer cola is 60% greater than the number of students who prefer milk. If 800 students prefer milk, how many students prefer cola?

A. 480

B. 1120

C. 1280

D. 2000

> At first glance, my brain is slightly befuddled. More students prefer cola than prefer milk. Maybe the first possible answer will help un-befuddle my mind. Yes, 480 is too small, because it is not larger than 800. So, the answer must be B, C, or

D. I need to think more carefully about "how many more" students prefer cola. Yes, it is 60% more (than the 800 who prefer milk). Hmm. What is 60% larger than 800. Aha, that is easy enough. I can easily do 60% of 800 in my head, and add the result to 800 in my head. The answer is C.

Alternatively, I could just use an approximation that 60% is a little more than a half. So, the answer needs to be a little more than 800 plus half of 800. That eliminates answers B (too small) and D (way too large), and I am done.

In both of these examples, I am playing a type of test taking game. The techniques I use are a type of math maturity that a fifth or sixth grade student might well have developed given appropriate instruction and practice. And therein lies a major problem in our math education system. Students are not given appropriate instruction and practice of this type of analysis and thinking because it would require too much class time.

Finally, consider the 3rd sample test question.

If the length and width of a rectangle are doubled, what is the effect on the area? It becomes

A. two times as great.
B. three times as great.
C. four times as great.
D. eight times as great.

The statement of the question suggests that the answer is independent of any particular rectangle. So, if I draw a rectangle and then one that is twice as long and twice as wide, all I need to de is to compare the areas. This is particularly easy if I have graph paper or a ruler. I do not need to know a formula for the area of a rectangle.

Drawing a relatively precise diagram can be useful in solving a variety of multiple choice test questions. Contrast this question with the following questions that require "constructed" solutions.

A. Prove that if the length and width of a rectangle are doubled, the resulting rectangle has four times the area of the initial rectangle.

B. Prove that if the length and width of a rectangle are doubled, the resulting rectangle has twice the perimeter of the initial rectangle.

Multiple Choice Versus Constructed Response

Consider the following quote:

"In the book of life, the answers aren't in the back." (Charles M. Schulz; American cartoonist speaking through the voice of his comic strip character Charlie Brown; 1922–2000.)

Now, here is a slight change to that statement:

> "Life in our world is not a multiple choice test with 4 or 5 options per question, and usually only one being *the* correct answer." (Multiple authors.)

There has been considerable research on constructed response (also called extended response) assessment. The first chapter of Tankersley (2007) is available free on the Web and provides a good introduction to use of constructed responses in assessment. Here is the first paragraph of Karen Tankersley's book:

> Early in their school careers, students learn that the teacher has the "right" answers to questions asked in the classroom. Successful students learn that their "job" is to try to figure out that "right" answer and to provide it for the teacher. Students who are able to do this quickly and accurately are perceived as brighter and are rewarded with higher grades and more positive feedback. Students who have difficulty in perceiving the answer the teacher is seeking may well be viewed as less competent and are less tolerated. **In far too many classrooms, teachers do not require students to think deeply or move beyond the basic knowledge and comprehension level.** Even those students who are perceived as bright, capable learners are seldom asked questions like "How do you know?" or "How did you get that answer?" or "Why do you think so?" or "Show me proof that answer is correct." This lack of cognitive follow-through in our classrooms leads to shallow thinking and encourages students to simply try to guess what the teacher is thinking during instruction rather than really cognitively engage in deep thinking and learning. [Bold added for emphasis.]

The Website (Lujan and Lujan, n.d.) looks specifically at constructed responses in math assessment. The 1992 math assessment made use of a combination of multiple choice, short-response, and extended-response questions. Quoting from that Lujan and Lujan:

> The Office of Education Research and Improvement (OERI) summarized the gathered NAEP data as follows. **Approximately one third to two-thirds of the students provided incorrect responses to extended-response questions indicating little evidence of understanding the mathematical concepts involved or the questions being asked.** Most students who appeared to understand the problems had difficulty explaining their work (ETS, 1994). The need for effective instruction and practice involving open-ended mathematical problems was evident. [Bold added for emphasis.]

Many countries outside the United States make little or no use of multiple-choice questions in their national assessments. In the United States there is an increasing movement toward having both constructed responses and multiple-choice questions on state and national math assessments. See, for example, http://cesa5mathscience.wikispaces.com/Constructed+Response.

Tutoring provides an environment in which it is feasible to routinely make use of extended response formative assessment.

Using a Calculator and Formulas

Mathematicians and many math teachers believe that students are becoming overly reliant on calculators. They observe students doing key presses without thinking. They are disturbed when a student uses a calculator to multiply or divide by 10. They are disturbed when a student makes a keying error and gets an answer that is "obviously" incorrect—but does not detect this error. They are disturbed when students make an error in their thinking and get a "really obviously" wrong answer—and do not detect that their answer is incorrect. They say, "Students lack number sense."

Here is a very rough "rule of thumb." In high stakes tests where a calculator is allowed, almost all of the problems can be solved without the use of a calculator and without use of a lot of paper and pencil calculation. (Of course, this requires good mental arithmetic and estimation skills as well as number sense.) If students are provided with a collection of formulas and a calculator, what does the test designer want to test? The test designer's goal is not to test memorization of formulas or skill in using a calculator. Rather, the test designer is trying to test the type of math thinking that is needed in math problem solving.

This suggests that students should become fluent users of a calculator and should develop understanding of how to make use of formulas. Memorization of formulas is not as important as learning to recognize problem situations in which common studied formulas are an aid in solving the problem. Part of "gaming" such high stakes tests is to know when formulas will be provided along with the test, and learn to recognize which formula or formulas will be helpful in solving a particular problem.

Students should develop number sense and estimation skills that help them to detect errors in: a) calculation, b) the logical steps and processes used in solving a problem, and c) actually carrying out steps in solving a problem. All of this should be a routine component of day-to-day instruction in math class. While it can be viewed as teaching to the test, it can also be viewed a very important aspect of teaching and learning math for use outside the world of math tests.

Tutoring Tips, Ideas, and Suggestions

Bob Albrecht's analysis of his tutees' performance on sample test questions indicates that even when provided with formulas, students often do not recognize which formula or formulas to use. When tutoring a student over a long period of time, consider the following idea.

Think of a formula as a concise, short math story. For example, $A = LW$ is a story about the three characters A, L, and W. These three characters appear in so many math stories, that we learn abbreviations for their names. A is a character named Area, L is a character named Length, and W is a character named Width. $A = LW$ is a shorthand version of Area equals Length times Width. The formula $P = 2L + 2W$ is about the three characters P (Perimeter), L (Length) and W (Width). $P = 2L + 2W$ is a shorthand version of Perimeter equals 2 times Length plus 2 times Width. Other often-appearing characters are r (Radius), d (Diameter), C (Circumference), and h (Height).

Each math formula (each short story) has a name. For example, the formula $C = 2\pi r$ is named "Circumference of a Circle." Two of its characters (C and r) can differ in size from one story (one math problem) to the next. However, π is a constant, and its value remains the same throughout all stories it appears in.

Working with the tutee, the textbook the tutee is using, and the standards that are being assessed by teacher-made and standardized tests, gradually accumulate a list of the most frequently used formulas your tutee is expected to recognize and be able to use. (The list can be short, because you are not trying to have your tutee memorize all formulas in the book! Rather, you are helping your tutee learn that a formula is a short story about a modest number of characters that appear is a variety of stories, and that the stories have names.)

As one part of each tutoring session, do one or both of the following:

1. Provide your tutee with one of the formulas. Ask the tutee to "think out loud" about what the formula means, what type of problem it helps solve, and possible difficulties in making use of the formula.
2. Provide a problem that the tutee "should" be able to solve and that requires a constructed response that draws on use of one or more of the formulas that have been collected. Let the tutee make use of the collected list of formulas. Observe your tutee's performance. As appropriate, intervene with questions and help.

