Improving Elementary School Math Education:

Some Roles of Brain/Mind Science and Computers

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Preface

> Mathematics is one of humanity's great achievements. By enhancing the capabilities of the human mind, mathematics has facilitated the development of science, technology, engineering, business, and government. (Kilpatrick, Swafford, and Findell, 2002)

This book is designed for use in the preservice and inservice education of elementary school teachers. The goal of the book is to improve the quality of math education that elementary school students are receiving.

Improving Math Education

Many people feel that math education is not as successful as they would like, and that it is not as successful as it could be. There is ample evidence that our math education system—and indeed, our entire educational system—can be much improved. There is continuing pressure on schools and teachers to improve math education.

As you read this book, you will find it helpful to have ready access to the Web. Math education practitioners and researchers know a lot about how to improve math education. This book contains a large number of links to Web resources that support and expand upon the assertions the book contains. It also contains quite a few poignant quotations. Here is a example of a quotation from a Website:

> For most students, school mathematics is an endless sequence of memorizing and forgetting facts and procedures that make little sense to them. Though the same topics are taught and retaught year after year, the students do not learn them. Numerous scientific studies have shown that traditional methods of teaching mathematics not only are ineffective but also seriously stunt the growth of students' mathematical reasoning and problem-solving skills. Traditional methods ignore recommendations by professional organizations in mathematics education, and they ignore modern scientific research on how children learn mathematics (Battista, 1999).

Michael Battista is one of the leading math educators in this country. His 1999 article provides an excellent summary of some of the things that are wrong with our math education system.

There are many ways to improve math education. This book focuses on three of them:

1. Appropriately using our rapidly growing knowledge of brain science, mind science, and other aspects of the Craft and Science of Teaching and Learning.
2. Appropriately using Information and Communication Technology (ICT).
3. Better teaching. Now, as in the past, teachers play a central role in math education. This book will help you to become a better teacher of mathematics.

About Me (the Author of This Book)

I have been a teacher of teachers for most of my professional career. I have specialized in the areas of computers-in-education and in math education. However, over the past decade I have also spent a lot of time and effort studying and teaching about the field of brain and mind science as it applies to teaching and learning. You can learn more about me at http://iae-pedia.org/David_Moursund .

This book combines my interests in brain/mind science, computers-in-education, and math education. I have used much of the material in a variety of courses that I have taught and workshops that I have led. However, I have not previously attempted to put all of these ideas together into a coherent whole.

Quite likely you have accessed this book from one of my Web sites, and you have noticed that the book is being made available at no cost. Over the years, I have authored or co-authored more than 30 commercially published books and a number of Websites. Nowadays, I make most of my writings available free on the Web. You can access such free materials at http://iae-pedia.org/David_Moursund_Books/.

Brain, Mind, and Computers—Cognitive Science

The typical human adult brain is a very complex organ that weights about three pounds. One can study the brain as an organ, much as one studies the heart, liver, and so on. However, a person's brain (more correctly, brain and rest of the person's body) "produces" or has a mind and consciousness. For many years, the study of the mind fell in the provenance of psychologists, while the study of the brain fell in the provenance of biologists, physicians, and neuroscientists. Now, the fields of brain study and mind study are drawing closer together. In this book we will often talk about brain/mind science to denote the combined discipline of brain science and mind science.

In 1956 a number of brain/mind scientists and computer scientists got together and essentially defined a new field—cognitive science. Cognitive science includes computer modeling of the brain/mind, and the study of brain/mind from an information processing point of view. Quite a bit of this book focuses on applications of cognitive science to the teaching and learning of mathematics.

Getting Better at Teaching Mathematics

Elementary school teachers typically teach a wide range of subjects, including Language Arts, Mathematics, Science, and Social Science. The elementary school teacher is responsible for a very wide range of content areas, a very wide range of student levels of current knowledge and understanding, and a very wide range of student interests. Being a good and successful teacher is a tremendous challenge, and there is always room for improvement!

As you might expect, progress in brain and mind science is providing us with ways to improve curriculum content, pedagogy, and assessment in all of the elementary school subject areas and grade levels. The same statement holds true for computers. Throughout this book we use the term Information and Communication Technology (ICT) rather than the term computers, since ICT is a broader and more inclusive term. Thus, many of the ideas in this book are applicable throughout the entire elementary school curriculum. However, the emphasis is on math education.

I assume that you want to be a good teacher. If you are already a good teacher, I assume you want to be a still better teacher. These assumptions constitute two of the prerequisites that I assumed as I wrote this book. I am not assuming any special or high level background and interest in math. Also, this book does not assume any particular knowledge about brain/mind science or ICT.

The third prerequisite I am assuming is that you have a good mind. This book is designed to challenge your mind—to make you think. This will cause your brain to create more connections among its neurons, and thus make you smarter!

As you read this book, you will likely have suggestions for its improvement. Please send your comments and ideas to moursund@uoregon.edu.

David Moursund

March 2004

Chapter 0
Some Big Ideas

"The saddest aspect of life right now is that science gathers knowledge faster than society gathers wisdom." (Isaac Asimov; Russian-born American author and biochemist; 1920–1992.)

"We are what we repeatedly do. Excellence, therefore, is not an act but a habit." (Aristotle; Greek philosopher; 384 BC–322 BC.)

You may think it a bit strange that the first chapter in this book is labeled Chapter 0. When asked to count by 1's, most people respond with 1, 2, 3, etc. However, many mathematicians will respond with 0, 1, 2, 3, etc. This book has a Chapter 0 because at one time in my life I was thoroughly enculturated into the world of mathematicians.

This chapter contains a brief introduction to a few of the BIG IDEAS in the book. My hope is that as you read this chapter, it will encourage you to continue reading the subsequent chapters.

Progress in Past Years

Improving math education has been a high priority in our educational system for many years. During the past three decades there has been:

- Substantial research on ways to improve the effectiveness of math curriculum, instruction, and assessment.
- Standards developed by the National Council of Teachers of Mathematics.
- Significant changes in the commercially available materials to support the teaching of mathematics. Quite a bit of the new material is based on large-scale projects funded by the National Science Foundation.
- A steady increase in the average IQ of students. (This topic is discussed in Chapter 4.)
- Many major efforts to improve our overall educational system, with special emphasis on math and science education, since the 1957 launch of the Russian satellite named Sputnik.
- Substantial progress in brain science (neuroscience), mind science (psychology), and cognitive science (a combination of computer & information science, neuroscience, psychology, and other related disciplines). .
- Huge improvements in the capabilities and availability of information and communication technology systems.

You might think that the combination of all of these things would have led to significant improvements in student learning of math. However, take a look at Figure 0.1 (Campbell, et al., 2002). This reports longitudinal data from the National Assessment of Educational Progress in Reading, Mathematics, and Science for students at three different grade levels. As you can see, there has been relatively little change in each of these three major components in our educational system.

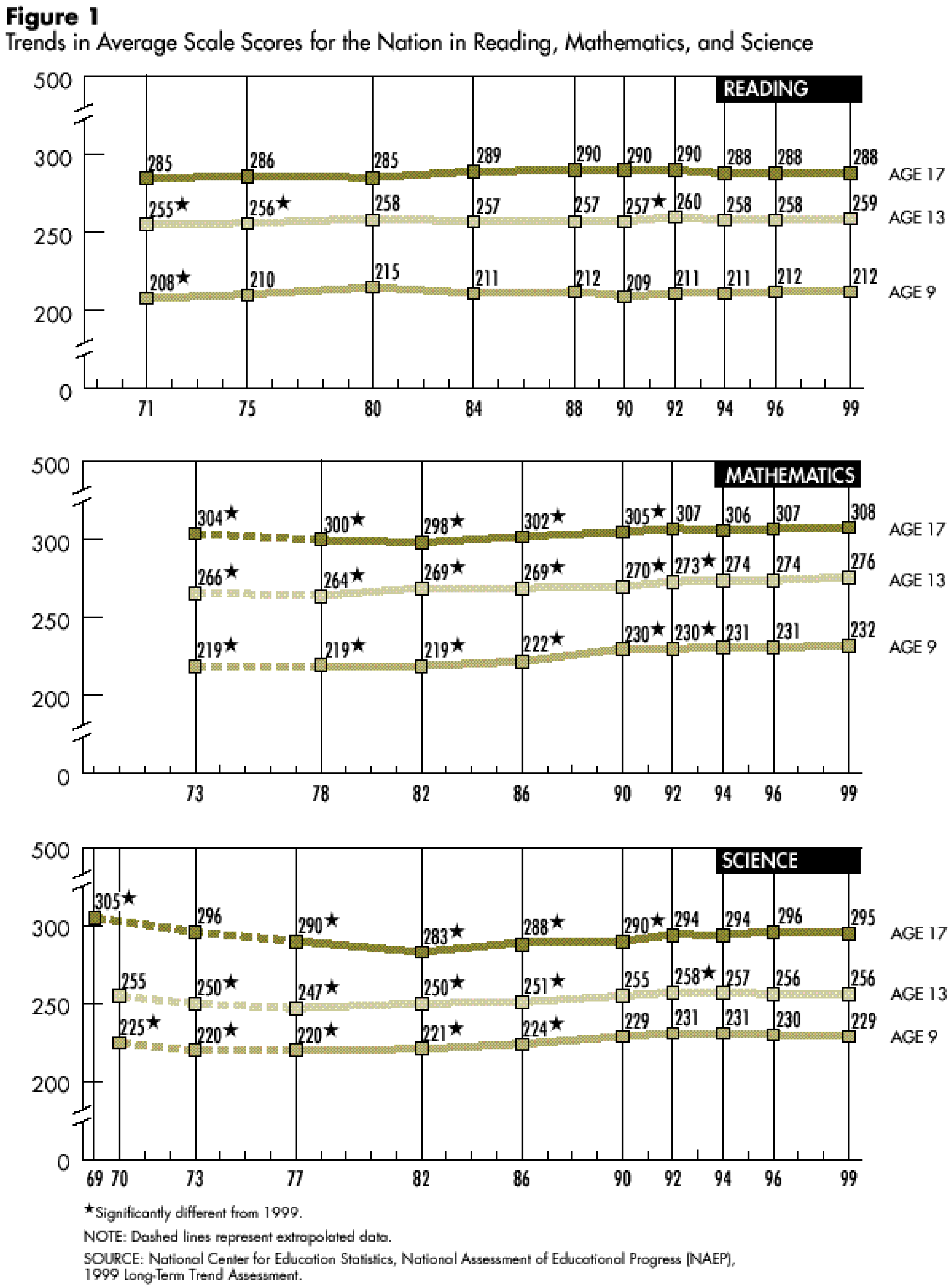

Figure 0.1. Trends in average scores for the US in reading, math, and science.

Increasing Mathematical Maturity—THE Goal in Math Education

You know that math is a large and complex discipline. You know that there are many different goals in math education. However, it is possible to encompass the goals of math education in a short sentence. **The goal of math education is development of the**

“mathematical maturity” of the learner. For some reason unknown to me, mathematicians use the term *mathematical maturity* when other people might use the term *mathematical expertise.* In this book, I take the two terms to mean the same thing.

A student’s mathematical maturity ”is a combination of five components. These are knowledge, understanding, and skill:

1. Within and about the content of the discipline of mathematics.
2. That facilitates transfer of learning both within the discipline of math and to other disciplines.
3. In learning math.
4. In communicating and thinking using the language of mathematics.
5. In formulating (posing, extracting) math problems, math questions, and math tasks that are components of the discipline of mathematics and other disciplines.

You might prefer to use the term mathematical expertise in place of mathematical maturity. The word “expertise” suggests a progression of moving up from being a novice, gaining increasing expertise over time through study and practice.

Big Idea # 1: The goal of helping each student to gain an increasing level of mathematical maturity (mathematical expertise) serves to unify and to provide direction to math education at all grade levels and for all students.

Teaching and Learning Math—and Other Disciplines

Math is but one of the disciplines in which we want students to gain a functional, useful level of knowledge and skills. The teaching and learning of math shares much in common with the teaching and learning of other disciplines. However, math is different from each other discipline. Thus, teaching and learning math is somewhat different than teaching and learning each other discipline. As a student progresses through school, his or her progress in gaining expertise in the various disciplines studied will vary among disciplines. For a specific discipline, the content knowledge, pedagogy knowledge, interests, and so on of the teacher make a significant difference in student learning.

Big Idea # 2: You can become a more effective teacher of math (and, of course, each other discipline that your teach). As you read this book and become a more effective teacher of math, much of what your learn will be helpful in becoming a more effective teacher of other disciplines. That is, much of what you learn can be transferred to teaching other disciplines.

Brain Versus Computer

In the early days of computers, people often referred to such machines as *electronic brains*. Even now, more than 50 years later, many people still use this term. Certainly a human brain and a computer have some characteristics in common. However:

- Computers are very good at carrying out tasks in a mechanical, “non-thinking” manner. They are millions of times as fast as humans in tasks such as doing arithmetic calculations or searching through millions of pages of text to find occurrences of a certain set of words. Moreover, they can do such tasks without making any errors.

- Human brains are very good at doing the thinking and orchestrating the process in many different very complex tasks such as carrying on a conversation with a person or reading for understanding. A human being has a mind. A human's brain/mind capability for "understanding" is far beyond the capabilities of the most advanced computers we currently have.

Big Idea # 3: There are many things that computers can do much better than human brains, and there are many things that human brains can do much better than computers. Our math educational system can be significantly improved by building on the relative strengths of brains and computers, and decreasing the emphasis on attempting to "train" students to compete with computers.

Improving Education

Formal education (schooling) began about 5,200 years ago when the Sumerians developed reading, writing, and arithmetic. For 5,200 years, people have been working to improve the effectiveness of schooling. The collected knowledge in how to do this is called the Craft and Science of Teaching and Learning.

Very roughly speaking, we can divide attempts to improving schooling into two categories:

1. Those that focus on what teachers, students, parents and other people involved in schooling know and do. For example, teacher education is much more extensive (requiring more years of schooling) than it was a hundred years ago, and this contributes to students getting a better education.
2. Those that focus on materials and ideas that can be widely reproduced and distributed. For example, a "modern" curriculum can be designed and incorporated into widely distributed student texts and teacher materials.

Big Idea # 4: The approaches 1 and 2 are both being strongly influenced by progress in brain/mind science and progress in computers. Brain/mind science and computers are important components of the Craft and Science of Teaching and Learning.

Goals of Education

People have widely varying ideas on what the goals of education should be. However, there is considerable agreement on two ideas:

1. Students should learn in a manner that facilitates their using their knowledge at later times and in differing situations. That is, students should learn in a manner that facilitates transfer of learning.
2. Students should learn to learn, both in general and in the specific disciplines they study in school. This process includes learning about themselves as learners, how to make effective use of their specific relative strengths, and how to make appropriate accommodations for their specific relative weaknesses.