Final Remarks

This chapter discusses teaching and tutoring to the test. In "the best of all possible worlds," instructional content, teaching processes, and assessment would be carefully aligned and would be authentic to major goals such as preparing students to appropriately deal with the math they are currently encountering both in and outside school, and the math they will encounter in the future. A good math tutor can help in this endeavor.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. Summarize some of your insights and personal experiences on the topic:

   Ways to "game" a multiple choice or True/False test—that is, ways to guess an answer with a better than pure chance outcomes.
3. Summarize some of your insights and personal experiences on the topics listed below. Do a compare and contrast of your answers.

   A. Ways you feel a student should be taught about how to study for a test.

   B. Ways to study for a test other than focusing on understanding.

   C. Ways to study for a test focusing on understanding.

Chapter 10

Peer Tutoring

"When you teach, you learn twice." (Seneca; Roman philosopher and advocate of cooperative learning; 4 BC–65 AD.)

"It takes a whole village to raise a child." (African Proverb.)

Students talk to each other about many different things. Sometimes a conversation is about courses, assignments, tests, and so on. In a math course, a brief conversation might be one in which a student is seeking help on a particular math problem.

These "conversations" are examples of informal peer tutoring. A tutee asks a tutor for information or help. The tutor responds in a cooperative, helpful manner.

However, there is much more to peer tutoring. As discussed in this chapter, peer tutoring can be a well-organized component of an educational program. Cooperative learning provides a well-researched and excellent example. In some schools certain students receive training on how to be effective in helping other students in a general-purpose "homework help" or more formal tutoring environment. In schools, cross-age tutoring, in which older students tutor younger students, has proven effective in benefiting both the tutors and the tutees. In home environments, older children often provide informal tutoring to younger children.

A good tutor understands the problems and tasks faced by tutees. The tutor empathizes with tutees and can draw on somewhat similar problems faced by other tutees. In a peer tutor/tutee arrangement, both might be taking the same class or the tutor might have taken the class recently. The peer tutor has more detailed knowledge about the class and the expectations being placed on students than does a non-peer tutor.

Many people who have done peer-tutoring comment on how much they learn in the process. Look back at the first quote at the beginning of this chapter. Similar comments come from teachers at all grade levels. A person learns a great deal through teaching a subject.

In summary, in peer tutoring we emphasize two things:

1. The tutor and tutee taking advantage of their shared leaning experiences and their understandings of challenges they have faced and are facing in their informal and formal educational systems.

2. The tutee and the tutor each gaining knowledge and experience through working together.

Tutoring Scenario—Play Together, Learn Together

Becoming a Better Math Tutor

As parents and grandparents, your authors have enjoyed watching their children and grandchildren learn from each other. Often this is in a "play together, learn together" environment. It often involves other children from the neighborhood.

It is fun to watch such play and learning. An occasional intervention can help increase the learning that is going on. Such interventions are not direct instruction. Rather, they are hints, suggestions, providing additional resources, and so on. For example, suppose that the children are playing with toy cars. With the aid of various building materials (such as paper, craft sticks, and tape) they might be led in the direction of building bridges for their cars to drive over. Paper and crayons might lead the children into creating some roadside scenery.

A roadway might be created. Hmm. Children might take turns moving their cars. This might be in a game-like "Mother, may I" type of car movements, or a game in which dice rolls are used to determining how far a player gets to move his or her car.

Over the years, one of your authors (Bob) has actively participated in developing play to learn together environments in school, after school and other environments. He has written extensively about his experiences and how successful they have been. See http://www.amazon.com/s/ref=nb_sb_noss?url=node%3D154606011&field-keywords=Bob+Albrecht&x=0&y=0.

Ordinary Peer-to-Peer Tutoring

One of your authors (Bob Albrecht) was asked to work with some secondary school students who do math peer tutoring. He noted this request to his co-author (Dave Moursund). In the short peer tutor tutoring time that Bob had available, he tried out the ideas that he thought best fitted the situation. The peer tutors were doing part of the "service" component of requirements to graduate from their high school.

Think about the challenge Bob faced. Suppose that you had an hour to help a high school student who is relatively well qualified in math to be ready to start tutoring other high school students in math. What would you want this student soon-to-be tutor to learn from you?

Here a few suggestions. (The four suggests that follow are written at about an 8.7 Flesch-Kincaid reading level. Feel free to copy this material and provide it to potential peer tutors.)

Being a Peer Tutor

1. Reassure the tutor in training that he or she knows a lot of math relative to the tutee. When the tutee asks for help on a math problem, here are four potential situations:

    A. You (the tutor) can solve the problem "off the top of your head." If not,

    B. You can likely use your tutee's math book to quickly review how to solve this type of problem. If not,

    C. Using your overall math knowledge and skills, and by talking over the problem with your tutee, you will be able to "figure out" how to solve the problem. If not,

D. You know who to ask for help and/or you know where and how to find help from other books or from the Web. It is all right to admit that you are unable to help your tutee on a particular problem.

2. Help your tutor-in training to learn the following short outline for tutoring:

   A. When asked for help on a problem or question, begin by finding our what is actually troubling your tutee. What does your tutee know about solving this type of problem and what has your tutee tried? This conversation with your tutee gives you time to refresh your memories about this type of problem and allows you to begin to formulate a plan of instruction and help. Also, you want to help your tutee to learn to think about these questions before asking for help. Quite likely you will find that your tutee has forgotten and/or never learned some of the key prerequisite math needed to solve the problem. This will lead you into reviewing needed prerequisites.

   B. Keep in mind that your tutee believes that the goal is to get the problem solved but that your goal is to help your tutee learn some math. Most of the focused help you will provide your tutee is to be on learning and understanding math. Your tutee expects that the problem will get solved, so you need to have that happen. However, **do not solve the problem for your tutee.** Insist that your tutee be actively engaged and make significant contributions toward getting the problem solved.

   C. If time permits after a problem is solved, do a little “debriefing” with your tutee. Ask your tutee to explain what he or she has learned in exploring the problem with your help.

3. Be professionally respectful and courteous to your tutee. Keep in mind that most of the information that passed between your and your tutee is confidential. Feel free to debrief your tutoring sessions with your immediate supervisor. However, avoid passing on information your tutee would feel is confidential unless you and your supervisor agree that this is quite important to do.

4. After you have finished your tutoring activities for the day, do a self-assessment and reflect about your experience. If possible, also debrief with your supervisor. What did you learn about the business of tutoring and about yourself as a tutor? What can you do to become a better tutor?

Cooperative Learning

Cooperative learning, in which small groups of students help each other to learn, can be thought of as a type of peer tutoring.

Roger T. and David W. Johnson are world leaders in cooperative learning. Their 1988 article summarizes some of the key ideas (Johnson, 1988). Quoting from their article:

> There are three basic ways students can interact with each other as they learn. They can *compete* to see who is "best"; they can work *individualistically* on their

> own toward a goal without paying attention to other students; or **they can work *cooperatively* with a vested interest in each other's learning as well as their own.** [Bold added for emphasis.]

When Johnson and Johnson wrote these words, the dominant mode of instruction in US schools was competition. In their 1988 article they say:

> The research indicates that a vast majority of students in the United States view school as a competitive enterprise where you try to do better than the other students. This competitive expectation is already fairly widespread when students enter school and grows stronger as they progress through school.

This situation still exists. Johnson and Johnson have long argued that this is a poor approach to education because it promotes the idea of winners and losers.

> **Brief Personal Story.** I (Dave Moursund) am reminded of when I was a child. I greatly enjoyed sports, and I participated with the neighborhood children in basketball, football, and softball. I had fun. I got better as I grew physically and mentally, and as I practiced a lot.
>
> However, I was never very good. A number of my peers were physically more gifted than me, and they got better at the sports faster than I did. In no sense did this make me a loser. I was an essential member of the teams. (We were not a heavily populated neighborhood. Every player counted!) By merely being on a team I helped increase the size of the team and I made contributions to both the team play and the play of the better team members. (You can visualize me in a softball game standing out in right field!)
>
> As I grew older, I realized I lacked the physical abilities to compete for a position on the high school sports teams. At the same time, I became aware that I had mental gifts and could be successful in some academic areas. Of course, that is another story ….