Big Idea #5: Transfer of learning and learning to learn are two important components of the Craft and Science of Teaching and Learning. They are areas in which practitioners and researchers have made considerable progress in recent years. We now know how to substantially improve how well we accomplish 1 and 2. Appendix A contains a list of goals of education.

Individual Differences

The human brain is very complex, no two brains are the same, and there are large differences among the brains of students. The individual differences come from a combination of nature and nurture. A simple minded way to think about this is to consider identical twins, separated at birth, placed in different home environments, cultures, communities, schools, and so on. For a "nature" point of view, the two children share a lot in common and have the same genes. The nurture aspects of their upbringing may differ substantially.

Constructivism is an important learning theory that explores and helps explain how students learn by building on the knowledge that they already have. This theory helps explain the success of tutoring, small classes, and instruction especially designed for the current developmental level, knowledge, and skills of a learner.

Big Idea # 6: We know that there are individual differences among our students, and we know the values of providing curriculum, instruction, and assessment that is appropriate to the knowledge and skills of the learner. Highly interactive intelligent computer-assisted learning (HIICAL) is a term that describes the best of modern computer-assisted instruction. Such computer-assisted instruction represents our best current progress in computerizing our insights into constructivism and other aspects of the Craft and Science of Teaching and Learning. Appropriate use of HIICAL can substantially improve student learning.

Mathematics as a Language

You know that each discipline has special vocabulary and symbol sets, and often assigns special meaning to words that have more commonly used meanings. Math does this more than most other discipline, and many people agree that it is appropriate to speak of math as a language, or to speak of the language of math. Thus, a student is faced by the task of learning to read, write, speak, listen, and think math.

Big Idea # 7: One of the major goals in education is for students to gain increasing communication and understanding skills in reading, writing, speaking, listening, and thinking in one or more "natural languages" used for general communication. The same ideas hold for learning math. However, our current math curriculum, weak in this area.

Math Manipulatives: Moving from Concrete to Abstract

Much of the power of mathematics lies in its abstractness. The mathematical sentence 2 + 3 = 5 can be thought of as an abstract mathematical model that is applicable to a wide range of situations—such as grouping together people, toys, or apples. You know about the four level Piagetian developmental scale: sensorimotor, preoperational, concrete operations, and formal operations. Much of mathematics is at the formal operations end of the scale.

Math manipulatives —whether they are physically concrete objects, or computer displays of such objects—provide an important aid in helping students move from the concrete to the abstract. Such concrete manipulatives are useful at all levels of math education.

Big Idea # 8: Math manipulative are an important aid to learning math at all levels, and computers add an important new dimension to such aids to learning math.

Problem Posing and Problem Solving

In this book we will take the term "problem posing" to include a broad range of activities such as asking questions, proposing tasks to be accomplished, formulating decision-making situations, and posing problems to be explored and possibly solved. We will take the term "problem solving" to encompass the full range of activities that contribute to answering questions, accomplishing tasks, making "good" decisions, and solving problems. We note that:

1. With these broad definitions of problem posing and problem solving, each discipline includes a major focus on posing and solving problems.
2. Mathematics is a powerful aid to problem posing and problem solving throughout the school curriculum.
3. Computers are a powerful aid to solving math problems and problems in other disciplines.
4. Progress in brain/mind science has the potential to increase our understanding of how the brain/mind works as it poses and solves problems, and how to improve its abilities to do this.

It is often useful to think about curriculum and instruction on a scale that moves from lower-order cognitive skills to higher-order cognitive skills. We understand that both lower-order and higher-order knowledge and skills are necessary in posing and solving problems. In recent years there has been considerable agreement (but, by no means complete agreement) that our schools should place more emphasis on the higher-order end of the scale.

Big Idea # 9: Every discipline (not just math) includes a major focus on problem posing and problem solving. By appropriately teaching for transfer, problem posing and problem solving ideas taught in one discipline (such as math) will help increase student problem posing and problem solving knowledge and skills in other disciplines.

Roles of Computers in Math Education

In this book, we take the term "computers" to encompass the entire field of Information and Communication Technology (ICT). The Internet (which includes the Web) is a very important component of ICT. Calculators, Personal Digital Assistants, still and video digital cameras, cell telephones, laptop computers, desktop computers, and supercomputers are all part of ICT.

This book explores three important aspects of ICT in math education.

1. ICT as part of the discipline of mathematics and content in the math curriculum.
2. ICT as an aid to teaching, learning, and assessment in math education.
3. ICT as an aid to using and doing math both in the discipline of mathematics and in other disciplines.

Big Idea # 10: ICT is a very important component of math education and a student's mathematical maturity (mathematical expertise). Knowledge and skills in the math-related aspects of ICT are of great importance to a person seeking to be an effective teacher of mathematics. Appendix B contains a list of goals for ICT in education.

An Analogy with Learning to Read/Reading to Learn

In our current educational system, about 70-percent of students learn to read well enough by the end of the third grade so that they can them focus on reading to learn. As students continue to progress through school, reading to learn becomes an increasingly large component of the instructional delivery system.

As noted earlier in this chapter, it is appropriate to think of math as a language. Thus, it is appropriate to think about the idea of learning to read math and then reading to learn math. Our current math educational system is weak in the area of learning the language of mathematics to a level that it readily facilitates learning math and uses of math in other disciplines.

Big Idea # 11: We can learn a lot about the teaching and learning of math by studying the teaching and learning of reading and writing.

Learning "Chunks" with Understanding

Research on short-term ("working") memory indicates that for most people the size of this memory is about 7 ± 2 chunks. This means, for example, that a typical person can read or hear a seven-digit telephone number and remember it long enough to key into a telephone keypad. When I was a child, my home phone number was the first two letters of the word diamond, followed by five digits. Thus, to remember the number (which I still do, to this day) I needed to remember only six chunks. But, I had to be able to decipher the first chunk, the word "diamond."

The human brain can memorize sequences of nonsense syllables or words. However, the typical person is not very good at this, and such rote-memorized data or information tends to quickly fade from memory.

On the other hand, the human brain is very good at learning meaningful chunks. Think about the five chunks: add, subtract, multiply, divide, and square root. Probably these chunks have different meanings to me than they do for you. As an example, for me, the chunk "multiplication" covers multiplication of positive and negative integers, fractions, decimal fractions, irrational numbers, complex numbers, functions (such as trigonometric and polynomial), matrices, and so on.

What does the chunk '"square root" mean to you? As you think about this, think about the extent to which your understanding of this chunk is dependent on having memorized and practiced a paper and pencil algorithm for calculating square root. Are you adept at paper and pencil calculation of square roots?

The brief discussion given above suggests:

1. Learning chunks with understanding is a very important aspect both of learning and in making use of short-term memory.
2. There is a significant difference between memorizing and practicing a computational algorithm and in learning with understanding the concept(s) of the "chunk" associated with that algorithm.
3. We now have machines (such as calculators and computers) that can carry out algorithms with great speed and accuracy.

Big Idea # 12: Our math education system can be substantially improved by taking advantage of our steadily increasing understanding of how the mind/brain deals with math (such as the ideas of chunking listed above), and steady improvements in ICT facilities.

Auxiliary Brain/Mind

The development of reading and writing was **VERY SIGNIFICANT.** In essence, reading and writing provide short term and long-term storage for personal use and that can be shared with others. Data and information can be stored and retrieved with great fidelity.

> The strongest memory is not as strong as the weakest ink. (Confucius, 551-479 B.C.)

Writing onto paper provides a passive storage of data and information. The "using" of such data and information is done by a human's brain/mind. Contrast this with the computer storage of data and information.

Computers add a new dimension to the storage and retrieval of data and information. Computers can process (carry out operations on) data and information. Thus, one can think of a computer as a more powerful auxiliary brain/mind than is provided by static storage on paper or other hardcopy medium.

Big Idea # 13: ICT provides us with a type of auxiliary brain/mind. The power, capability, and value of this auxiliary brain/mind continue to grow rapidly. Certainly this is one of the most important ideas in education at the current time.

Concluding Remarks

Thirteen big ideas… You might be thinking to yourself, "That's simple enough. I'll memorize the list, pass the test, and them move on in my teaching career." Unfortunately, that won't help much in making you into a better teacher or helping your students get a better education.

Our educational system is faced by the continuing challenge of translating theory into practice. Each individual teacher faces this challenge. You, personally, can improve our educational system by understanding the underlying theory of the 13 big ideas, and then translating them into your everyday practice as a teacher. As you get better at this translation process, and as you increase your expertise in the areas of these big ideas, you will get to be a better teacher and your students will get a better education.

Recommendations Emerging from Chapter 0

Each chapter of this book ends with a short list of recommendations. You can become a better teacher of mathematics by understanding these recommendations and by implementing some of them into your everyday teaching of math. The recommendations in Chapter 0 are numbered 0.1, 0.2, etc. those in Chapter 1 are numbered 1.1, 1.2, etc. The last chapter of the book contains a short summary of the most important recommendations.

0.1 When you are teaching math, think carefully about what you are doing and could be doing to help your students learn to make effective use of math throughout the curriculum—and then implement some of your "could be doing" ideas.

0.2 When you are teaching disciplines other than math, think carefully about what you are doing and could be doing to help your students learn about roles of math in these disciplines—and then implement some of your "could be doing" ideas.

0.3 Give increased though and effort to translating educational theory into routine everyday practice.

Activities and Questions for Chapter 0

Each chapter ends with some activities and questions. These can be used for self-study. They are also useful for small group and whole class discussions in workshops and courses. Occasionally a faculty member might want to assign one of these as "homework."

1. Select one of the Big Ideas given in this chapter. Explain in your own words what this Big Idea means to you. Then discuss the nature and extent to which you incorporate or pay attention to this Big Idea in your current teaching of math.

2. Select the Big Idea in this chapter that seems most important from your point of view, and the one that seems least important from your point of view. Explain the process that you used to do this selection. In doing this, be sure to point out aspects of your two choices that make one more important and the other less important from your point of view.

3. Consider the chunk, *auxiliary brain/mind.* Think about your understanding of this chunk from the point of view reading and writing using a static, hardcopy medium such as paper. Then think about your understanding of this chunk from the point of view of reading, writing, and the automation of some processing activities using a dynamic (computer) medium. Do a compare and contrast of your thoughts, feelings, level of understanding, and so on of these two different aspects of *auxiliary brain/mind.*

Chapter 1
Four Key Questions

Mankind owes to the child the best it has to give. (United Nations Declaration of the Rights of the Child, 1959)

Civilization advances by extending the number of important operations which we can perform without thinking of them. (Alfred North Whitehead)

The goal of this book is to help improve the mathematics education students receive while in elementary school. This chapter explores the question, "What is mathematics?" It also raises some additional questions that are explored in later chapters.

This chapter also contains some general background information that will prove useful in later parts of the book.

Improving Math Education

What questions occur to you as you think about the goal of improving math education? As I think about this goal, four important questions occur to me.

1. What is mathematics?
2. What are the major goals for math education in elementary school?
3. What are some general ways to improve math education in elementary school?
4. What can you (personally) do to help improve the mathematics education students receive while in elementary school?

Table 1.1 Questions to help guide thinking about improving math education.

The 1st is addressed in this chapter, while the 2nd and 3rd questions are discussed in later parts of the book. You, personally, will need to answer the 4th question.

Reflective Reading

What did you think about when you read the first of the four questions? Did you stop reading and attempt to form an answer to the question? Did you try to imagine yourself attempting to give an answer in various situations such as when talking to a young student, when talking to a parent, or when talking to a fellow teacher? Or, did you sort of "bleep over" the question, proceeding quickly to reading the next three questions?

Reading a book about math and math education is a lot different than reading a novel. I like to read and I read a lot. I read some things quite rapidly, and I read some other things quite slowly. When I read "scholarly, academic" materials I tend to read slowly, in a reflective manner. I pause frequently to think about what I am reading. I attempt to figure out what the sentences and paragraphs mean. I actively work to construct meaning—what the writing means to me, personally. I think about how I might incorporate the information into my teaching, writing, and conversations.

Educators have some fancy words to describe this activity. These include the terms:

- **constructivism**—building meaning and understanding based on your current knowledge and understanding;
- **metacognition**—thinking about your own thinking;
- **reflective reading**—functioning in a reflective manner when reading; being deeply mentally engages in a "higher-order" thinking manner while reading; questioning and challenging the information that is being presented and the assertions that are being made; carrying on mental arguments with the author.

This is a short book. If you read it like you would read a novel, you will likely finish the whole book in a couple of hours. However, if you read reflectively, pausing frequently to actively engage in metacognition and in the process of constructivism, you will read much more slowly. **In doing so, you will be functioning like a mathematician and a good math educator. You will be demonstrating progress you have made in increasing your mathematical maturity.**

If the previous paragraph has not shamed you into rereading the four questions, using reflective reading, then perhaps you will do so just to please me. I am reminded of the adage, "You can lead a horse to water, but you can't make it drink." I am trying to whet your thirst for knowledge that will help you to be a better teacher of mathematics. Please, please begin practicing your reflective reading knowledge and skills. Make a commitment to helping your students become better reflective readers. Progress in this endeavor will help improve the quality of education that your students receive.

Very Brief History of Invention of Mathematics

The Web contains a huge amount of information about the history of mathematics (History topics index, n.d.). About 5,200 years ago the Sumerians developed reading, writing, and arithmetic (Vajda, 2001). It is no coincidence that reading, writing, and arithmetic were developed simultaneously. The Sumerians were faced by the problems of growing population, growing bureaucracy, and growing business. They needed reading, writing, and arithmetic.

> As the societies of the Sumerian city-states advanced in the fourth millennium B.C., large-scale trade and other economic activities were increasingly hampered by the lack of a permanent record of transactions. At first, the Sumerians employed stone and clay tokens, which represented various goods and numerical values, to keep track of their mercantile dealings. Around 3200 B.C. these tokens were replaced by markings made on clay tablets and written language was born. The first *cuneiform* writings consisted of pictograms, which were drawings of the items represented. Shortly thereafter ideograms, or abstract symbols, were also employed. These allowed the Sumerians to symbolize ideas as well as concrete things. Cuneiform was soon used to record all important activities, from the sale of land to marriage and adoption contracts (Invention of writing, n.d.).

The 3 Rs are an aid to the human mind. (You can think of them as mind tools. A number of people also talk about computers as mind tools.) The 3 Rs are a way to communicate over time and distance. They provide powerful aids to representing and solving problems.

The human mind is adept at learning to communicate orally. A person gains considerable skill at oral communication by merely growing up in an environment in which people communicate this way. Reading and writing of this oral communication language made it possible to create permanent records of what people were communicating orally. This facilitated an accumulation and sharing of knowledge that eventually greatly changed the societies of our planet.

However, the human mind has much less natural talent to learn to deal with precise quantities and with representations of precise quantities. Thus, from early on people worked to develop aids to the mind to increase its ability to deal with number, quantity, distance, time, and so on. Writing proved to be a powerful aid to such endeavors. With the help of writing, a person can carry out manipulations on numbers that are well beyond what a typical mind can do without some sort of external aid. (Try doing multidigit long division in your head!) Writing, as an aid to mathematics, facilitated the development of "higher" forms of math, such as geometry and algebra. It also facilitated the steady accumulation of mathematical knowledge.

To summarize, the reading and writing of natural language and the reading and writing of mathematics developed simultaneously. The goal in both cases was to develop aids to representing and solving certain types of problems of government and business. Over time, the availability of a mathematics language facilitated the development of powerful tools for representing and solving a wide range of math-related problems that could not previously be solved. Math has proven to be so useful and important that it is part of the core curriculum in elementary schools throughout the world.

> "Mathematics is the queen of the sciences, and arithmetic the queen of mathematics."(Carl Friedrich Gauss, 1777-1855) [Note from Moursund: In this statement, "arithmetic" is what we now call "number theory" and is a much broader topic than arithmetical computation.]

What is Mathematics?

Imagine yourself as a student in one of my preservice or inservice elementary school teacher education classes, and I have just asked you, "What is mathematics?" What would you say? Perhaps you would talk about counting, doing arithmetic, and measuring distance, time, angles, and areas. Perhaps you would talk about solving math problems, such as word problems. Perhaps you would talk about tasks that many students find challenging, such as multiplication and division of multidigit numbers, working with decimals, and working with fractions. You might talk about geometry, algebra, probability, statistics, and calculus.

Or, perhaps you would give a really sophisticated answer such as the one from Michael Battista (1999) quoted below:

> Mathematics is first and foremost a form of reasoning. In the context of reasoning analytically about particular types of quantitative and spatial phenomena, mathematics consists of thinking in a logical manner, formulating and testing conjectures, making sense of things, and forming and justifying judgments, inferences, and conclusions. We do mathematics when we recognize and describe patterns; construct physical and/or conceptual models of phenomena; create symbol systems to help us represent, manipulate, and reflect on ideas; and invent procedures to solve problems.

Battista is a leading math educator, and his answer is similar to what many leading math educators would provide. Spend some time thinking about how his answer differs from your personal answer. (That is, continue to practice your reflective reading!)

Here is a somewhat different way to think about developing an answer to the question, "What is mathematics?" You know that math is but one of a number of disciplines that students study in school. A discipline can be defined by a combination of:

• The types of problems, tasks, and activities it addresses.
• Its accumulated accomplishments (results, achievements, products, performances, scope, power, uses, and so on).
• Its history, culture, and language (including notation and special vocabulary).
• Its tools, methodologies, and types of evidence and arguments used in solving problems, accomplishing tasks, etc.

Table 1.2 Four defining aspects of a discipline.

The list in table 1.2 helps to illustrate why it is difficult to give a short answer to the question, "What is mathematics?" For example, what do we mean by the culture of mathematics? Here is a good example of an answer by Alan Schoenfeld (1992):

> I remember discussing with some colleagues, early in our careers, what it was like to be a mathematician. Despite obvious individual differences, we had all developed what might be called the mathematician's point of view—a certain way of thinking about mathematics, of its value, of how it is done, etc. What we had picked up was much more than a set of skills; it was a way of viewing the world, and our work. We came to realize that we had undergone a process of acculturation, in which we had become members of, and had accepted the values of, a particular community. As the result of a protracted apprenticeship into mathematics, we had become mathematicians in a deep sense (by dint of world view) as well as by definition (what we were trained in, and did for a living).

Notice the emphasis on becoming enculturated into the mathematical community. As a student studies math year after year in school, the student should be building an understanding of math aspects of the four bulleted items in Table 1.2. This understanding gains additional meaning when it includes comparing and contrasting math with other disciplines that the student is studying.

Recommendations Emerging from Chapter 1

Each chapter of this book ends with a short list of recommendations. You can become a better teacher of mathematics by understanding these recommendations and by implementing some of them into your everyday teaching of math. The recommendations in Chapter 1 are numbered 1.1, 1.2, 1.3, etc.

1.1 The concept of reflective reading is important in all "scholarly, academic" reading. Practice it for yourself, and help your students to master it. (Note that this recommendation applies to all curriculum areas, not just math.)

1.2 The reading and writing of natural language and the reading and writing of math developed simultaneously and are thoroughly intertwined. You know a lot about helping students learn reading and writing of natural language. Give careful thought about how this knowledge transfers to the task of helping students learn to read and write math—and then routinely apply your increasing insights about the teaching of math as a written language.

1.3 Math is a broad and deep discipline that humans have been developing for more than 5,000 years. One of your goals as a teacher is to help your students gain increased understanding of each discipline that you teach. As you develop your daily lesson plans in math and the other disciplines you teach, think about how they contribute to students gaining increased understanding of these disciplines. Consciously work to increase your understanding of these disciplines and your students' understanding of these disciplines.

Activities and Questions for Chapter 1

Each chapter ends with some activities and questions. These can be used for self-study. They are also useful for small group and whole class discussions in workshops and courses. Occasionally a faculty member might want to assign one of these as "homework."

1. Think about your own math education in terms of the four bulleted items in Figure 1.2. Give a brief summary of what you know and understand for each of the four bulleted items.
2. Repeat (1) above for some other discipline that you teach. Then do a compare and contrast analysis of the depth and breadth of your understanding of the two disciplines.
3. In this chapter, I asserted that math is a language.
   a. Think about the meaning of "language" and then put together some good arguments for and against the idea that math is a language.
   b. Think about some of the things that you know about how to help a student learn reading, writing, speaking, listening and thinking in a "natural language." Then think about how these ideas might carry over to helping a student to learn to communicate effectively in mathematics.
4. Suppose that you were responsible for creating two quiz questions designed to measure your fellow students' understanding of key ideas in this chapter. Make up two higher-order questions that require deep thinking and understanding to answer.

Chapter 2
Goals of Education and Math Education

> An educated mind is, as it were, composed of all the minds of preceding ages." (Bernard Le Bovier Fontenelle, mathematical historian, 1657-1757)
>
> Man's mind, once stretched by a new idea, never regains its original dimensions. (Oliver Wendell Holmes, American Jurist, 1841-1935)

Any improvement in math education needs to be measured against an agreed upon set of goals for math education. Different people and different groups of people (different stakeholder groups) have differing opinions as to the appropriate goals for math education.

This chapter has two main parts. The first part is a discussion of the overall goals of education. The assumption is that the goals of math education need to be consistent with and supportive of the overall goals of education. The second part is a discussion of current goals of math education from the point of view of the National Council of Teachers of Mathematics (NCTM, n.d.) Later chapters will discuss how brain/mind science and computers contribute to items 1-2 given above.

Enduring Goals of Education

From the point of view of a particular stakeholder group, we improve math education by some appropriate combination of:

1. Removing or placing less emphasis on goals that are of declining importance in the group’s opinion.
2. Adding or placing more emphasis on goals that are of increasing importance in the group’s opinion.
3. Better accomplishing the goals that the stakeholder group agrees on.

This observation suggests that educational goals likely undergo considerable change over time. You might wonder if there are some enduring goals.

David Perkins' 1992 book contains an excellent overview of education and a wide variety of attempts to improve our educational system. He analyzes these attempted improvements in terms of how well they have contributed to accomplishing the following three major and enduring goals of education (Perkins, 1992, p5):

1. Acquisition and retention of knowledge and skills.
2. Understanding of one's acquired knowledge and skills.

3. Active use of one's acquired knowledge and skills. (Transfer of learning. Ability to apply one's learning to new settings. Ability to analyze and solve novel problems.)

These three general goals—acquisition & retention, understanding, and use of knowledge & skills—help guide formal educational systems throughout the world. They are widely accepted goals that have endured over the years. They provide a solid starting point for the analysis of any existing or proposed educational system. We want students to have a great deal of learning and application experience—both in school and outside of school—in each of these three goal areas. (A more extensive list of goals in education is given in Appendix A.)

You will notice that these goals do not point to any specific content areas. For example, these goals do not mention reading and writing. Obviously Perkins' list of goals needs to be supplemented with lists of objectives that support the goals. Each such objective can be analyzed from the point of view of how well it supports accomplishing Perkins' three goals.

In some sense, one can view these three goals as constituting a hierarchy moving from lower-order to higher-order knowledge and skills. This is illustrated in Figure 2.1. Of course, the terms low-order, medium-order, and high-order are not precisely defined. Also, the various stakeholder groups that set goals for education tend to disagree among themselves as to how much emphasis to place on each.

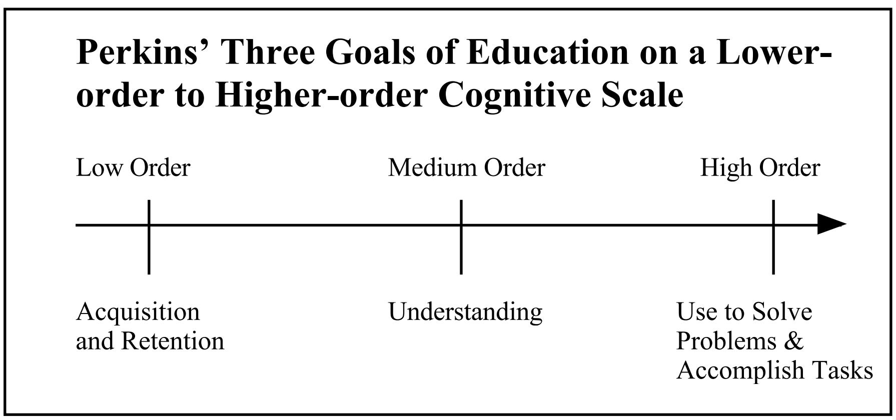

Figure 2.1. Scale: lower-order to higher-order goals of education.

Perkins' first goal can be thought of as having students gain and retain lower-order knowledge and skills. In simple terms, we want students to memorize and retain some data and information. People have the ability to memorize a great deal of data and information with little understanding (knowledge) of what they are memorizing. It is relatively easy to assess lower-order knowledge and skills. However, we also know that students (including you and I) have a strong propensity to forget what we have memorized.

The second goal focuses on understanding. What is your understanding of what it means for you or some other human to understand something? Are you good at self-assessing the understanding that you gain by reading a book such as this one, or by listening to a lecture on a topic? As a teacher, are you good at assessing the nature and extent of the understanding your students are gaining?

Pay special attention to the third goal. There, the emphasis is on problem solving and other higher-order knowledge and skill activities. You know that computer systems can solve or help solve a wide variety of problems. How does a computer's "higher-order, problem-solving knowledge and skills" compare with a human's higher-order and problem-solving knowledge and skills?

This last question is particularly important to our educational system. It is clear that computer systems can do some things better than people, and that people can do some things better than computer systems. The capabilities computer systems are continuing to change quite rapidly. Thus, our educational system is faced by the challenge of coping with a rapidly moving and quite powerful change agent (Moursund, 2004).

Goals of Math Education

The National Council of Teachers of mathematics (NCTM) is this country's largest professional society devoted to PreK-12 math education. NCTM's Standards for school mathematics describe the mathematical understanding, knowledge, and skills that students should acquire from prekindergarten through grade 12. Although each of these Standards applies to all grades, the relative emphasis on particular Standards will vary across the grade bands.

Ten standards are listed below. Aside from a change in format, these are quoted from NCTM (n.d.).

Five Content Standards

S1. Number and Operations

Instructional programs from prekindergarten through grade 12 should enable all students to:

1.1 understand numbers, ways of representing numbers, relationships among numbers, and number systems;

1.2 understand meanings of operations and how they relate to one another;

1.3 compute fluently and make reasonable estimates.

Number pervades all areas of mathematics. The other four Content Standards as well as all five Process Standards are grounded in number.

S2. Algebra

Instructional programs from prekindergarten through grade 12 should enable all students to:

2.1 understand patterns, relations, and functions;

2.2 represent and analyze mathematical situations and structures using algebraic symbols;

2.3 use mathematical models to represent and understand quantitative relationships;

2.4 analyze change in various contexts.

Algebra encompasses the relationships among quantities, the use of symbols, the modeling of phenomena, and the mathematical study of change.

S3. Geometry

Instructional programs from prekindergarten through grade 12 should enable all students to:

3.1 analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;

3.2 specify locations and describe spatial relationships using coordinate geometry and other representational systems;

3.3 apply transformations and use symmetry to analyze mathematical situations;

3.4 use visualization, spatial reasoning, and geometric modeling to solve problems.

Geometry and spatial sense are fundamental components of mathematics learning. They offer ways to interpret and reflect on our physical environment and can serve as tools for the study of other topics in mathematics and science.

S4. Measurement

Instructional programs from prekindergarten through grade 12 should enable all students to:

4.1 understand measurable attributes of objects and the units, systems, and processes of measurement;

4.2 apply appropriate techniques, tools, and formulas to determine measurements.

The study of measurement is crucial in the preK–12 mathematics curriculum because of its practicality and pervasiveness in so many aspects of everyday life. The study of measurement also provides an opportunity for learning about other areas of mathematics, such as number operations, geometric ideas, statistical concepts, and notions of function.

S5. Data Analysis and Probability

Instructional programs from prekindergarten through grade 12 should enable all students to:

5.1 formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them;

5.2 select and use appropriate statistical methods to analyze data; develop and evaluate inferences and predictions that are based on data; understand and apply basic concepts of probability.

To reason statistically—which is essential to be an informed citizen, employee, and consumer—students need to learn about data analysis and related aspects of probability.

Five Process Standards

S6. Problem Solving

Instructional programs from prekindergarten through grade 12 should enable all students to:

6.1 build new mathematical knowledge through problem solving;

6.2 solve problems that arise in mathematics and in other contexts;

6.3 apply and adapt a variety of appropriate strategies to solve problems;

6.4 monitor and reflect on the process of mathematical problem solving.

Problem solving is an integral part of all mathematics learning. In everyday life and in the workplace, being able to solve problems can lead to great advantages. However, solving problems is not only a goal of learning mathematics but also a major means of doing so. Problem solving should not be an isolated part of the curriculum but should involve all Content Standards.

S7. Reasoning and Proof

Instructional programs from prekindergarten through grade 12 should enable all students to:

7.1 recognize reasoning and proof as fundamental aspects of mathematics;

7.2 make and investigate mathematical conjectures;

7.3 develop and evaluate mathematical arguments and proofs;

7.4 select and use various types of reasoning and methods of proof.