The goal in cooperative learning is to create environments in which the success of a team of learners depends on the active participation and learning success of each individual member. Here is an example. Students in cooperative learning groups work on learning a list of spelling words. On the weekly whole-class test, each team member gets a score that reflects both the level of success of the whole team and the level of success of the individual. In this situation it behooves the stronger spellers on the team to help the weaker spellers make good progress.

The Johnson and Johnson article contains a summary of the research literature. Here is a piece of their summary:

> With several colleagues, we recently did a meta-analysis on all the research studies that compare cooperation, competition and individualistic learning (122 studies from 1924 to 1980). The results indicated that cooperation seems to be much more powerful in producing achievement than the other interaction patterns and the results hold for several subject areas and a range of age groups from elementary school through adult.

Alfie Kohn is a strong proponent of cooperative learning. In Chapter 11 of his book *Punishment by Rewards* he says:

> One of the most exciting developments in modern education goes by the name of cooperative (or collaborative) learning and has children working in pairs or small groups. An impressive collection of studies has shown that participation in well-functioning cooperative groups leads students to feel more positive about themselves, about each other, and about the subject they're studying. Students also learn more effectively on a variety of measures when they can learn with each other instead of against each other or apart from each other. Cooperative learning works with kindergartners and graduate students, with students who struggle to understand and students who pick things up instantly; it works for math and science, language skills and social studies, fine arts and foreign languages. (Kohn, 1993.)

Project-based learning (PBL) done in teams is somewhat related to cooperative learning. However, there are usually considerable differences. In PBL, a team works over a period of time developing a written and/or multimedia presentation, a product, or a performance. Each team in a class may well be working on a different project, and a team has considerable choice in details of what it covers in a project. There is no class-wide exam. Teams often experience the difficulty that one or more members contribute little, while the others do most of the work. While team members learn from each other, this is not necessarily stated as a clear and measurable goal.

> A robotics club is a good example of a project-based learning environment. Members work together to create a robot that can perform designated tasks. More experienced members of the club help less experienced members. See http://iae-pedia.org/Robotics_and_Education and http://i-a-e.org/iae-blog/robotics-and-robotics-contests-in-precollege-education.html.

Paired Peer Tutoring

Paired peer tutoring involves dividing students in a class into pairs. In each pair, one student plays the role of tutor and the other the role of tutee. Then they switch roles.

Reciprocal Teaching

One form of paired peer tutoring is called reciprocal teaching. The document (NCERL, n.d.) on Reciprocal Teaching provides details and summarizes the research supporting this form of tutoring in the teaching of reading. The teacher plays a central role in this type of tutoring. Quoting from the document:

> Reciprocal teaching refers to an instructional activity that takes place in the form of a dialogue between teachers and students regarding segments of text. The dialogue is structured by the use of four strategies: summarizing, question generating, clarifying, and predicting. The teacher and students take turns assuming the role of teacher in leading this dialogue.

> The purpose of reciprocal teaching is to facilitate a group effort between teacher and students **as well as among students** in the task of bringing meaning to the text. [Bold added for emphasis.]

The idea reciprocal tutoring was originally developed for use in math and reading instruction for children in elementary school with various learning difficulties. More recently, the idea has been used with a wider range of students.

Peer-assisted Learning Strategies (PALS)

Here are two quotes about use of a paired peer tutoring program called Peer-assisted Learning Strategies (PALS) in math education grades 2-6 (CES, 2011).

> **PALS Math** entails coaching and practice for students in grades K-6. Students practice skills on game boards (at kindergarten and first grade) and worksheets (at grades 2-6) that hone skills and concepts at each grade level. The coach uses a sheet that contains questions that guide the player. Coaching lasts about 15–20 minutes.
>
> At grades 2-6, PALS math also uses mixed-problem practices worksheets that include the problem type the students just worked on as well as easier types of problems. Students work independently on these worksheets, after which they exchange and score the practice sheets. Practice lasts 5–10 minutes.
>
> **Research.** Over the past 15 years, PALS Reading and Math has been evaluated repeatedly. Studies show that "mainstreamed students with learning disabilities, low-achieving students, and high-achieving students make greater progress in PALS Reading and Math classrooms than their respective counterparts in non-PALS classes." [More research information is available at http://www.promisingpractices.net/program.asp?programid=143.]

In paired peer math tutoring, each child gets an opportunity to communicate in the language of math. Each child gets an opportunity to learn how a peer functions in math. The activities are highly prescribed. It is common to pair a stronger student with a weaker student. However, remember that the two students switch roles from time to time.

The activities are carefully designed so that a "tutor" who is less mathematically capable that his or her "tutee" can provide useful feedback. Thus, both tutors and tutees are actively engaged and experience success.

Notice the statement about the use of worksheets that team members score for each other. This idea is often used in math classes. Students "trade papers" and then correct each other's papers as the teacher reads off the correct answers. While correct/incorrect can be a useful type of feedback, it is a far cry from the type of feedback that a math savvy tutor is able to provide when helping a tutee with a math assignment. Your authors view this paper-grading situation as one in which use of computer-assisted learning is very desirable. Our suggestion is that computers should replace this component of the PALS peer tutoring system.

Cross-age Tutoring

Cross-age tutoring is a type of paired tutoring in which the tutor is significantly older than the tutee. For example, one might pair up fifth grade students and first grade students.

The article (Kalkowski, n.d.) covers both peer and cross-age tutoring. Quoting from the Introduction: http://educationnorthwest.org/webfm_send/499

It is likely that peer and cross-age tutoring have been part of human existence since hunter-gatherer times. As Jenkins and Jenkins write, "Tutorial instruction (parents teaching their offspring how to make a fire and to hunt and adolescents instructing younger siblings about edible berries and roots) was probably the first pedagogy among primitive societies" (1987, p. 64).

There is a substantial research base on cross-age tutoring. Here is the abstract of a 36-page 2005 ERIC article (Robinson et al., 2005):

> This review of the literature on peer and cross-age tutoring emphasizes programs in mathematics and suggests that such programs have positive academic outcomes for African American and other minority students as well as for White students who participate as tutors, as tutees, or both. Such programs also appear to have a positive impact on a variety of attitudinal and socioemotional outcomes, such as students' attitudes towards school, their self-concepts, and their sense of academic efficacy.

A 2004 draft copy of the (Robinson et al., 2005) paper is available free online at http://scalemsp.wceruw.org/files/research/Products/RobinsonSchofield_PeerCrossageTutoringOutcomes.pdf. The literature that is reviewed in this paper provides strong support for the claim that cross-age tutoring in math is quite beneficial for both tutees and their tutors. Here is a quote about the effects on tutees:

> Tutor academic achievement. Importantly, many studies have shown that academic achievement also improves for the tutor (e.g., Gartner & Riessman, 1994). Both Cohen et al.'s 1982) meta-analysis and Britz's et al.'s (1989) review noted academic gains for tutors. However, a much smaller proportion of the studies included in these reviews examined academic outcomes for tutors than academic gains for tutees. Of those studies, though, 87% noted tutor academic improvement in Cohen's sample, and all six of the studies that examined tutor outcomes in Britz's analysis noted mathematical gains among tutors. The finding that tutors improve academically have also been supported by more recent research (e.g., Early, 1998; Topping et al., 2004; White, 2000)

Tutoring Tip for All Teachers

Informal peer tutoring (providing help to fellow students) is a fact of life in education. More formal peer tutoring is common.

Both informal and formal peer tutoring can be improved by formal—in class—training/education of the tutors and tutees. Here are two ways to think about this:

1. Through such instruction you give credence to the practices, help tutees get better at asking for the specific help they need, and help the tutors get better at providing deeper, higher-order answers.

2. When your students become adults and possibly have children of their own, they will be able to make use of what they know about being a tutor or tutee in helping their own children to learn.