Systematic reasoning is a defining feature of mathematics. Exploring, justifying, and using mathematical conjectures are common to all content areas and, with different levels of rigor, all grade levels. Through the use of reasoning, students learn that mathematics makes sense. Reasoning and proof must be a consistent part of student's mathematical experiences in prekindergarten through grade 12.

S8. Communication

Instructional programs from prekindergarten through grade 12 should enable all students to:

8.1 organize and consolidate their mathematical thinking though communication;

8.2 communicate their mathematical thinking coherently and clearly to peers, teachers, and others;

8.3 analyze and evaluate the mathematical thinking and strategies of others;

8.4 use the language of mathematics to express mathematical ideas precisely.

As students are asked to communicate about the mathematics they are studying—to justify their reasoning to a classmate or to formulate a question about something that is puzzling—they gain insights into their thinking. In order to communicate their thinking to others, students naturally reflect on their learning and organize and consolidate their thinking about mathematics.

S9. Connections

Instructional programs from prekindergarten through grade 12 should enable all students to:

9.1 recognize and use connections among mathematical ideas;

9.2 understand how mathematical ideas interconnect and build on one another to produce a coherent whole;

9.3 recognize and apply mathematics in contexts outside of mathematics.

Mathematics is an integrated field of study, even though it is often partitioned into separate topics. Students from prekindergarten through grade 12 should see and experience the rich interplay among mathematical topics, between mathematics and other subjects, and between mathematics and their own interests. Viewing mathematics as a whole also helps students learn that mathematics is not a set of isolated skills and arbitrary rules.

S10. Representation

Instructional programs from prekindergarten through grade 12 should enable all students to:

10.1 create and use representations to organize, record, and communicate mathematical ideas;

10.2 select, apply, and translate among mathematical representations to solve problems;

10.3 use representations to model and interpret physical, social, and mathematical phenomena.

Representations are necessary to students' understanding of mathematical concepts and relationships. Representations allow students to communicate mathematical approaches, arguments, and understanding to themselves and to others. They allow students to recognize connections among related concepts and apply mathematics to realistic problems.

Observations About the NCTM Standards

The NCTM Standards consist of 33 goals distributed among five Content Standards and five Process Standards. Notice the active verbs used to start each goal statement: understand (5 times), use (4 times), analyze (3 times), apply (3 times), recognize (3 times), and select (3 times). **Notice that "compute" is used just once!** A number of other terms are used just one.

The amount of detail provided in the previous section does not allow a careful classification of the 33 goals into lower-order, middle-order, and higher-order knowledge and skills. However, it is evident that the emphasis in on middle-order and higher-order knowledge and skills, and that problem solving is mentioned frequently. The NCTM Standards also emphasize communication and using math to represent and model problems. Finally, the NCTM Standards include an emphasis on using math to help represent and solve problems in other disciplines, and thinking about math as an interdisciplinary tool.

It is interesting to look at the list of goals and see they fit with the definition of a discipline given in Table 1.2 and repeated here as Table 2.2 for your convenience. From my point of view, the NCTM Standards seem to place little emphasis on the history and culture of mathematics. The emphasis given to the types of problems addressed and the accumulated accomplishments seems to be only within the context of the specific mathematical topics covered. As a consequence of this, a student might complete high school and have gained little insight into any mathematical accomplishments of the past 5,000 years!

- The types of problems, tasks, and activities it addresses.
- Its accumulated accomplishments (results, achievements, products, performances, scope, power, uses, and so on).
- Its history, culture, and language (including notation and special vocabulary).
- Its tools, methodologies, and types of evidence and arguments used in solving problems, accomplishing tasks, etc.

Table 2.2 Four defining aspects of a discipline.

As a final comment in this section, it is interesting to compare the three overall goals of education stated by Perkins with the 33 goals given in the NCTM Standards. You might see that the NCTM Standards contain the essence of Perkins' three goals, but provide substantial more detail of what these three goals mean within the specific discipline of mathematics.

More generally, each academic discipline has developed a detailed set of standards for its discipline. Such detail is needed in order to then specify scope and sequence or benchmarks for each grade level, and then to specify day to day lesson plans. As a preservice or inservice teacher you can easily hold in mind the three goals of education specified by Perkins. However, it is unlikely that you can hold in mind the 33 goals in the NCTM Standards or the huge number of other goals for the other disciplines that you teach.

My personal solution to this difficulty is to develop an understanding of the nature and extent of my expertise in the various disciplines I deal with in my professional work. In essence, I think carefully about what I know and can do relative to what I believe I "should" know and be able to do. I also compare what I know and can do to what my peers know and can do.

For example, I think about my mathematical expertise versus my brain/mind science expertise. I have Ph.D. in mathematics, and I have never taken a course in brain/mind science. On the other hand, in recent years I have spent much more time studying and thinking about brain/mind science than I have spent on mathematics. I am now confident that I know more about brain/mind science than most preservice and inservice PreK-12 teachers, and also more than most of my colleagues who teacher teachers. However, I recognize that some of my colleagues know much more about brain/mind science than I do.

The expertise scale in Figure 2.2 is useful to me. See if it helps you. For each discipline that you teach, you can think of where you fall on the expertise scale, and you can think about whether this level of mastery of the discipline is appropriate to the goal of being a good teacher of the discipline. We will talk more about being a good math teacher in a later chapter.

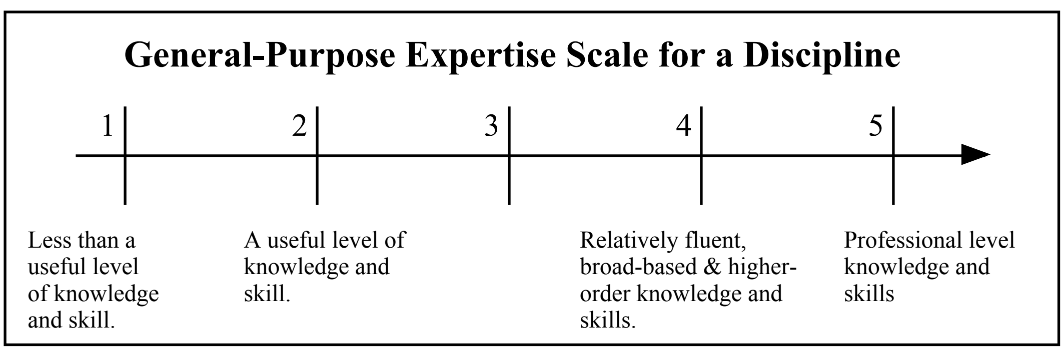

Figure 2.3 General-purpose expertise scale.

More on "What is mathematics?"

In this section we provide two more answers to the question, "What is mathematics."

Alan Schoenfeld is one of the leading math educators in the US. He says:

> Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to

> determine the nature or principles of regularities in systems defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world objects ("applied mathematics"). The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure—mathematical sense-making (Schoenfeld, 1992).

This definition is the type that one mathematician tends to write in attempting to communicate with another mathematician. Think of it as a statement from one person who is high on the mathematical expertise scale to another mathematician who is high on this scale. But then, think about it in terms of what might be involved in you and your students moving up the mathematical expertise scale. Note, for example:

- "The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation." Later in this book we talk about the Piagetian developmental scale. The tools of mathematics are at the high end of this developmental scale.
- The emphasis on learning to think mathematically, and the difference between learning to use the tools and learning to think mathematically.

Our current math education system is not very successful in helping students to make sense of mathematics and to think mathematically.

The following quotation is from the book Everybody Counts (MSEB, 1989, p. 84):

> Mathematics is a living subject which seeks to understand patterns that permeate both the world around us and the mind within us. Although the language of mathematics is based on rules that must be learned, it is important for motivation that students move beyond rules to be able to express things in the language of mathematics. This transformation suggests changes both in curricular content and instructional style. It involves renewed effort to focus on:
>
> - Seeking solutions, not just memorizing procedures;
> - Exploring patterns, not just memorizing formulas;
> - Formulating conjectures, not just doing exercises.

Notice the strong emphasis on problem posing (for example, formulating conjectures) and problem solving (seeking solutions). The *Everyone Counts* book focuses on the idea of "mathematics as an exploratory, dynamic, evolving discipline rather than as a rigid, absolute, closed body of laws to be memorized."

Concluding Remarks

Mathematics is a large discipline, with great breadth and depth. As a teacher of math, your goal is to help you students increase their level of mathematical maturity—their level of math expertise. Perhaps you have heard the statement:

> If you don't know where you are going, you're likely to end up somewhere else. (Lawrence J. Peter, of "Peter's Principles" fame.

Think about what this means in terms of math education. Apply the idea both to students and to teachers. One of the weaknesses of our elementary school math education system is that many students and many teachers don't know where they are going.

As you read and think about the various answers to, "What is mathematics?", you can construct an answer that is meaningful to you. As you draw on your answer while creating and teaching math lesson plans, you can help your students to construct answers that are appropriate to their current levels of mathematical maturity.

Recommendations Emerging from Chapter 2

Each chapter of this book ends with a short list of recommendations. You can become a better teacher of mathematics by understanding these recommendations and by implementing some of them into your everyday teaching of math.

2.1 Construct a personally understandable and useful answer to the question, What is mathematics?" Explore this question with your colleagues and your students. I suspect that you will be surprised by the shallowness of the answers you will get from your colleagues and students.

2.2 When you develop a lesson plan in any discipline, think about how your learning goals fit in with and contribute to Perkin's three goals of education. Then think about the relative emphasis the lesson places on lower-order, medium-order, and higher-order knowledge and skills. Be sure that you are satisfied with the balance in the lesson plan.

2.3 To be a good teacher in a discipline, one must have an "appropriate" understanding of the content of the discipline. For example, one might expect an elementary school teacher to understand mathematics at the level specified by the content and process goals of the NCTE Standards for PreK-12 mathematics. Analyze your strengths and weaknesses in each of the 33 goals. Develop a systematic plan of action for addressing your areas of weakness that seem most important to your teaching.

Activities and Questions for Chapter 2

1. This chapter talks about lower-order, medium-order, and higher-order knowledge and skills, but it doesn't define these terms. Select a grade level that you teach or are preparing to teach. Then:

   A. Define the three terms for math at that grade level, making sure that you give examples to make your definition clear.

   B. Select some other discipline at this grade level, and define the three terms for that discipline.

   C. Compare and contrast your answers to 1A and 1B, and draw some general conclusions.

2. Appendix A of this book contains a much longer list of goals of education than is provided by Perkins. Analyze the longer list. Then discuss the

usefulness of Perkin’s list versus the usefulness of the longer list in developing and teaching math lessons.

3. Select one Content Goal and one Process Goal from the NCTM Standards that you feel are particularly important from your point of view. Give brief arguments for the particular importance of these two goals.

Chapter 3
Teaching and Learning

... pedagogy is what our species does best. We are teachers, and we want to teach while sitting around the campfire rather than being continually present during our offspring's trial-and-error experiences. Michael S. Gazzaniga (1998, p 8)

Chance favors only the prepared mind. (Louis Pasteur; French chemist and microbiologist; 1822–1895.)

Humans have been teaching and learning in formal "school" settings for more than 5,000 years. During this time they have accumulated a huge amount of information about the Craft and Science of Teaching and Learning. This chapter covers three general topics that are part of the background information needed in later chapters.

1. **Transfer of learning.**
2. **Learning theory.**
3. **Lower-order and higher-order knowledge and skills.**

Transfer of Learning

Transfer of learning is a continuing challenge to our educational system. We want students to be able to use their learning in a wide variety of settings that they will encounter after gaining the learning. The National Science Foundation held an invited workshop in March of 2002 to map out a research agenda in this area. The following is quoted from a write-up on that workshop (Mestre, 2002).

> We define transfer of learning (hereafter transfer) broadly to mean the ability to apply knowledge or procedures learned in one context to new contexts. A distinction is commonly made between near and far transfer. The former consists of transfer from initial learning that is situated in a given setting to ones that are closely related. Far transfer refers both to the ability to use what was learned in one setting to a different one as well as the ability to solve novel problems that share a common structure with the knowledge initially acquired.

Notice the emphasis on solving novel problems. Chapter 5 of this book focuses on problem solving. Later sections of the current chapter discuss near and far transfer, and situated learning.

There is a lot of research literature on transfer of learning. As with research in other aspects of education, one needs to explore this research in terms of:

1. Is it good research? An excellent discussion on what constitutes good educational research is available in *Good Educational Research* (2003).
2. How can we translate theory into practice? How does a teacher teach for transfer and how does a student learn for transfer? These two questions are

especially important in math education, where our level of success is not very good.

3. What additional research is needed? What are important questions that we don't know the answers to?

In brief summary, the NSF workshop suggested that some good research has been done, that we are not good at translating theory into practice, and that a huge amount of research remains to be done.

Two years before the NSF workshop, Barnett & Ceci (2002) said: "Despite a century's worth of research, spanning over 5,000 articles, chapters, and books, the claims and counterclaims surrounding the question of whether far transfer occurs are no nearer resolution today than at the turn of the previous century."

Here are a few key ideas that the research tells us:

1. One of the common reasons why transfer of learning does not occur is that the students have not learned enough and understood it well enough. Far transfer is rooted in learning for understanding.
2. Rote memorization and practice to a high level of automaticity are keys to near transfer. We know a lot about teaching and learning for a high level of automaticity—in number facts, keyboarding, and many other areas. Computers are a useful aid in such teaching and learning.
3. Teaching via rote memorization is a very poor approach to achieving far transfer.
4. It is important to teach in a manner that facilitates learning to learn. Knowledge and skill in learning is amenable to far transferable.
5. The context or situation in which learning occurs has a significant impact on far transfer. This helps explain difficulties students have in transferring knowledge gained in a math class to the types of setting they encounter in other classes or outside of school.
6. Many of the ways that we use to "teach to the test" are poor in producing far transfer of learning other than transfer "to the test."
7. A sequential block approach in schooling is a significance hindrance to far transfer of learning. This block approach is common in two settings:
   A. In presenting a subject such as math, the material is taught and learning is assessed in a form: Topic 1, Test on Topic 1; Topic 2, Test on Topic 3, Test on Topic 3; etc. There is relatively little integration of the topics, except perhaps in an end of unit or end of term test.
   B. The school day is divided into blocks of time devoted to different disciplines. Each discipline gets its block of time. There is very little teaching or assessment effort that cuts across the disciplines.

Near and Far Transfer

The term near transfer is used to describe situations in which transfer of learning occurs automatically, without conscious thought. Transfer that requires conscious, thoughtful analysis is called far transfer. And, of course, there are a myriad of situations between these two extremes.

The human brain is an analogue storage and processing organ. It is very good at pattern matching—in recognizing without conscious thought that one situation (event, face, pattern, problem, etc) is nearly the same as one that has been previously encountered and dealt with. A very young baby learns to recognize his or her mother's face, and transfers this learning to accommodate changes in time, place, facial makeup, hairdo, and so on.

B.F. Skinner and others developed behaviorism, a stimulus-response learning theory. They amply demonstrated that mice, rats, pigeons, and other animals can be trained to recognize a stimulus and carry out a learned response. That is, it is possible to train for near transfer, whether the trainee is a mouse or a person. Even though a number of newer learning theories have been developed, behaviorism is still an important learning theory.

Near transfer is an important aspect of math education. Our educational system has decided that the one-digit addition and multiplication facts are so important that they should be part of a student's near transfer repertoire.

It turns out that most human brains are capable of this learning task. However, it takes many students a very larger amount of time to achieve the needed level of subconscious automaticity. Moreover, some of this learned automaticity degrades over time unless it is regularly used. (Remember, the human brain is an analogue storage and processing device, not a digital computer.) There are many other demands in our educational system for students to gain a high level of automaticity. There is not sufficient time in the school day for all of these demands to be met.

Moreover, our educational system has set much higher learning goals than are achievable by this behaviorist approach. We want students to gain higher-order knowledge and skills that they can apply in novel problem-solving situations. In recent years a new "low-road, high-road" theory of transfer has been developed, and it is quite useful in education.

Low-Road, High-Road Theory of Transfer

The Perkins and Salomon (1992) low-road/high-road theory of transfer of learning provides a good foundation for understanding transfer of learning and teaching for transfer. This theory is a modern alternative to the near and far transfer theory. In my opinion, it is a more useful theory, as it provides better insight into how to teach for transfer. In brief summary:

- Low-road transfer focuses on learning for "subconscious quick response automaticity—a stimulus-response type of learning.
- High-road transfer focuses on: cognitive understanding; purposeful and conscious analysis; mindfulness; and application of strategies that cut across disciplines.

Here is an example of low-road transfer in the teaching of reading. A goal in reading instruction is for student to be able to recognize some written "sight word" quickly without conscious thought, linking the printed symbols with "meaning" stored in the neurons in his or her brain. An important aspect of low-road transfer is that it can take a great deal of time and effort to achieve the needed level of automaticity. However, once achieved, much of this automaticity is maintained after a significant period of time (such as a summer) of non-use.

In high-road transfer, there is deliberate mindful abstraction of an idea that can transfer, and then conscious and deliberate application of the idea when faced by a problem where the idea may be useful.

Here is an example of high-road transfer. Suppose that in math you are teaching students the strategy of breaking a large problem into a collection of more manageable smaller problems. You name this strategy "Breaking a big problem into smaller problems." You have students practice it with a number of different math problems. You then have them practice the same strategy in a number of different disciplines.

You might wonder why I didn't pick number facts (such as multiplication of one-digit integers) as the example to illustrate low-road transfer. It seems to me that single digit multiplication is a more complex example than sight words. Here are two reasons for this:

- If we have students memorize 8 x 7, we know that the student still faces the challenge of recognizing that this is the same as "eight times seven" and VIII times VII.
- In the world outside of school books, the need to calculate 8 x 7 is almost always buried in or contained in some problem situation. (That is why we include word problems in the curriculum.) Contrast this with the need to read a word that is clearly displayed.
- Typically when a student is memorizing a word, the student already has some understanding of the meaning of the word. This is not typically the case for when a student is memorizing a number fact.

As I think about number facts versus sight words, I begin to get some insight into the difficulties of learning math versus the difficulties of learn to read. A typical student learning to read already knows how to speak and listen, and understands oral communication. In essence, that is not the situation faced by a student learning math.

Situated Learning

Situated learning is a theory that indicates that what one learns is highly dependent on the situation (the environment, the culture, the context, etc.) in which the learning is situated. This is closely related to transfer of learning. Increased transfer is facilitated by having the "situation" of the learning by reasonably similar to the "situation" in which the learning is to be applied.

For example, consider a student learning math in a classroom environment that mainly makes use of worksheets, with lots of pages of printed computational tasks. For days, the student works on additional facts and simple addition. At a later time, for days, the student works on multiplication facts and simple multiplication. Now consider this student in a situation outside of school in which it might be appropriate to use some of the math knowledge and skills that were being taught. The outside of class environment is a lot different than the classroom environment. This is a significant detriment to transfer of learning.

Or, think about a classroom setting that places major emphasis on students learning to solve word problems. Contrast this environment with a typical outside of class environment in which a student encounters a situation in which it is desirable to pose a math problem (in his or her head) and then solve the problem (perhaps mentally). The problem posing and then solving situation rooted in as real world environment is quite a bit different than the classroom environment when the worksheet or book provides the problem, and the problem may not be a meaningful component of the students outside of school environment.

Situated learning theory is supportive of case study, problem-based learning, and project-based learning. All three of these teaching approaches include creating learning environments that tend to be like those found outside of the classroom.

Some Learning Theories

There are a number of theories of how people learn, and these theories can be used as the basis for designing curriculum. This section briefly discusses several of these theories.

Behavioral Learning Theory

In very simple terms, behavioral learning theory is a stimulus/response learning theory. It has had a major impact on our educational system. For example, people think about memorization based on use of flash cards as a behavioral approach to teaching and learning. In that sense, behavioralism can be viewed as a vehicle to support learning for low-road transfer.

Behavioral learning theory has a long history. A few of the key people in this field include Edward Lee Thorndike (1874-1949), John Watson (1878-1958), and B.F. Skinner (1904-1990). In more recent times, learning theory researchers have focused more of their attention on cognitive learning theories—learning theories that include the conscious higher-order thinking capabilities of the learner. However, cognitive learning theories emerged at about the same time as and coexisted with behavioralism.

Constructivism

Constructivism is a learning theory that states that new knowledge and skills are built upon one's current knowledge and skills. What that sentence is easy to memorize and seems self-evident, it is a major challenge to effectively implement constructivist-based learning theory. That is because each person has different knowledge and skills.

Constructivism is not a new learning theory. The origins of constructivist learning theory are rooted in the work of people such as John Dewey (1859-1952), Jean Piaget (1896-1980) and Lev Vygotsky (1896-1934). However, in recent years constructivism has emerged as one of the key ideas in teaching and learning.

Each learner brings different knowledge and skills to a new learning task. As a preservice or inservice teacher you know that a typical classroom of students faces you with tremendous differences in previous knowledge and skills, learning styles, interests, and so on.

This presents tremendous challenges both to teachers and to learners. Such challenges are especially evident in Special Education. In Special Education a great deal of time and effort goes into developing an Individual Education Plan (IEP) for a student. The IEP pays careful attention to the learner's current level of knowledge and skill. It then crafts educational goals and a plan of teaching/learning that is specific for the learner.

We can gain some additional insight into constructivism by looking at some research results produced by Benjamin Bloom. His research showed that with appropriate one-on-one tutoring, the typical "C" student could learn at the level of an "A" student. That is, such tutoring can produce a two-sigma improvement (two standard deviations improvement) in student performance on tests over the material being taught (Bloom, 1954).

Gestalt Theory

Gestalt theory, developed by Max Wertheimer, is a focus on the “whole” rather than the parts (Wertheimer, 1924). Gestalt theory focuses on understanding and on problem solving. As a teacher, you are undoubtedly familiar with evaluating a student’s writing in a holistic manner (perhaps using a rubric). In this approach to evaluation you don’t get bogged down in the small details, such as quality of the handwriting or an occasional error in spelling or grammar. Rather, you focus on how well the student is addressing the problem of effective communication.

The same idea holds for math.

> Suppose a mathematician shows you a proposition and you begin to "classify" it. This proposition, you say, is of such and such type, belongs in this or that historical category, and so on. Is that how the mathematician works?
>
> "Why, you haven't grasped the thing at all," the mathematician will exclaim. "See here, this formula is not an independent, closed fact that can be dealt with for itself alone. You must see its dynamic functional relationship to the whole from which it was lifted or you will never understand it." (Wertheimer, 1924)

Gestalt theory supports discovery learning and project-based learning. It says that learning should not be the rote memorization of tasks. Teachers should not give students problems that can be solved by applying a series of steps learned by rote.

Metacognition

Metacognition is defined as "thinking about thinking." It is a term developed by John Flavell in the mid 1970s.

> Metacognition refers to one's knowledge concerning one's own cognitive processes or anything related to them, e.g. the learning-relevant properties of information or data. For example, I am engaging in metacognition... if I notice that I am having more trouble learning A than B; if it strikes me that I should double-check C before accepting it as a fact; if it occurs to me that I should scrutinize each and every alternative in a multiple- choice task before deciding which is the best one.... Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of those processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete [problem solving] goal or objective. (Flavell, 1976, p. 232)

The term has also come to include the knowledge of one's own cognitive and affective processes and states, and the ability to consciously and deliberately monitor and regulate those processes and states.

Nowadays, metacognition is considered an important idea at all levels of education and in all disciplines. Alan H. Schoenfeld, a University Professor in Cognition and Development, is a leading expert on metacognition in mathematics. Schoenfeld (1992) provides an extensive discussion of problem solving, metacognition, and sense-making in mathematics. These three topics are thoroughly intertwined. In brief summary, sense-making—gaining understanding—lies at the heart of learning mathematics. Metacognition is a valuable aid to sense-making. As one progresses in learning math, he or she can tackle increasingly difficult, non-routine, problems.

> How accurate are you in describing your own thinking? ... [G]ood problem solving calls for using efficiently what you know: if you don't have a good sense of what you know, you may find it difficult to be an efficient problem solver (Schoenfeld, 1987, p. 190).

There has been quite a bit of research on metacognition. One of the general findings is that a significant part of effective learning is to be aware of, and in control of, one's own learning. There are a number of ways to view this finding. For example, it ties in with constructivism. It helps to emphasize the difference between teaching and learning. It is a statement about student-centered learning. It can serve as argument supporting the need for research on the extent to which our schooling process facilitates students having time for metacognition.

Information Processing Theory

Information processing theory draws together ideas on what we know about how the brain processes information. As an example of such knowledge, George Miller (1956) published a paper discussing the ability of a typical person's short-term memory to deal with seven plus or minus two chunks of information at one time. A chunk is any meaningful unit such as digits, words, chess positions, or a person's face. For example, a person might view the letter string *p i g* as three chunks, one letter each. A letter is a familiar chunk if the person is familiar with the alphabet. Or, the person might view this as the word pig, a single chunk.

Such a process of chunking takes some thinking (some mental processing, some encoding and decoding). For example, suppose I am at a conference far from home and I want to telephone my department's secretary. I think of the needed phone number as seven chunks:

- Long distance (which I can translate into the digit 1).
- My area code (which I can translate into 541).
- My university's prefix (which I can translate into 346)
- Four digits that are specific to the secretary's phone.

If I am going to memorize the phone number of a new faculty member at my university, all I really need to do is memorize the last four digits. I can reconstruct the remainder of the phone number from chunks that have meaning to me and that I have previously memorized.

Information processing learning theories focus on four aspects of information processing in a person's brain and mind:

1. Encoding: Information is input through our sensing organs (which provide very short term memory, a fraction of a second up to two seconds, depending on the sense organ) and is attended to. Input information that is not attended to disappears (is forgotten).
2. Short term memory. This has a quite small capacity and stores information for a short period of time, perhaps up to 18 seconds.
3. Long term memory. This has a very large capacity and can store information for an extended period of time.
4. Retrieval: The information is found at the appropriate time, and reactivated for use on a current task. While this is sometimes an easy process, it is sometimes not so easy. Think about your ability to quickly remember the

name of a person that you know. As people age, most get much slower at such retrieval.

Information processing learning theories look at each of the four components listed above, and then suggest teaching and learning processes that can lead to better learning. For example, the "attended to" part of (1) is the focus of "attention theory." What can a teacher do to help get a student to focus his or her attention on the materials being presented? What can a learner do to focus his or her attention on the important aspects of what is to be learned?

The limited size and duration of short term memory suggests that care should be taken to not overload a learner's short term memory, and that both the teacher and the student needs to understand moving chunks of information from short term memory into long term memory.

Storage in long-term memory is highly dependent on the ideas of constructivism and of meaning/understanding. Retrieval is also highly dependent on meaning and understanding.

In terms of these four ideas, the human brain is substantially different than a computer. It is possible to very rapidly input huge amounts of information into a computer storage device. Such information can be stored for a very long time. Retrieval from a computer is not like retrieval from a human brain. Use of a search engine such as Google provides some insight into this. I provide Google with a short sequence of words that describe information that I want to retrieve from the Web. In well under a second, Google provides me with a large number of Web sites that may meet my needs. However, most of these Websites won't prove useful—they don't really make sense in terms of my needs. As I attempt to retrieve information from my brain, sense making is a key issue, and I tend to retrieve information that makes sense in the context that I want to explore.

In summary, when it comes to pure storage and retrieval of data, a computer is far better than a human brain. When it comes to storage and retrieval of information that makes sense to a person, a person's brain may be far better than a computer. Sense making is absolutely critical to learning! Learning without understanding puts one in direct competition with computers, and computers are far better than humans in this endeavor.

Lower-Order and Higher-Order Knowledge and Skills

Educators often talk about students gaining *lower-order knowledge and skills*, and *higher-order knowledge and skills.* In this section we will consider three different approaches to thinking about and possibly defining these terms.

Bloom's Taxonomy

In 1956, a group of educational psychologists headed by Benjamin Bloom developed a classification of levels of intellectual behavior important in learning. Table 3.3 contains some basic information about Bloom's Taxonomy (Bloom's Taxonomy, n.d.)

Taxonomy Terms	Definition and Indicators
Knowledge	Knowledge of dates, places, events, major ideas, and facts. Questions at this level frequently use terms such as list, define, tell, describe, identify, show, label, collect, examine, tabulate, quote, name, who, what, when, where, and so on that can be answered by rote memorization.

Comprehension	Comprehend, understand, and associate meaning with the knowledge you have. Translate knowledge to a new context. Interpret facts. Compare, contrast, order, and group data. Identify and understand cause and effect relationships. Questions at this level frequently make use of terms such as summarize, describe, interpret, contrast, predict, associate, distinguish, estimate, differentiate, discuss, identify causes and predict consequences.
Application	Use your knowledge and comprehension to solve new and novel problems, and to accomplish new and novel tasks. Questions at this level frequently make use of terms such as apply, demonstrate, calculate, complete, illustrate, show, solve, examine, modify, relate, change, classify, experiment, and discover.
Analysis	Use your knowledge, comprehension, and application to find (identify, see) patterns and relationships, organize the parts, and identify related components. Questions at this level frequently make use of terms such as analyze, separate, order, explain, connect, classify, arrange, divide, compare, contrast, select, explain, infer.
Synthesis	Use your knowledge, comprehension, application, and analysis to create new ideas, generalize (perhaps drawing on several different fields), solve complex problems, make meaningful predictions, and draw conclusions. Questions at this level frequently make use of terms such as combine, integrate, modify, rearrange, substitute, plan, create, discover, design, invent, compose, formulate, prepare, generalize, and rewrite.
Evaluation	Drawing upon all of the above: compare, contrast, and discriminate between ideas; assess value and correctness of theories; make choices based on argument; verify value of evidence; and recognize subjectivity. Questions at this level frequently make use of terms such as assess, decide, rank, grade, test, measure, recommend, convince, select, judge, explain, discriminate, support, and conclude.