As a starting point, facilitate a whole class discussion with your students about how they help each other learn and how they learn from each other. Use the words *tutor* and *tutee* as you and your class explore what is going on when students help each other. Gather suggestions from your students on how to improve the effectiveness of tutors and tutees.

Final Remarks

The three types of peer tutoring presented in this chapter all have the "flavor" of making education more cooperative and collaborative. This is in contrast with individual students fending for themselves individually and/or in a competitive mode.

We live in a word in which both competition and collaboration are commonplace. Your authors feel it is quite important for students to learn about both sides (and, the in-betweens) of cooperation/competition.

Self-Assessment and Group Discussions

This book is designed for self-study, for use in workshops, and for use in courses. Each chapter ends with a small number of questions designed to "tickle your mind" and promote discussion. The discussion can be you talking to yourself, a discussion with other tutors, or a discussion among small groups of people in a workshop or course.

1. Name one idea discussed in the chapter that seems particularly relevant and interesting to you. Explain why the idea seems important to you.
2. Examine some of your own experiences in helping your fellow students and in being helped by your fellow students. Think of some ways to make these experiences more pleasant and useful.
3. Examine some of your personal experiences in learning in very competitive environments and in learning in very cooperative/collaborative environments. In this compare/contrast, draw some conclusions that fit your current views of how students should be taught.

Chapter 11

Additional Resources and Final Remarks

> "The strongest memory is not as strong as the weakest ink." (Confucius; Chinese thinker and social philosopher; 551 BC – 479 BC.)

A tutor is a teacher. Just like a classroom teacher, a tutor collects and develops aids to teaching and to personal learning. This chapter provides some resources that will help a tutor build a personal collection of aids to tutoring.

Our traditional formal system of education has served the world reasonably well for thousands of years. It has changed over time to accommodate the growing educational needs of the increasing number of students to be educated. It is a system that tends to preserve "traditional" content, teaching methods, and assessment methodologies.

The Information Age is a major challenge to the education systems of the world. The reason is the increasing pace of change in accumulated knowledge, tools to help in development and accumulation of knowledge, and tools to aid in dissemination and use of this knowledge.

In some sense, a tutor is "caught between a rock and a hard place." The rock and the hard place are our traditional educational system and the rapidly changing world outside of our education system. The resources presented in this chapter can help a tutor to bridge the gap—to appropriately aid students who are living their lives in a world that is rapidly changing.

Tutoring Scenario—Tutoring Homework Assignments

One of the most common tutoring activities is helping a student with homework. A tutee may need help on specific problems or further help in learning the content taught in a recent class.

Suppose—just suppose—that a tutor could be successful in enticing a tutee to do some math explorations just for fun. Is it possible for a tutor to provide a tutee with a math-related idea or topic that is so intrinsically motivational that the tutee will—just for fun—do some follow-up work on the topic?

Well…here is a suggestion. Don't call it homework! Call it *home play* or use some other term that emphasizes playing to learn and having fun while learning. Spend a little bit of a tutee's precious tutoring time exploring a topic that might be intrinsically motivating to the tutee. Feel free to experiment. When you happen to find a topic that is so intrinsically motivating that your tutee does some follow-up exploration, celebrate the success!

Bob Albrecht likes to play this "game." Drawing on ideas in this book-and especially, some of the materials from this chapter—you can improve your repertoire of potentially math-related ideas that your tutees will find intrinsically motivating.

Electronic Digital Filing Cabinet

Here are two main categories of tutoring-related things we hope that you will want to collect:

1. Math manipulatives, measuring devices, books, calculators, and other "physical" aids to learning and using math.
2. Materials that can be stored in a computer system. This includes print, audio, and video material, and software. It also includes links to such materials. In collecting links, think partially in terms of what your tutees might want to access and use on a Smart phone.

This Digital Filing Cabinet section focuses on the second category. A personal Digital Filing Cabinet (DFC) consists of computer files of information that you have collected and organized to fit your personal needs. When stored electronically, the materials are easily modified and easily shared with others. Here are two useful references:

- General overview: http://iae-pedia.org/Digital_Filing_Cabinet/Overview.
- Math education DFC: http://iae-pedia.org/Math_Education_Digital_Filing_Cabinet.

A DFC can contain articles, worksheets, problem sets, and links to material. For example, http://iae-pedia.org/Free_Math_Software is a link to a list of free math software.

Computer-Assisted Learning

There are many good free CAL materials available on the Web. Some of these are given in this section.

Drill and Practice

Drill and practice—with immediate feedback on correct and incorrect answers—is an effective aid to learning.

AAA Math provides a broad range of math drill and practice materials. The following provide specific grade level materials. If you are tutoring a 4th grader, you can click on the 4th grade link, quickly evaluate it, and decide whether it is useful in your tutoring.

AAAMath–1st grade http://www.aaaknow.com/grade1.htm

AAAMath–2nd grade http://www.aaastudy.com/grade2.htm

AAAMath–3rd grade http://www.aaastudy.com/grade3.htm

AAAMath–4th grade http://www.aaastudy.com/grade4.htm

AAAMath–5th grade http://www.aaastudy.com/grade5.htm

AAAMath–6th grade http://www.aaastudy.com/grade6.htm

AAAMath–7th grade http://www.aaastudy.com/grade7.htm

AAAMath–8th grade http://www.aaastudy.com/grade8.htm

The AAAMath resources can be used to put together a collection of materials focusing on a specific topic. As an example, we know that many students entering their first algebra course are poorly prepared in doing arithmetic with fractions. This is a major stumbling block in algebra. Here is a collection of fraction materials from AAAMath.

Adding fractions with the same denominator http://www.aaastudy.com/fra57ax2.htm

Adding fractions with different denominators http://www.aaastudy.com/fra66kx2.htm

Adding Mixed Numbers http://www.aaastudy.com/fra66dx2.htm

Subtraction with the same denominators http://www.aaastudy.com/fra57bx2.htm

Subtraction with different denominators http://www.aaastudy.com/fra66lx2.htm

Subtracting mixed numbers http://www.aaastudy.com/fra66ex2.htm

Multiplying fractions http://www.aaastudy.com/fra66mx2.htm

Multiplying fractions by whole numbers http://www.aaastudy.com/fra66nx2.htm

Multiplying mixed numbers http://www.aaastudy.com/fra-mul-mixed.htm

Dividing fractions by fractions http://www.aaastudy.com/fra66px2.htm

Dividing fractions by whole numbers http://www.aaastudy.com/fra66ox2.htm

Divide mixed numbers http://www.aaastudy.com/fra-div-mixed.htm

Here is another easy-to-use drill-and-practice site: Math Flashcards http://www.thatquiz.org/tq-1/.

We like this site. Your tutee can practice addition, subtraction, multiplication, and division of integers. He or she can choose several ways of presenting problems. We suggest that you explore this site and try all of the options in the menu on the left side of the page.

Math Function Machines

Math function machines are designed to help students learn about functions. There are many math function machines available on the Web. Some are free, and some are not. In a typical function machine, the user provides input to the machine (to the function) and the machine provides the output. The goal is to figure out the math function the machine is using. A modest amount of instruction can get a student started in using a math function machine.

Here are three examples:

- Function Machine from National Laboratory of Virtual Manipulatives. http://nlvm.usu.edu/en/nav/frames_asid_191_g_3_t_2.html. NLVM provides excellent virtual math manipulatives and a free 60-day trial. See http://nlvm.usu.edu/en/nav/vlibrary.html.
- Stop that creature! http://pbskids.org/cyberchase/games/functions/functions.html.
- Function Machine http://www.mathplayground.com/functionmachine.html.

Tutorial Materials

There are many free Web materials designed specifically to help teach various topics. Here are some examples:

The Kahn Academy offers a huge and growing number of short videos on topics in Arithmetic, Algebra 1, Algebra 2, Banking & Money, and Calculus. It also has over 100 self-paced exercises and assessments covering everything from arithmetic to physics, finance, and history.