Table 3.3. Bloom's taxonomy.

Even though it was developed nearly 50 years ago, Boom's taxonomy is still a quite valuable way to look at the range of lower-order to higher-order knowledge and skills. Notice that even at the 2nd level (comprehension) there is an emphasis on transfer of learning—using one's knowledge in new contexts. At the 3rd level (application) the learner is expected to transfer knowledge and comprehension to novel problem situations and tasks.

Data Processing Taxonomy

The field of Computer and Information Science has given rise to the four-point data processing taxonomy scale given in Figure 3.4.

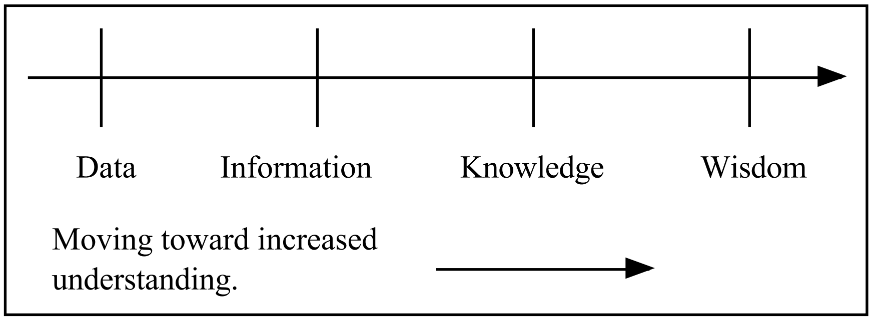

Figure 3.4. Data, information, knowledge, wisdom taxonomy.

The first part of this taxonomy comes from the early days of electronic digital computers. As computers first came into use in businesses, they were thought of as data processing machines. The focus was on the input, storage, and processing of raw, unprocessed data to produce simple documents such as invoices and payroll checks. However, it soon became evident that computers could analyze data to produce informative reports (information). For example, an analysis of the ZIP codes and dollar amounts on invoices provides information about the number of sales and average size of sales in various postal zones. This information might be used to help design a marketing campaign.

In more recent times, many businesses have found that computers can be used to process information to produce knowledge, somewhat in the manner that a person draws together diverse information to gain knowledge about a topic and then makes recommendations about possible actions to take based on this knowledge.

One can memorize data, and parrot it back. One processes data (organizes it into meaningful chunks or arrangements) to produce information. Of course a student can memorize and parrot pieces of information with little understanding or ability to make use of the information. Knowledge is a step further along on the taxonomy. It involves understanding and ability to make use of the data and information to answer questions, solve problems, make decisions, and so on. Wisdom has to do with using one's knowledge in a responsible (wise) manner. Some people are now asking about the nature or extent of wisdom that can be programmed into a computer.

Robert Sternberg has taken the position that wisdom can and should be taught in schools, even at the elementary school level.

> I define wisdom as the application of intelligence and experience toward the attainment of a common good. This attainment involves a balance among (a) intrapersonal (one's own), (b) interpersonal (other people's), and (c) extrapersonal (more than personal, such as institutional) interests, over the short and long terms. Thus, wise people look out not just for themselves, but for all toward whom they have any responsibility. (Sternberg, 2002.)

One of the central issues in defining the terms data, information, knowledge, and wisdom is the role of understanding and meaning making. Each of us tends to have his or her own definition of terms **understanding** and **meaning**. Perhaps you feel that only a human brain can have understanding. However, it is interesting to explore the possibility that a computer system might have some type of understanding. We will return to this topic later in this book as we explore some roles of computers in math education.

For many years, it has been common to say that an electronic digital computer is a machine designed for the input, storage, processing, and output of data and information. Although computers or a central processing unit in a computer are often called "brains," it is clear that there are huge differences between a human brain and a computer.

Expertise Level of Learner

Figure 3.5 illustrates a learner who is at a certain point on an expertise scale in one specific area of expertise. This learner has knowledge and skills that place him or her at this point. From the point of view of this learner, additional knowledge and skills needed to move up the scale are higher-order. The knowledge and skills that have been learned well are now lower-order.

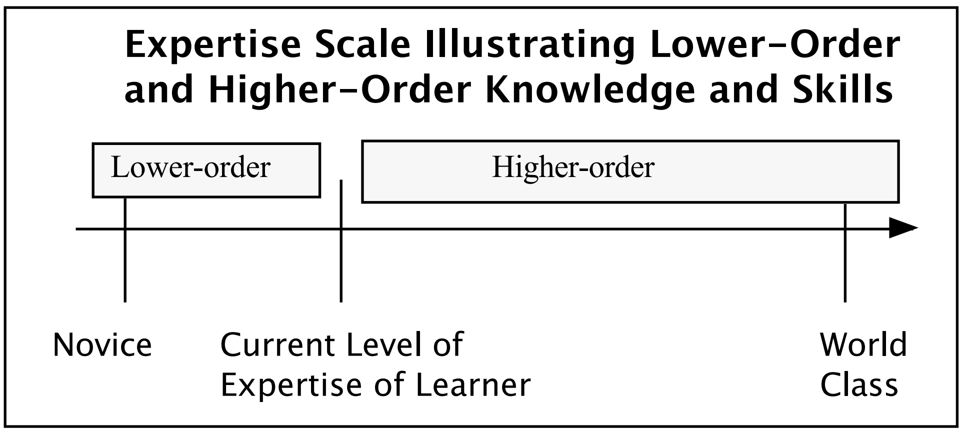

Figure 3.5. Diagram illustrating lower-order and higher-order.

Notice how this fits in with constructivism. Constructivism suggests that instruction and expected learning should be pitched approximately at the level of the large black dot in Figure 3.6.

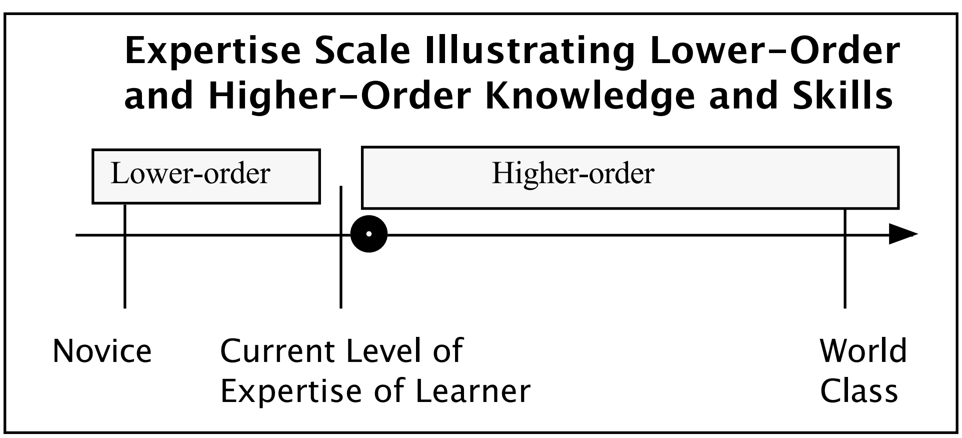

Figure 3.6. Constructivism suggests instruction at the level of the large black dot.

However, this analysis does not help us much as we design and implement a math lesson. Three obvious difficulties are:

1. There are many different combinations of knowledge and skill that can lead to a student having a particular level of expertise in a specific area. Thus, even if all students get exactly the same test score on a test of prerequisite knowledge (that is, all demonstrate the same level of expertise as measured by the test) and all of the students have taken the same previous math courses, the students may differ widely in their specific combinations of knowledge and skill, as well as their ability to transfer their knowledge and skill to the new math topic.
2. The knowledge and skills leading to a particular level of expertise have been acquired throughout a student's lifetime. Expertise depends on many variables such as habits of mind, attitudes, perseverance, amounts and frequency of practice (experience), and lots of other things that are not measured on the test.
3. Logical/mathematical is one of the eight multiple intelligences identified by Howard Gardner. Students vary considerably in their logical/mathematical level of intelligence. Spatial intelligence is also on Gardner's list, and it is very important in learning and using math.

This type of analysis suggests that one of the goals in education should be for each student to learn a lot about himself or herself as a learner in math and in other subject areas. Math education should include a significant emphasis on learning one's own strengths and weaknesses in learning to learn math. As a learner gains in mathematical maturity, he or she should take increasing responsibility for his or her learning.

Concluding Remarks

This chapter contains general background information about transfer of learning, various learning theories, and the idea of lower-order and higher-order knowledge and skills. The content of this chapter is applicable to curriculum, instruction, and assessment at all grade levels and in all subject areas.

When this content is examined just from a math education point of view, we see a number of weaknesses in our math education system. In brief summary, our math education system places far too much emphasis on lower-order knowledge and skills. This contributes to relatively slow student progress in understanding math and being able to transfer their math knowledge sand skills outside the context or situation in which it is learned.

Recommendations Emerging from Chapter 3

3.1 There are a number of different learning theories. As a teacher, you need to understand basic ideas of behavioral learning theories and of cognitive learning theories. As you design and implement a math lesson, give careful consideration to the emphasis you are placing on automatic (non thinking) types of learning and on thinking and understanding types of learning.

3.2 Although a teacher teaches and creates a "situated" learning environment, it is a student who constructs knowledge. As a teacher, you want your students to construct knowledge that they can use in the future—both in school and outside of school. As you create a math lesson, pay careful attention to how your instruction is consistent with and supportive of ideas of situated learning, constructivism, and low-road/high-road transfer.

3.3 Constructivism and developmental theory are intertwined in student learning. If the math content being taught is too much below or too much above a student's mathematical maturity, little learning will occur.

Activities and Questions for Chapter 3

1. Think about your personal level of mathematical maturity and how it affects how you teach math. Are you at a formal operations level in your mathematical maturity? Provide some examples of how your level of mathematical maturity seems to affect how you teach math and the math learning expectations you place on your students.
2. Think about a variety of your low-road (near transfer) capabilities. (For example, you may be an excellent touch keyboarder.) How did you acquire these capabilities? Perhaps you can estimate how much time it took for each capability. Do you notice a degradation in your near transfer capability over time, if you do not use the knowledge and skill?
3. Think about what your understanding of high-road transfer was before you read this chapter. Is this an idea that you have been taught, and that has been explicitly used in the instruction you have received in a variety of courses? If you have not had much previous experience with high-road transfer, select a few areas in which you are good at far transfer, and explore them in terms of the high-road theory of transfer.
4. Select a math topic that you have taught or are preparing to teach. Do a very careful introspection of your knowledge and understanding of what you think a typical student should know and what a typical student actually does know before beginning the study of this topic. Think about ways to determine if students have this knowledge and understanding. Think about the learning that will occur from your instruction if students lack this prerequisite knowledge and understanding.
5. Select a course or long unit of math instruction that you have taken in the past. Do a careful analysis of the nature of this instruction from a Situated Learning, transfer of learning point of view. Your analysis should include a focus on learning for the next course, learning for transfer to non-math courses, learning for transfer to general "real world" outside of school setting, and learning for transfer to being a math teacher.

Chapter 4
Brain/Mind Science

Cogito, ergo sum. I think, therefore I am. (René Descartes, 1596-1650)

Intelligence is what you use when you don't know what to do. (Jean Piaget)

The mind is not a vessel to be filled but a fire to be kindled."(Plutarch; Roman historian; 46 AD–120 AD.)

The fields of brain science and mind science are now becoming thoroughly intertwined. In this book we use the term brain/mind science to designate the combined discipline. The human brain is a very complex organ and it has considerable plasticity. One's brain is changed by learning, as well as a number of other things such as disease, injury, drugs, and aging.

Significant progress in mind science has occurred over the past 150 years. In the past decade, braining imaging technology has developed to a level where rapid and significant progress is occurring in brain science.

This chapter provides a brief introduction to brain/mind science and its contributions to teaching and learning. The main emphasis is on math. However, there is also quite a bit of emphasis on reading, since approximately 70% of students who have reading difficulties also have math difficulties.

What is Brain Science?

Brain science is now one of the "buzz words" in education. Many people use the term in an all-inclusive manner that covers both the science of the mind (psychology) and the science of the brain (neuroscience). However, the work in psychology on the science of the mind goes back more than a hundred years, while significant progress in neuroscience is quite recent. In this book we use the term brain/mind science to designate the discipline that focuses on the study of the brain and the mind.

John T. Bruer is president of the James S. McDonnell Foundation. He has written extensively about brain/mind science and the McDonnell Foundation has provided substantial funding for research in this area. An excellent introduction to the field is available in Bruer (1999). In this article, Bruer talks about a long-standing schism between research in the science of the mind (psychology) and research in the science of the brain (neuroscience).

> It is only in the past 15 years or so that these theoretical barriers have fallen. Now scientists called cognitive neuroscientists are beginning to study how our neural hardware might run our mental software, how brain structures support mental functions, how our neural circuits enable us to think and learn. This is an exciting and new scientific endeavor, but it is also a very young one. As a result we know

> relatively little about learning, thinking, and remembering at the level of brain areas, neural circuits, or synapses; we know very little about how the brain thinks, remembers, and learns (Bruer, 1999).

The Human Brain

Current research suggests that Homo sapiens developed about 200,000 years ago. The oldest fossil evidence for anatomically modern humans is about 130,000 years old (Homo sapiens, n.d.).

An average adult brain weighs about three pounds and contains more than 100 billion neurons. These neurons communicate with each other via a network averaging about 5,000 dendrites per neuron. The number 100 billion is an impressively large number. However, think about the hard drive storage on a modern microcomputer. Such storage capacity is now measured in gigabytes—billions of bytes. The price per gigabyte of disk storage is now under a dollar. However, it is totally incorrect to equate neurons with bytes of storage. Storage in the brain is done via the dendrites

> The human brain controls memory, vision, learning, thought, consciousness and other activities. By means of electrochemical impulses the brain directly controls conscious or voluntary behavior. It also monitors, through feedback circuitry, most involuntary behavior and influences automatic activities of the internal organs.
>
> During fetal development the foundations of the mind are laid as billions of neurons form appropriate connections and patterns. No aspect of this complicated structure has been left to chance. The basic wiring plan is encoded in the genes.
>
> …
>
> The brain's billions of neurons connect with one another in complex networks. All physical and mental functioning depends on the establishment and maintenance of neuron networks. (Elert, n.d.).