Kahn Academy (n.d.) *Watch. Practice. Learn almost anything—for free.* Retrieved 8/18/2011 from http://www.khanacademy.org/.

The Math Forum at Drexel University contains a wide range of instructional materials and aids to learning. The Math Forum home page http://mathforum.org/ has tabs you can click that send you to places that are relevant and useful, such as: Math Help (Ask Dr. Math); Puzzles and Problems; and Resources & Tools.

Video Materials Mainly for Tutors

The following materials are mainly aimed at adults. They can help improve your general education. Some of your more advanced tutees may enjoy viewing TED videos as "home play."

IAE (n.d.). Math education free videos. *Information Age Education* Retrieved 5/18/2011 from http://iae-pedia.org/Math_Education_Free_Videos

TED Talks (Various Dates). There are now more than 900 Technology, Entertainment, Design. (TED) free videos. Retrieved 5/19/2011 from http://www.ted.com/talks.

These short videos cover a huge range of topics. Twenty-three are "tagged" with the word **math.** For a complete list of tags and access to the tagged videos, see http://www.ted.com/talks/tags. Access the 23 math-related talks at http://www.ted.com/talks/tags/math. Here are titles we think will be useful to both tutors and tutees.

David Hoffman shares his Sputnik mania.

Evan Grant: Making sound visible through cymatics.

Marcus du Sautoy: Symmetry, reality's riddle.

Michael Moschen juggles rhythm and motion.

Robert Lang folds way-new origami.

Scott Kim takes apart the art of puzzles.

Math Software

The Web provides access to many free "I can do it or solve it for you" resources. Examples include graphing functions, solving equations, and performing statistical calculations.

GeoGebra http://www.geogebra.org/cms/ is free software similar to Geometer's Sketchpad. It is a handy tool for learning and teaching/tutoring high school geometry. You can use it to do the stuff that you see in a typical high school geometry textbook.

Graph 4.3 http://www.padowan.dk/graph/ is an open source application used to draw mathematical graphs in a coordinate system.

Free Math Software http://iae-pedia.org/Free_Math_Software provides brief descriptions and links to a variety of math problem-solving tools. You and some of your tutees may find these tools useful.

Wolfram Alpha http://www.wolframalpha.com. For a very broad range of problems, you provide the problem to Wolfram Alpha, it solves the problem, and then shows how it solved the problem. It can do exact arithmetic with large multidigit numbers

Calculators

Online Calculators

A WEB search of free online calculators will provide you with many different types of online calculators.

Martindale's Calculators On-line Center http://www.martindalecenter.com/Calculators.html provides links to over 24,150 Calculators & Spreadsheets. As a tutor, you may find some specific types of calculators that will interest one of your tutees.

Free Online Graphing Calculator http://www.rentcalculators.org/stheli.html. Click on Free online graphing calculator to access the online calculator. This site also provides links to guidebooks you can download for the TI-83, TI-84, TI-86, and TI-89 Graphing Calculators.

Carbon Footprint

Carbon Footprint Calculator http://www.carbonfootprint.com/calculator.aspx.

Calculating Your Carbon Footprint http://www.motherearthnews.com/Healthy-People-Healthy-Planet/Carbon-Footprint-Calculator.aspx.

Dictionaries, Formulas, and Facts

Math

The Web contains many math information resources such as dictionaries and lists of commonly used formulas.

A Maths Dictionary for Kids http://amathsdictionaryforkids.com/dictionary.html.

Algebra Cheat Sheet http://tutorial.math.lamar.edu/pdf/Algebra_Cheat_Sheet.pdf.

Geometry & Trig Reference Area http://coolmath.com/reference/geometry-trigonometry-reference.html.

Geometry Formulas http://www.scienceu.com/geometry/facts/formulas/.

Mathematics Tables and Formulas http://www.sosmath.com/tables/tables.html.

Math Dictionary http://www.shodor.org/interactivate/dictionary/.

Math Reference Tables http://math2.org

General Dictionary and Thesaurus

Dictionary.com http://dictionary.reference.com/.

Thesaurus.com http://thesaurus.com/.

Brain Teasers

A Web search of the term *math brain teasers* produces a large number of hits.

The site http://www.cut-the-knot.org/Curriculum/index.shtml contains interesting examples. Here is one of them quoted from the Website:

1089 and a Property of 3-digit Numbers

Number 1089 is a centerpiece of a curious mathematical trick used to stun the uninitiated with the performer's math prowess.

Take any 3-digit number, say, 732 and write it backwards: 237. [Editor's note. The trick does not work if the number is a palindrome—that is, if the number is the same when it is written backwards.] Subtract the smaller of the two numbers (237 in our case) from the larger (732). With our selection, we obtain the number 495. Write this one backwards too and compute the sum: 495 + 594. Here comes the surprise: regardless of your original selection, the final result will always be 1089!

Your tutee may ask you to explain how the 1089 trick works. Be prepared! Go to Why the Answer Is Always 1089 http://www.mathsisfun.com/1089algebra.html.

The following site contains a large number of brain teasers. http://www.brainteasercentral.com/list.php?pagenum=0&catid=2.

Physical and Virtual Manipulatives

Physical blocks, spinners, dice, geoboards and other math manipulatives have long been important aids to teaching and learning math. Computers now make it possible to have virtual manipulatives that one views on a computer screen and manipulative by issuing instructions to a computer.

The Didax Math Manipulative Information Center at http://www.didax.com/manipulatives/ provides a nice introduction to math manipulatives.

The following article provides an introduction to the research base for physical and virtual math manipulatives.

Clements, D.H. (1999). 'Concrete' manipulatives, concrete ideas. *Contemporary Issues in Early Childhood. 1(1), 45-60.* Retrieved 5/18/08. http://www.didax.com/articles/concrete-manipulatives-concrete-ideas.cfm.

There are a huge number of virtual manipulatives that can be accessed through the Web. The following Website provides a list of places to access virtual manipulatives:

Southern Oregon University (2009). Virtual manipulatives. Retrieved 7/13/2011 from http://www.soesd.k12.or.us/files/manipulinks.pdf.

The National Library of Virtual Manipulatives is a frequently mentioned site. A 60-day free trial version of a large number of virtual manipulatives can be downloaded. A single user site license can be purchased for $39.95.

NLVM (2011). National Library of Virtual Manipulatives. Retrieved 5/19/2011 from http://nlvm.usu.edu/en/nav/siteinfo.html.

The Math Forum @ Drexel has been mentioned earlier in this chapter. One of its links is to Algebra Visualization Tools at http://www.home.earthlink.net/~fossmountdesign/Algebra.html. Users can change parameters by sliding a controller along a scale and instantly seeing the effect that has on the graph.

Data

There are many sources of data available free on the Web. They are fun and they provide lots of possibilities for math exploration.

Data Clocks

There are many types of data clocks available on the Web. They show frequently updated data such as population . Here are some examples:

Poodwaddle World Clock http://www.poodwaddle.com/clocks/worldclock/. Displays real time data (to 0.1 second) such as population, energy, financial, environmental, food production, et cetera, et cetera.

U.S. & World Population Clocks http://www.census.gov/main/www/popclock.html. Displays the current (to the nearest minute) U.S. and World population.

The Official US Time http://www.time.gov/. Displays the current official U.S. time in the time zone you select by clicking on a map of the U.S.A.

Physical Constants, Measurement Units, Conversions

Fundamental Physical Constants http://physics.nist.gov/cuu/Constants/index.html.

International System of Units (SI) http://physics.nist.gov/cuu/Units/index.html.

United States Customary Units http://en.wikipedia.org/wiki/United_States_Customary_System

Online Conversions http://www.onlineconversion.com/.

Solar System Data

Planetary Fact Sheets http://nssdc.gsfc.nasa.gov/planetary/planetfact.html.

Nine Planets http://nineplanets.org/.

Solar System Live http://www.fourmilab.ch/solar/solar.html.

U.S. and World Data

CIA World Fact Book https://www.cia.gov/library/publications/the-world-factbook/.