The human brain is immensely complex, and even the brains of identical twins are not identical. Moreover, the human brain is continually changing, because learning produces change in the brain. Finally, we know that the human brain has great plasticity, allowing major changes in the human brain (often thought of as rewiring) to occur over time, even in adults.

Here is a poignant example. In recent years, researchers have discovered that a small percentage of children are severely speech delayed because the phoneme processors in their brains function too slowly. This understanding led to the development of some highly interactive intelligent computer-assisted learning (HIICAL) software specifically designed to help in the rewiring of such children's brains (Fast ForWord, n.d.). This intervention appears to help about 85% of the children to speed up their phoneme processors so that they can understand ordinary speech. Similar software is used in working with people who receive cochlear implants, as they learn to regain a useful level of hearing.

In discussing the development of a brain, it is common to talk about "nature" and "nurture." At the very beginning of their development, identical twins essentially have the same genes, which we think of as contributions from "nature." Even while in the womb, however, these are significant differences in "nurture," and so by birth the brains of identical twins have significant differences.

We now have the technology to study how differences in genes between two people contribute to major differences between the people, such as one being dyslexic and another not being dyslexic. What is happening is that progress in study of the human genome is combining with progress in braining imaging to identify specific genes and functioning of parts of the brain that relate to student difficulties in learning to read (dyslexia) and learning to calculate (dyscalculia).

As a preservice or inservice teacher, you know that there are large differences among the children you teach. For some students, the differences from "average" are so large that the students are identified as having various types of learning disabilities. While estimates of the percentage of students with significant learning disabilities (LD) vary considerably, it may well be that more than 20-percent of students fall into this category.

And, of course, you know that some students learn much better and faster than others. The definitions used for Talented and Gifted vary considerably, but it is relatively common to use definitions that are met by five to ten percent of the students (Moursund, 2004a, Chapter 8; Neag Center, n.d.).

Intelligence Quotient (IQ)

For many years, psychologists studying the human brain/mind have tried to measure its capabilities. Quite a bit of this work has focused on defining "intelligence" and measuring a person's intelligence.

> The concept that intelligence could be or should be tested began with a nineteenth-century British scientist, Sir Francis Galton. Galton was known as a dabbler in many different fields, including biology and early forms of psychology. After the shake-up from the 1859 publishing of Charles Darwin's "The Origin of Species" (read the e-book now!), Galton spent the majority of his time trying to discover the relationship between heredity and human ability (History of I.Q., n.d.).

Ways to measure intelligence were first developed more than 120 years ago, and this continues to be an active field of research and development. A very simplified summary of the current situation consists of:

1. There are a variety of IQ tests that produce one number or a small collection of numbers as measures of a person's intelligence.
2. The "one number" approach (the general intelligence, or "g" factor) can be traced back to Charles Spearman who proposed the idea in 1904, and it still has considerable prominence.
3. Expert estimates suggest that anywhere between 30 and 80 percent of the variation in IQ scores is determined by genetic factors, with 50 to 60 percent being the most commonly accepted range (Niabett, 1998). The 50 to 60 percent figure corresponds to a correlation of about .71 to .77.
4. There have been a number of studies of possible genetic differences that might affect IQ between "White" Americans and "African" Americans. In an

analysis of this research literature, Niabett (1998) reports, “The studies most directly relevant to the question of whether the Black/White IQ gap is genetic in origin provide no evidence for a correlation between IQ and African (rather than European) ancestry.” That is, the differences are due to “nurture,” not genetics.

5. Many people have proposed and discussed the idea of multiple intelligences. In the past two decades the work of Howard Gardner has helped to publicize this idea. Logical-mathematical and spatial are two of the eight Multiple Intelligences identified by Howard Gardner (Gardner, n.d.).
6. Various measures of intelligence correlate well with the rates of student learning. Thus, students in school at the lower end of the IQ scale tend to learn approximately half as fast as student in the middle of the IQ scale, while students at the higher end of the IQ scale tend to learn approximately twice as fast as those in the middle of the IQ scale.
7. Intelligence is a product of a combination of nature and nurture. Over the past few decades, IQ has been increasing at a significant pace (Sternberg, 1997).

Intelligence, and Increasing IQ

Robert Sternberg is a prolific researcher and author in the field of intelligence. The following is quoted from Sternberg (1997):

> Technology is changing society in many ways—some quite unexpected. It's been credited with much of the dramatic rise in IQ scores over the past 30 years.
>
> …
>
> With all the moaning and groaning we constantly hear about the way schools educate our children, we often lose sight of an important and startling fact: intelligence, as measured by so-called intelligence quotients, or IQs, has been increasing over the past 30 years, and the increases are large—about 20 points of IQ per generation for tests of fluid intelligence such as the Raven Progressive Matrices, which require flexible thinking with relatively abstract and novel kinds of problems.

Fluid intelligence refers to one’s ability to solve novel problems that do not depend on formal school and acculturation.

This phenomenon of increasing IQ has lead to re-norming of IQ tests, so that 100 remains at the midpoint of the scale. There seems to be considerable agreement among researchers that increasing IQ is a result of richer cognitive environments. Here is a brief report about brain research done on rats.

> Another piece of the puzzle was provided by Bill Greenough of the University of Illinois. He exposed one group of rats to a stimulating environment—toys, colors, playmates, exercise devices, challenges. A comparison group of rats was housed in routine laboratory cages with little stimulation.
>
> When Greenough looked at the brains of the animals in the two groups he found the key to building brain power. **The animals living in the stimulating**

> **environment had 25 percent more connections between their brain cells than the control rats, and they were a lot smarter** (Kotulak, 1996). [Bold added for emphasis.]

Significant changes in the brain go on throughout one's lifetime. It is well known that you can "teach an old dog new tricks." We want to raise our children in intellectually rich home and school environments.

Rate of Learning

Elementary school teachers know that there are large differences in how fast various students learn. Research indicates that this difference may be as large as a factor of five (MacDonald, n.d.). Stated in simpler terms, this means that a typical class may have one or more students that learn less than half as fast as the average, and one or more that learn more than twice as fast as average. The combination of one-half and twice produces a factor of four between the slower and faster learners.

You know, of course, that students differ significantly in their interests, their areas of relative strength, and their areas of relative weakness. Gardner's and other researchers' work on multiple intelligences suggest that a student's intelligence in different areas may vary considerably. As an example, my logical/mathematical IQ is well above average, but my spatial IQ is below average. And, my children decided that I am "tune deaf," which was a nice way of suggesting that my musical IQ leaves much to be desired.

There has been quite a lot of research on the math learning of students classified as having general learning disabilities. The term learning disability (LD) has been given a legal definition:

> The regulations for Public Law (P.L.) 101-476, the Individuals with Disabilities Education Act (IDEA), formerly P.L. 94-142, the Education of the Handicapped Act (EHA), define a learning disability as a "disorder in one or more of the basic psychological processes involved in understanding or in using spoken or written language, which may manifest itself in an imperfect ability to listen, think, speak, read, write, spell or to do mathematical calculations."
>
> The Federal definition further states that learning disabilities include "such conditions as perceptual disabilities, brain injury, minimal brain dysfunction, dyslexia, and developmental aphasia." **According to the law, learning disabilities do not include learning problems that are primarily the result of visual, hearing, or motor disabilities; mental retardation; or environmental, cultural, or economic disadvantage. Definitions of learning disabilities also vary among states.** [Bold added for emphasis.] Accessed 3/4/04: http://www.kidsource.com/NICHCY/learning_disabilities.html.

The bold faced part in the above quote suggests some of the difficulties that educators face. In essence, from a teacher point of view two different students may have nearly identical learning difficulties. However, one is classified as LD and is eligible for special services. The other is not classified as LD, and so extra funding may not be available.

Students with learning disabilities tend to learn math much slower than students without such disabilities. Here are a few poignant points from Miller and Mercer (1997).

1. LD students experience difficulty in learning computation, problem solving, and other math starting at the earliest grade levels and continuing throughout their schooling.
2. LD students tend to make one-half of a grade level of math learning progress per school year.
3. The math learning of LD students tends to plateau at some place at the 4^{th} to 5^{th} grade levels as they continue through secondary school. After that, the rate of forgetting tends to equal the rate of learning.

You will notice that (2) above is consistent with information given in Chapter 4 about general rates of learning. Point (3) presents an interesting challenge to top-down school, standards-based school reform efforts such as No Child Left Behind.

In Chapter 3 we briefly discussed constructivism. Differences in students' rates of learning play havoc with a teacher's attempts to teach in a constructivist manner. Consider, for example, students who make .75 of a year of math learning progress per year (as compared to average) and those who make 1.25 years of math learning progress as compared to average. Suppose that these rates of learning math begin at birth and continue year after year. Table 4.1 illustrates this situation.

Age of learner	**Comment**	**Math age level of "slow" math learner**	**Math age level of "average" math learner**	**Math age level of "fast" math learner**
1		0.75	1	1.25
2		1.5	2	2.5
3		2.25	3	3.75
4		3	4	5
5	Kindergarten	3.75	5	6.25
6	Grade 1	4.5	6	7.5
7	Grade 2	5.25	7	8.75
8	Grade 3	6	8	10
9	Grade 4	6.75	9	11.25
10	Grade 5	7.5	10	12.5

Table 4.1 Slow, average, and fast learners of math.

Notice the difference in mathematical "math age level" between the slow math learners and the fast math learners when they enter school as the kindergarten or first grade level. If the kindergarten or first grade teacher tends to aim the math curriculum at the middle of the class, this instruction will be way over the heads of some students, and it will be boring and unproductive for other students. One solution to this difficulty is providing a great deal of individualization of instruction. Computer-assisted learning can be helpful in this endeavor.

Reading and the Brain

This short section is about students learning to read a natural language. A paragraph at the end of the section relates this section to learning math.

A normal human brain is "wired" to be able to learn to a natural language. Throughout the world children learn to understand spoken language and to talk—without going to school! Indeed, if raised in a bilingual or trilingual environment, children become bilingual or trilingual.

The situation for learning to read is certainly not the same as the situation for learning to speak and listen. It takes years of informal instruction, formal instruction, and practice to develop a reasonable level of skill in reading. One benchmark for progress in learning to read is making a transition from learning to read to *reading to learn.* In the current education system in the United States, approximately 70-percent of students reach or exceed this stage by the end of the third grade. There students tend to transition relatively smoothly into a fourth grade and higher grade level curriculum that places more and more emphasis on reading to learn. In our current educational system, the expectation is that by approximately the seventh grade students will be using reading as their dominant aid to learning.

The 70-percent figure stated above means, however, that approximately 30-perenct of students have not yet met the reading to learn benchmark by the end of the third grade. Some of these students are diagnosed as being dyslexic. Historically, this term was applied to students with normal or above normal IQ who had a great deal of difficulty in learning to read. Recent brain research has discovered that the brains of many students are wired differently than those of students who make "normal" progress in leaning to read (Shaywitz, 2003). Sally Shaywitz estimates that perhaps as many as 20-percent of all children have a significant level of dyslexia.

There are neurological explanations for why some students have reading difficulties. At the current time brain scientists are just beginning to identify some of the genetic sources of reading difficulty. Here is an example of such findings, quoted from the abstract of Mikko Taipale et al. (2003). You will notice that the "language" of a gene researcher is quite a bit different than the language of a typical elementary school teacher.

> We report here the characterization of a gene, *DYX1C1* near the *DYX1* locus in chromosome 15q21, that is disrupted by a translocation t(2;15)(q11;q21) segregating coincidentally with dyslexia. … We conclude that *DYX1C1* should be regarded as a candidate gene for developmental dyslexia. Detailed study of its function may open a path to understanding a complex process of development and maturation of the human brain.

Math is a language. We want students to learn to read, write, speak, listen, and think in this language. The brain/mind research on learning mathematics is not as extensive as the research on learning a natural language. As I have studied the math literature, I have looked for a parallel to the idea of learning to read and then reading to learn. At what grade level do we expect students to have progressed far enough in reading and doing math so that they begin to "read and do math" to learn math? My current knowledge of math education produces the answer, "I don't know, and our math education system does not set specific goals in this are."

Math and the Brain

Brain imaging techniques now provide us with information about which parts of the brain are involved in accomplishing different sorts of tasks, such as reading versus doing math.For example:

> Through separate studies involving behavioral experiments and brain-imaging techniques, the researchers found that a distinctly different part of the brain is used to come up with an exact sum, such as 54 plus 78, than to estimate which of two numbers is closer to the right answer. Developing the latter skill may be more important for budding mathematicians.

> …
>
> In addition to shedding light on how mathematicians' brains work, the researchers' results may have implications for math education. If the results of these studies on adults also apply to children, the studies imply that children who are drilled in rote arithmetic are learning skills far removed from those that enrich mathematical intuition, Professor Spelke said.
>
> "Down the road, educators may look harder at the importance of developing children's number sense"—for example, their ability to determine a ballpark answer rather than a specific answer, she said. Number sense is considered by some to be a higher-level understanding of mathematics than rote problem-solving (Halber,1999).

By now, you (the reader) may be getting tired of reading over and over again statements about rote memory and understanding as they apply to problem solving. Be assured, however, that you will read still more as you continue in this book and read other math education books and esearch literature. I am presenting you with multiple perspectives and sources of evidence on this aspect of math education. My thesis is that our rote memory approach to math education, when accompanied with little understanding on the part of students, is a poor way to approach trying to achieve current goals of math education. However, some rote memory learning in math is essential.

There are a number of similarities learning the language of mathematics and learning a natural language. It is not surprising that difficulty in learning to read is reasonably strongly linked to difficulty in learning math.

The following quoted material hints at the dyslexia-math learning situation. The material was posted on *08/24/03* to a discussion board by a mother of identical twin sons.

> Hi. I have a strange problem that I hope someone can offer some good advice for—I have identical twin sons who are 17-year-old seniors in high school. They have struggled with reading all through school, but have worked very hard to succeed. One son has maintained a 4.0 GPA all through high school and the other has a 3.6 GPA.
>
> They take difficult classes, like honors Chemistry, Physics, honors Algebra, etc. In addition, both are varsity athletes and are involved in many school activities, such as peer counseling, outdoor lab leaders, key club, etc.
>
> …
>
> The problem is they can NOT succeed on timed reading tests—we even paid a private tutor to help them improve their chances on the SAT—but neither of them can get above 1000 on the SAT …
>
> I have always suspected they were dyslexic (their father and sister both are)—so I just had them tested and they came back as "significantly dyslexic." **For example, on the untimed math concepts test, one of them scored in the 99 percentile, but scored only in the 1 percentile in the timed math test.** (Boldface added for emphasis.) (Accessed 2/13/04: http://www.voy.com/32297/2/2128.html.)