FedStats http://www.fedstats.gov/.

Stats about all U.S. Cities http://www.city-data.com/.

Telephone Area Codes http://www.allareacodes.com/.

U.S. Census Bureau http://www.census.gov/.

United States Atlas http://www.infoplease.com/atlas/unitedstates.html.

World Atlas http://www.worldatlas.com/.

World Population Data Sheet 2011 http://prb.org/Publications/Datasheets/2011/world-population-data-sheet.aspx

Zip Codes https://www.zip-codes.com.

Large Hadron Collider

Large Hadron Rap http://www.youtube.com/watch?v=j50ZssEojtM.

Large Hadron Rap Lyrics https://www.msu.edu/~mcalpin9/lhc_rap/largehadron.html.

Alpinekat at the LHCFest http://www.youtube.com/watch?v=yLTemkssghg.

Large Hadron Collider–Wikipedia http://en.wikipedia.org/wiki/Large_Hadron_Collider.

Final Remarks—the Future

Today's students are growing up in a world that is quite different from the one children faced a decade or two ago. The big change is summarized by the term *Networking and Information Technology* (NIT). Students routinely communicate with each other using NIT facilities voice (cell phone), texting, email, and social networking systems. They routinely use the Web to access video, music, and interactive games. The Web is steadily growing in size and we see steady improvements in aids to finding and making use of the information we want to find.

There is another important ongoing NIT change. Computers and computerized devices are getting smarter. The combination of human brain and computer brain gets better through:

1. Informal and formal education of humans.
2. Improvements in the hardware and software capabilities of computer brains.

Our formal education system is losing ground in its attempts to provide students with an education that thoroughly integrates use of human and computer brains. Our informal education system—for people of all ages is doing better in this integration process because it is less bound by the education traditions of the past. People at play and at work learn to use NIT to help them do things that they want to do or need to do. (For example, it is not "cheating" when an adult at work uses NIT to help solve the problems and accomplish the tasks of the job.)

We people who are interested in education are living in very interesting times. The total collected human knowledge is growing very rapidly. Many of the world's problems are global and are being addressed by teams of researchers located in many different countries. Daily progress is being reported and shared on a worldwide basis.

Becoming a Better Math Tutor

Students growing up in this world face a lifetime of needing to deal with complex personal, local, regional, national, and global problems. They also face the challenges of a rapid pace of change in the sciences and technology available to help address these problems.

> Each of us has the potential to be an actively engaged lifelong learner. Each of use has the potential to be of routine help to others in these learning processes. Think of your everyday life activities as including being both a tutor and a tutee. Be actively engaged in helping the world, others, and yourself.

Appendix 1

Advice to Tutees

Note for Tutors

An article for tutees starts on the next page. It has a Flesch-Kincaid reading level of about grade 5.5. It is printed using a slightly larger type size than the rest of the book.

Some of your tutees will be able to read and understand the content. Others will lack this level of knowledge and skill to read by learning. So, here is a three-part suggestion:

1. If you are working with a tutee you believe can benefit by reading this document, provide a copy to the tutee and suggest (encourage) the tutee to read it. You might, for example, provide a tutee with a copy during the first tutoring session. Explain that you want the tutee to read a certain part of the document before the next tutoring session and to bring it to the session. Suggest that the tutee mark places in the document that require further explanation. This will give you a chance to find out if the tutee will take the personal responsibility of reading the document. Whether the tutee follows through or not, the next tutoring session can spend some time discussing ideas in the document.
2. If you are working with a tutee with reading skills below the level this document requires, then spend some of the tutoring time helping your tutee learn the most important ideas in the document. Come back to these ideas from time to time—indeed, some could be revisited in every tutoring session.
3. Provide the tutee's parents or other adult caregivers with a copy of the article.

A handout for tutees begins on the next page. Remember, make use of this handout only if you are confident that your tutee has an adequate level of reading ability. Your goal in the tutoring sessions is to help your tutee gain in math content and math maturity knowledge and skills. However, be aware that a tutee may be having math difficulties partly because of reading difficulties.

The document is from Appendix 1 of the book:

Moursund, David and Albrecht, Robert (2011). *Becoming a better math tutor.* Eugene, OR: Information Age Education.

Advice to a Math Tutee

"Help me, Obi Wan Kenobi." Princess Leia from a Star Wars episode.

A tutor is a special type of teacher. A tutor often works with just one student at a time. This makes it possible for lots of interaction between the teacher and student. Such interaction helps a student to learn faster and better.

A tutor and a tutee are a team, working together to help the tutee learn.

A tutor's student is called a tutee. This article will help you (a tutee) learn to work with a tutor. Being a tutee is a lot different then being a student in a large class. A tutee can ask questions at any time during a tutoring session. A tutor can provide individual help when a tutee gets stuck. A tutor and a tutee are a team, working together to help a tutee learn.

Some Important Ideas

You know that school helps students learn reading, writing, and math. These are all hard subjects. It takes a lot of brainpower to learn to do reading, writing, and math. If you can read this article, it proves that you have the needed brainpower!

Some people learn faster than others. Some people are good at learning in large classes. Some learn best when they receive one-on-one help.

This article is about being a math tutee. Here are three really important goals to hold in mind:

1. Your goals as a math tutee are to learn some math and to get better at learning math.

2. To learn math means to learn how to do math. It means learning to use math to help solve problems in everyday life and in all areas that you study.
3. Math is a language. You can learn to read, write, speak, listen, and think in the language of math. You can learn to understand math. All of these activities are part of math maturity. Math tutoring will help you to increase your level of math maturity.

> It is the tutor's job to help you learn better and faster.
>
> It is your job to learn how to make use of this help and to use the help.
>
> Tutoring will help you learn math content. It will help you grow in math maturity and develop useful math Habits of Mind.

Why Learn Math?

Math is required in schools. Have you ever wondered why? After all, there are many other things that students could spend their time learning.

What is math? Think about this question. Can you give several different answers? Are your answers the same as your friends would give? Your math tutor will help you explore different answers to the "What is math?" question.

Teachers and parents give a number of different answers as to why math is required in school. Mainly, they say you need it because it is useful. They say things like:

1. You need it in your life outside of school. You use math in dealing with money. You use math to measure and think about time, distance, speed, length, area, weight, and so on. You use math in sports and in many computer games.
2. Math is important in many different school subjects. For example, math is used in all of the sciences. A type of math called statistics is useful in all subjects that gather data and work to figure out what the data means. Graphs are often used to help explain such data.
3. You will need to use math as you do adult things like run a household, hold a job, and be a responsible citizen. You need math

to understand using credit cards, borrowing money, making interest payments, and saving for retirement.

There are other possible answers. For example, continuing the list:

4. Math is an important part of human history and achievement. Thousands of people helped develop the math you study in school.
5. Many people have found that math is fun. They enjoy playing games and solving puzzles that involve math.
6. Mathematicians and many other people think of math as having beauty, somewhat like there is beauty in art, dance, and music. Think about the beauty in the math types of patterns of nature found in flowers, sunsets, rainbows, and clouds.

A math tutor can help you explore possible answers to the "why do I need to learn math" question. Some students are satisfied with answers such as "because the teacher and my parents tell me I have to learn it."

However, life is much more than just doing what adults tell you to do. Part of growing up is learning to provide answers for yourself. You, personally, need to find answers that are meaningful to you. Think about how math is part of your life now, and how it will always be part of your life.

Your Goals as a Tutee

In your first tutoring session, your tutor may ask questions such as:

- "Why are you here?"
- "What is math?"
- "What do you want to get out of these tutoring sessions?"
- "What can I (your tutor) do to make these sessions useful to you?"

Your tutor is looking for answers that will help him or her to best serve you. Of course, how you respond is up to you. Here is a type of answer to the first question that is not helpful.

> "I am here because this is third period. I am scheduled for tutoring during third period."

That is a smart aleck type of response. It disrespects your tutor.

Here are some more useful types of answer.

- "I need help on my math assignments. I get stuck on some problems and the teacher does not have time to answer my questions.

- "I need to do better on the state math test."
- "I am way behind other kids in the math class. I need help catching up."
- "I need to pass the math course in order to graduate."
- "I want to raise my math grade to a B (or, to an A).

Notice that each of these better responses states a goal. It is possible to measure progress in these types of goals. Both you and your tutor can tell if you are getting your math assignments done and turned in on time. You and your tutor can tell if you are getting better at answering the types of questions used on state math tests. You and your tutor can tell if you are catching up with the rest of the class.

Item 6 in the previous section said that there is fun and beauty in math. Thus, you might have goals such as:

- "I want to learn some fun math things."
- "I want to learn to see the beauty in math."
- "I want to learn how math has helped change history."

In summary, your tutoring sessions should be goal directed. You can help set the goals. You can help change the tutor's goals so they better fit your needs and interests.

Your Tutor's Goals

Your tutor will have goals. These can be divided into three categories:

1. Goals related to meeting your needs.
2. Goals related to being a professional, successful tutor. If a tutor is being paid, the tutor must meet the needs of the supervisor or employee. A volunteer tutor also needs to perform in a professional manner. Both paid and volunteer tutors need to gain personal satisfaction in their work.
3. Goals related to creating a professional, mutually respectful learning environment. Both the tutor and the tutee need to focus their attention. They need to be on task.

Being a Responsible, Attentive Student

Notice the third of these goals. You need to pay attention and to be on task. Your tutor needs to help you pay attention and to help you stay on task. Many students find that it is hard to stay on task when they are learning math. You can get better at this through practice!

Tutoring can help you learn to become a more responsible student. A responsible student puts energy into learning, both in class and outside of class. A responsible student does required reading and written assignments. A responsible student turns in homework on time. A responsible student comes to class and tutoring sessions with the needed tools (such as pencil, paper, and books).

A responsible student pays attention in class and while being tutored. You can learn to detect when you are not paying attention–when your mind is wandering from the learning task. With some instruction and practice you can learn to focus your attention on a learning task.

Notice that the word **math** is not used in the two paragraphs given above. Learning to be a responsible and attentive student is useful in studying any subject area. It is also applicable to the jobs and other tasks you will undertake in the future.

Tutoring Sessions are Different Than a Graded Class

Your tutor does not give you grades! Your tutor does not decide whether you pass or fail a subject. Being a tutor is quite different than being a regular teacher.

> Your tutor does not give you grades.
>
> Your tutor does not decide whether you pass or fail a test or a subject.
>
> A tutor's goal is to provide you with individual help that meets your needs.

The math you are learning now builds on the math that you covered in the past. As a math student, you face two major challenges:

1. You might not have learned some of the topics the teacher assumes you know. There are lots of reasons for this. For example, you may have changed schools, and your previous teachers did not cover the topic. You may have missed school because you were sick. The teacher may have covered a topic but you did not understand it.
2. You have forgotten some math you learned in the past. You have probably heard the expressions, "Use it or lose it." It is easy to forget math that is not part of your everyday life.

You and your tutor working together can figure out what you have forgotten or never learned. Your tutor can help you learn topics that you did not have a chance to learn. Your tutor can help you relearn topics that you have forgotten. It is important to gain skills in relearning topics you have forgotten.

Computer as Tutor

A computer's brain is a lot different than a person's brain. A computer can have a certain type of smartness. It is called artificial intelligence or machine intelligence. A computer can do some things much better than a person.

Computers can do certain types of tutoring. For example, a computer tutor works well in drill and practice situations. The computer tutor asks a question, accepts an answer, and provides feedback on whether the answer is correct. A drill and practice program of this sort can help a tutee gain speed and accuracy in math facts and mental arithmetic.

The computer tutor can keep records on what you know well and what you are struggling with. It can present review questions to help you maintain your skills. It can keep records that show your progress from week to week.

In a math class the teacher helps you cover material in a textbook. The teacher gives explanations and provides examples. Now, think about a computer version of the book and what the teacher does. A computer can use video to explain a topic and give examples. A computer can check for understanding by asking you questions and immediately processing your answers.

In addition, such a computer system allows you to move at your own pace. It is easy to stop and look at a video again. You can listen to an explanation as many times as you want. You can take a sample quiz as many times as you want. You can be in control!

Many companies are working to make better computer tutors. This is an important part of the future of math education. But, a computer is not a human being. There are many things that a human tutor can do that a computer tutor cannot do. A human tutor and a computer tutor working together can be a powerful aid to learning. As you talk to your tutor, find out whether you will get a chance to work both with your human tutor and with a computer tutor.

> A tutor, a tutee, and a computer working together are a powerful team for learning math or another subject.

There is quite a bit of free tutorial software available on the Web. Here are a few examples.

- A variety of lessons at http://www.math.com/students/practice.html.
- AAA Math arithmetic lessons at http://aaamath.com/. See the long list of topics in the menu on the left of the Website.
- Arithmetic quiz at http://www.thatquiz.org/tq-1/.
- Kahn Academy (n.d.) *Watch. Practice. Learn almost anything—for free.* Retrieved 6/14/2011 from http://www.khanacademy.org/.

You can search the Web for more free tutoring sites. You need to tell the search engine your particular interests. A search on *arithmetic practice* will lead you to a huge number of arithmetic tutor sites. A search of *algebra practice* will provide you with lots of algebra help. You might want to have your tutor provide you some help in learning to do such Web searches.

Final Remarks

Tutoring is a valuable learning experience. With the help of a tutor you can learn faster and better than you do in a class with many other students. That is because you can get immediate help on any difficulties you are having. You can talk over what is going well and what is going poorly.

Sometimes a student gets to have just a few tutoring sessions, and there are quite specific goals. For example, 10 tutoring sessions may help prepare you for a state test. Tutoring twice a week for a full year can add greatly to your math knowledge and maturity.

We encourage you to share this article with your parents and others. Spend time talking with them about your tutoring experiences.

Appendix 2

Things Parents Should Know About Tutoring

Note for Tutors

An article for parents, grandparents, and other adult caregiver starts on the next page. It has a Flesch-Kincaid reading level of about grade 8 or 9. It is printed using a slightly larger type size than the rest of the book.

It is useful to think of a tutoring team as having a number of potential members (components):

1. The tutee. Tutoring centers about a tutee, and the overall goal is to help improve the education of the tutee.
2. The "lead" tutor. This may be a paid professional, a volunteer, a same-age peer tutor, or a cross-age peer tutor.
3. Parent, grandparents, guardians, and/or other responsible adult. They may help provide both informal and formal tutoring.
4. The overall environment and in which the tutee lives, and people within the environment that have routine contact with the tutee. This include siblings, close friends, school counselors, personnel in religious institutions, and so on.
5. Computers, audio and video materials, edutainment, and other aids to learning.

You are familiar with the often-quoted statement, "You can lead a horse to water, but you can't make it drink." Similarly, we can provide a tutee with many and varied aids to learning, but we cannot make a tutee take appropriate advantage of these opportunities. One of the challenges of being a tutor is that there are so many different "players" in the game. The tutor often has little control over what the various players do.

This particular Appendix focuses on the adult caregivers in a tutee's life. It may well be that the only contact a tutor has with these people is through the tutee. (Of course, if you are doing the tutoring in a tutee's home, that is a different matter.) If it seems appropriate to you, consider providing your tutees with a copy of the document that starts on the next page, and request that it be given to the tutee's parents and/or other appropriate people. (Feel free to rewrite the document to better fit the situation.) The next time you meet with your tutee, you can inquire as to whether the document was delivered and whether there was any reaction to it. From time to time you can talk to your tutee about his or her home environment as it relates to the tutoring going on and what your tutee is learning.

The document is from Appendix 2 of the book:

Moursund, David and Albrecht, Robert (2011). *Becoming a better math tutor.* Eugene, OR: Information Age Education.

Advice to Parents of a Child Who Is Being Tutored in Math

A tutor is a special type of teacher who works with one student or a very small number of students at a time. A student who is being tutored is called a tutee.

This article is written for parents and others who are responsible for the day-to-day care of a child who is being tutored in math. We will use the expression "your child" even though you might be a grandparent, foster parent or other caregiver. The article will give you some insights into:

1. What goes on in a tutoring session?
2. How tutoring helps a tutee.
3. What you can do to help your child benefit from the tutoring.

What is Math?

Many people think of math just in terms of arithmetic. However, today's students also learn about algebra, geometry, probability, and statistics. The emphasis is on thinking, understanding, and problem solving.

While rote memory is important, the thinking and understanding are more important. We want students to be able to use their math knowledge and skills in everyday life. We want them to recognize when math can help solve a math-related problems such as shopping, borrowing money, and planning for one's future. We want them to understand the graphs and charts used in newspapers and other publications.

At the current time, school math focuses on preparing for tests. However, this is a very narrow-minded approach to math education. Outside of the math classroom, math does not consist of True/False and Multiple Choice Tests!

Tutoring can help a tutee on the routine drill and practice types of homework. But, a good tutor does much more. A good tutor helps a tutee to understand math and to get better at solving challenging math problems. A good tutor helps a tutee develop habits of mind that are needed to do well in math.

The Bottom Line

Tutoring works! With the help of a tutor, a student can learn faster and better.

Your first thought might be that tutoring is a simple thing. A tutor and tutee get together regularly. The tutor helps the tutee on homework ad on getting ready for tests.

However, there is much more than this. For example, how can you tell if your child has a well-qualified tutor? How can you tell if tutoring is helping your child?

You are an important member of your child's tutoring team. In the "big picture," it is helpful to think of five parts of a tutoring team:

1. **The tutee.** Tutoring centers about a tutee, and the overall goal is to help improve the education of the tutee.
2. **The "lead" tutor.** This will likely be a paid professional, or a volunteer. If the tutoring takes place in a school, the tutor will have a supervisor.
3. **Parents and/or other responsible adults.** They may help provide both informal and formal tutoring. In addition, they help to create a home environment that is conducive to learning.
4. **The tutee's environment.** This includes the overall environment and in which the tutee lives and the people in it. It includes your child's home environment, siblings, close friends, school counselors, personnel in religious institutions, and so on.
5. **Technology.** This includes computers, audio and video materials, books, and other aids to learning. It also includes entertainment television, music, and computer games.

Think about your roles. You are both a team member and you help to create the environment in which your child learns. A later section of this document will provide you with some advice.

Who Needs and Who Gets Tutoring

If we use a very broad definition of tutoring, then infants receive a lot of tutoring. In a normal home environment they receive lots of personal attention from caregivers. They learn their native language(s) through one-on-one help. They learn about their culture by being immersed in the culture and through individualized feedback from many different people in the culture.

Moreover, reading to your children, playing games with your children, working with your children on learning sports, and so on are all examples of tutoring. In some sense, whatever a parent is "into," such as music, reading, athletics, and so on gets communicated to their children.

However, let's talk about tutoring that is a supplement to schoolwork or a major component of schooling. Each child is unique. Our school system groups students into classes of perhaps 20 to 30 or more students. A teacher of such a large group of students cannot know the individual needs of each student. The teacher does not have the time to identify the specific strengths and weaknesses of each individual student. The teacher is required to cover the curriculum specified by the school, school district, and state.

New learning is built on what a child already knows. Often a student has not gained the prerequisite knowledge and skills needed to successfully learn a new topic. Instruction on the new topic is so far over a student's head that the student is confused, easily discouraged, and unable to keep up with the class.

There are many reasons why a student is not well prepared to learn a particular new topic. The student may have missed school when required previous knowledge and skills were being covered. The child may have changed schools, and the previous school may not have covered the topics.

The child may have specific learning challenges that can slow down progress in school. Here are a few examples:

- Vision, hearing, general health, nutrition, and other problems that have not been identified and/or adequately addressed.
- Dyscalculia is a particular type of math-learning challenge. Students with dyscalculia have difficulty learning to do arithmetic.
- Dyslexia, a particular type of reading problem. Many dyslexic students have trouble in learning math. This is partly because math is a language. Learning and doing math requires reading and writing in the language of math.
- ADD (Attention Deficit Disorder) or ADHD (Attention Deficit Hyperactive Disorder). This makes it difficult for the student to pay attention to (concentrate on) a topic being taught.

The list can be expanded. What is important for you to know is that a great many students can benefit from individualized help in school. A tutor can help a student to identify learning gaps and to address them. A tutor can provide individualized feedback to a student in a timely and personalized manner.

Tutoring is not only for students having learning difficulties. Students who are mentally or physically gifted often receive tutoring or special small-group instruction. This is common in the performing arts—for example, for a student taking music lessons.

Tutoring and/or very small classes are sometimes made available to very gifted students. Your school district may have a special program for talented and gifted students. Students who are especially gifted in math can easily learn math at twice the pace of an ordinary math class.

A Highly Qualified Tutor

A tutor is a teacher. As a parent, you will want to know about the qualifications and experience of your child's math tutor. Here is a list of some possible qualifications that a good math tutor might have:

1. **Math content knowledge.** Have good math problem solving knowledge and skills over the range of his or her math content knowledge.
2. **Math education experience.** Have considerably experience in helping students learn math. If your child has particular math-learning challenges, you want a tutor who is experience in dealing with such challenges. In any case, you want a tutor who understands both the theory and practice of teaching and learning math.
3. **Math Standards.** Know the school, district, and state math standards below, at, and somewhat above the level at which one is tutoring.
4. **Communication skills.** This includes areas such as:
    a. Being able to "reach out and make appropriate contact with" a tutee; and
    b. Being able to develop a personal, mutually trusting, human-to-human relationship with a tutee.
5. **Empathy.** Knowledge of "the human condition" of being a student with a challenging life in and outside of school. A good tutor can help a tutee build self-confidence as a learner.
6. **Learning.** A math tutor needs to be a learner in a variety of areas relevant to math education. Important areas include computer technology and brain science. Computers are an important tool in tutoring.
7. **Diversity.** A math tutor needs to be comfortable in working with students of different backgrounds, cultures, race, creed, and so on. In addition, a math tutor needs to be able to work with students

with dual or multiple learning-related exceptionalities, such as ADHD students who are cognitively gifted.

Roles of a Parent of a Child Being Tutored at School

Remember that we are using the term "parent" to include grandparents and other regular caregivers.

If your child has an Individual Education Plan (an IEP) at school, then you have a legal right to know details of tutoring your child receives through the IEP.

In any case, parents should know both the purpose of a child being tutored and the outcomes that can be expected. Parents should interact with their children in a manner that helps support the tutoring.

Here is an example. You ask your child, "What did you do in school today." You are probably used to answers that don't say much. This does not help you or your child.

Now, change this. You might say, "Hi Pat. Today you worked with your math tutor in school. Tell me about some of the things you and your tutor did."

Don't give up if you get a "nothing" type of answer. Ask some specific questions. "Tell me something that you learned. Explain it so that I can understand it." "Give me an example. Here is a pencil and piece of paper. Show me." "What are some uses of this math? Can you give me an example from what you do outside of school?" Your goal is to engage your child in a conversation about math and learning math.

This type of routine math-related interaction is very good for your child. It can also be very good for you, as it will help you learn what your child is learning. It will give you insight your child's math learning challenges and successes.

DO NOT spend time telling your child how difficult math was for you! You want to create a positive math-learning environment. Your child can learn math, and you what to help and encourage your child. You want to celebrate the successes your child is experiencing.

References

Most of the references listed here are from citations within the text. However, a few are additional resources that readers may find useful. The $3.00 Kindle version of this book includes 21-page annotated version of this set of references. You can purchase the book at http://www.amazon.com/s/ref=nb_sb_noss?url=node%3D154606011&field-keywords=David+Moursund&x=0&y=0.

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