As a reader, you should recognize that this quote is not from a research paper. Rather, it is part of a mother's plea for help. Research in brain/mind science is slowly producing such help!

Seven Plus of Minus Two

In a 1956 article, George Miller noted, "Everybody knows that there is a finite span of immediate memory and that for a lot of different kinds of test materials this span is about seven items in length." The article then goes on to explore how 7 ± 2 seems to be a magical quantity 7 ± 2 appearing in many different measures of human sensory and brain processing capabilities. The article includes a heavy emphasis on how to make more effective use of short-term memory by chunking information (putting a number of individual items into a chunk, that is then deal with as a single item).

It turns out that short term memory span is very important in problem solving and other higher-order cognitive tasks. Thus, there has been a lot of research on short-term memory and how to "enhance" its capabilities.

> The contrast of the terms *bit* and *chunk* also serves to highlight the fact that we are not very definite about what constitutes a chunk of information. For example, the memory span of five words that Hayes obtained when each word was drawn at random from a set of 1,000 English monosyllables might just as appropriately have been called a memory span of 15 phonemes, since each word had about three phonemes in it. Intuitively, it is clear that the subjects were recalling five words, not 15 phonemes, but the logical distinction is not immediately apparent. We are dealing here with a process of organizing or grouping the input into familiar units or chunks, and a great deal of learning has gone into the formation of these familiar units.
>
> In order to speak more precisely, therefore, we must recognize the importance of grouping or organizing the input sequence into units or chunks. Since the memory span is a fixed number of chunks, we can increase the number of bits of information that it contains simply by building larger and larger chunks, each chunk containing more information than before (Miller, 1956).

The building of "larger and larger chunks" is a fundamental concept in learning and problem solving. For example, suppose you want to memorize a long sequence of binary digits (a sequence of 0's and 1's). Table 4.2 contains conversions between binary numbers and base 10 numbers. Suppose as you view the string of binary digits to be memorized, you divide them into groups of three and then memorize the corresponding base 10 number. In that way, memorizing 21 binary digits is like memorizing 7 base 10 digits. However, this only works if you have a high level of automaticity in converting groups of three binary digits into a base 10 digit, and then back again.

Binary number	**Base 10 number**
000	0
001	1
010	2
011	3
100	4
101	5

110	6
111	7

Table 4.2 Binary to base 10 conversion table.

George Miller describes this chunking process as a recoding, or translation scheme. Miller (1956) notes, "Apparently the translation from one code to the other must be almost automatic or the subject will lose part of the next group while he is trying to remember the translation of the last group."

Math is a language with its own vocabulary. If the vocabulary being used in a "conversation" (for input orally or in writing) is sufficiently familiar to the receiver, then a great deal of information can be communicated in a small number of chunks. Without the automaticity, this cannot occur.

Moreover, for many students, math is learned by rote memory, with little or no understanding. For such students, a sequence of math words and symbols is much like a sequence of nonsense words and symbols. Such sequences are difficult to learn, difficult to chunk into smaller numbers of units, and difficult to recall from memory.

One of the most important ideas in learning mathematics (gaining in math maturity, math expertise) is learning chunks that have meaning. Storing and retrieving math information, and thinking, reading, writing, and talking in math involve rapid (automatic) chunking and unchunking.

The above discussion, when combined with some of the ideas in Chapter 3, pinpoint two important ideas for math education:

1. Rote memorization for automaticity both in near transfer (low-road transfer) and for the automaticity needed in chunking and unchunking.
2. Understanding needed in making use of chunks in both short-term and for storage and retrieval using long-term memory.

Both automaticity and understanding are essential. The issue in math education is achieving an appropriate balance between the two. What constitutes an appropriate balance varies from student to student. I remember carrying on a conversation a number of years ago with one to the top research mathematicians in this country. He indicated that he was terrible at doing arithmetic computations!

Piaget's Developmental Theory

Piaget's developmental theory discusses various stages of development and his work has proven to be quite important in education. Very roughly speaking, Piaget thought of these stages as being driven by "nature" rather than by "nurture." The brain of a newborn child is about 350 cc in size, and that of an adult is about 1,500 cc in size. This brain development is, to a great extent, programmed by genetics. Piaget's developmental theory is summarized in Table 4.3 (Huitt and Hummel, 1998).

Approximate Age	Stage	Major Developments
Birth to 2 years	Sensorimotor	Infants use sensory and motor capabilities to explore

		and gain understanding of their environments.
2 to 7 years	Preoperational	Children begin to use symbols. They respond to objects and events according to how they appear to be
7 to 11 years	Concrete operations	Children begin to think logically. In this stage (characterized by 7 types of conservation: number, length, liquid, mass, weight, area, volume), intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking develops (mental actions that are reversible).
11 years and beyond	Formal operations	Thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts. Only 35% of children in industrialized societies have achieved formal operations by the time they finish high school.

Table 4.3. Piaget's Stages of Cognitive Development

The Piagetian scale of cognitive development does not refer to any specific area of cognitive development. Here is a slight expansion of the bottom right corner of the table:

> **Formal Operations.** In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts. Early in the period there is a return to egocentric thought. **Only 35% of high school graduates in industrialized countries obtain formal operations; many people do not think formally during adulthood** (Huitt and Hummel, 1998). [Bold added for emphasis.]

I must admit that I was astounded when I first encountered this piece of information. Further Web research produced the following statement about college students (Gardiner, 1998):

> Many studies suggest our students' ability to reason with abstractions is strikingly limited, that a majority are not yet "formal operational."

The information given in the two quotes is consistent—we expect the percentage of college students at formal operations to be higher than percentage who are high school graduates. Such information suggests that much of attempt to have students learn while in school may be far above their developmental level. We will discuss this topic more in the next section.

Developmental Theory in Math

During the 1950s, Dutch educators Dina and Pierre van Hiele focused some of their research efforts on developing a Piagetian-type scale just for geometry (van Hiele, n.d.). It is a five-level scale, and it does not provide approximate age estimates. See Table 4.4.

Stage	Description
Level 0 (Visualization)	Students recognize figures as total entities (triangles, squares), but do not recognize properties of these figures (right angles in a square).
Level 1 (Analysis)	Students analyze component parts of the figures (opposite angles of

	parallelograms are congruent), but interrelationships between figures and properties cannot be explained.
Level 2 (Informal Deduction)	Students can establish interrelationships of properties within figures (in a quadrilateral, opposite sides being parallel necessitates opposite angles being congruent) and among figures (a square is a rectangle because it has all the properties of a rectangle). Informal proofs can be followed but students do not see how the logical order could be altered nor do they see how to construct a proof starting from different or unfamiliar premises.
Level 3 (Deduction)	At this level the significance of deduction as a way of establishing geometric theory within an axiom system is understood. The interrelationship and role of undefined terms, axioms, definitions, theorems and formal proof is seen. The possibility of developing a proof in more than one way is seen.
Level (Rigor)	Students at this level can compare different axiom systems (non-Euclidean geometry can be studied). Geometry is seen in the abstract with a high degree of rigor, even without concrete examples.

Table 4.4. Van Hiele developmental scale for geometry.

Traditionally, the majority of high school geometry courses is taught at Level 3. The van Hieles also identified some characteristics of their model, including the fact that a person must proceed through the levels in order, that the advancement from level to level depends more on content and mode of instruction than on age, and that each level has its own vocabulary and its own system of relations. The van Hieles proposed sequential phases of learning to help students move from one level to another.

It is interesting to compare Level 3 (Deduction) in the van Hiele scale with the top level (Formal Operations) of the Piaget scale. To me, it appears that these two levels are about the same. This suggests to me:

1. A formal proof-oriented secondary school geometry course is beyond the cognitive and geometric developmental level of the great majority of high school students. This statement becomes even more important if we consider students at the 9th or 10th grade level, when such a course is frequently taught.
2. It is likely that more advanced rigorous high school math courses are beyond the cognitive and mathematical developmental level of the great majority of high school students.

There has been some useful research in mathematical developmental theory. In this discussion we will be talking about children who do not have significant learning disorders.

Here is brief a summary of some of the things we know about mathematical development.

1. Humans (and a number of other animals) are born with a certain amount of innate mathematical knowledge/skill. Very young infants have the ability (in some sense) to "count" up to three (Piazza and Dehaene, 2003). Some other animals has counting abilities similar to young human children (Hauser, 2000).

2. Children are born with and/or soon develop a significant level of spatial sense, spatial reasoning, and so on. (Hunter-gathers who couldn't find their way home faced great perils.)
3. To a very large extent, mathematical development depends on "nurture." In this document we use the term "mathematical maturity" when talking about mathematical development.
4. A weak mathematical home environment and a weak and poorly taught math curriculum lead to very slow progress in mathematical development. A strong mathematical home environment and a strong and well-taught math curriculum lead to a much faster pace of development of mathematical maturity.
5. All of the students with mental capabilities that allow them to attend and participate in school can learn math. The statement, "I just can't do math." that ones hears so often might best be responded to with the statement, "Hogwash!
6. Logical/mathematical is one of the eight Multiple Intelligences identified by Howard Gardner. As with general intelligence or any specific type of intelligence, there are wide variations between the extremes one sees of students in school. Since math is a vertically structured discipline, variations in innate logical/mathematical ability tend to be amplified by our formal schooling process.
7. Many elementary school teachers have not achieved formal operations in their mathematical maturity. This is a significant detriment to their helping to move their students towards this level of mathematical maturity.
8. The discipline of mathematics has been steadily growing in depth and breadth for more than 5,000 years. From both a learner and a user point of view, "higher mathematics" tends to be quite abstract, and it is certainly at the formal operations level on the Piagetian developmental scale.

Several of these points are illustrated by the work of Seymour Papert (Papert, n.d.). Papert did five years of post doctorate work under the supervision of Jean Piaget. He then when on to lead in the development of the Logo programming language, did extensive research on children learning in a Logo environment, did research in artificial intelligence, and so on. Papert's observation is that in a Logo computer environment and with appropriate teachers and materials, quite young students can make rapid progress toward achieving formal operations in general, and in achieving formal operations in certain aspects of math.

Logo has a long history of use in elementary school, not only in the US, but in a number of other countries. While not as widely used as in the past, it still has a broad base of strong supporters. Research on the use of Logo in elementary schools suggests:

1. If the teacher has a good understanding of Logo, teaching problem solving, and teaching for transfer, then very good student learning occurs.

2. In the typical use of Logo in schools, the conditions specified in (1) do not hold. In that case, typically relatively little is achieved by having students make use of Logo.

Organizations such as the National Council of Teachers of Mathematics have developed quite detailed scope and sequence for the K-12 math curriculum. My observation is that some of the widely accepted scope and sequence is not consistent with our growing knowledge of development theory in mathematics. Here are a few summary statements in this area:

1. The human mind has trouble learning and understanding probability. Research suggests that learning for understanding in this topic requires students to be at a formal operations level (Soen, 1997). Thus, at the K-12 level, instruction in this topic is typically "over the heads" of the developmental level of most students.
2. Statement (1) also holds for ratio and proportion, and much of what we want students to learn about doing arithmetic with fractions.
3. The number line is a somewhat abstract concept. Many students entering the first grade do not have an understanding of the number line at a level that is consistent with what the curriculum is expecting, and this difficulty persists as expectations increase at higher grade levels.

One way to detect a mismatch between student math maturity and the math curriculum is to look for places where the students "just don't seem to get it" and many seem to take the memorize and regurgitate approach. If your best efforts to teaching for understanding seem to be unsuccessful, you may be encountering a situation in which the mathematical maturity of your students is inappropriate to the task of learning what you are trying to teach. In that situation, you are well advised to move back to topics that are at a more appropriate mathematical developmental level for your students, and use these topics to build increased mathematical maturity.

Numbers and Number Sense

Quoting a famous mathematician Leopold Kronecker (1823-1891): "God made the natural numbers; all else is the work of man." If we go back about 11,000 years ago, all people on earth were hunter-gatherers. It helped to have good spatial sense—to not get lost when out hunting. Thus, we can understand that humans tend to have some built-in ability to learn to deal with the geometry of being a hunter.

We know from research on very young babies that humans have a modest amount of built-in sense of number—roughly speaking, the ability to distinguish amoung the quantities 1, 2, and 3. Research conducted in tribes that have been isolated from the progresses of "modern" civilizations indicate significant language differences in the area of numbers and counting.

> As noted in the last chapter "Numbers and Counting", the history of numerical thought seems to proceed as follows. First, we discover numbers, which are discrete quantities. Second, we invent physical tokens (strings, stones, bones, etc.) to represent numbers. Third, we invent words and symbols to represent numbers. This last step presents the *problem of numeration*—how to represent numbers by words and symbols—and a *system of numeration* represents an attempt to solve this problem.

> Different cultures have addressed this problem in many different ways. For example, there are quite a few "primitive" languages in which the number-words include only ‘one’, ‘two’, and ‘many’, or even ‘one’ and ‘many’. Most languages, however, have a large variety of number words; for example, English has infinitely-many distinct number-words, as you can readily see by counting and noticing that, no matter how far you count, there will always be at least one more number-word standing at attention in case you call upon it (Hardegree, 2001).

I conclude that there are both nature and nurture components to a person’s math capabilities. However, nurture (via formal and informal education) seems to be the dominant component for most people.

Concluding Remarks

Collectively, the human race knows a lot about brain/mind science and how it relates to teaching and learning. Moreover, we are living at a time of rapid growth in the field of brain/mind science.

However, brain/mind science is a field where it is difficult to translate theory into practice. As the adage says, “"When you are up to your neck in alligators, it's hard to remember the original objective was to drain the swamp." When a teacher is facing a classroom full of young students, he or she tends to be in survival mode rather than in the mode of learning, understanding, and implementing current ideas from brain/mind science.

This provides an excellent opportunity to practice “chunking.” I would guess that that brain/mind science has some significant meaning to you. Consider brain/mind science as a single chunk that you hold in short term memory as you think about designing a lesson for your students. That still leaves you about 6 ± 2 chunks of short-term memory space to deal with the key ideas you need to think about as you develop the lesson. A variation of this, that makes use of low technology, is to write your self a note**, “Remember to take brain/mind science into consideration.”** that you place near the top of a page you are using to develop a lesson plan.

Recommendations Emerging from Chapter 4

4.1 Brain/mind science is a valuable component of the Science of Teaching and Learning. The pace of current progress in brain/mind science is a challenge to teachers and out educational system. Recommendation: develop and implement a plan for “keeping up” in the parts of brain/mind science that are directly relevant to being a good teacher.

4.2 When we combine what we know about cognitive developmental theory and rates of student learn with the idea of constructivism, we come to better understand some major weaknesses in our educational system. Recommendation: Work to increase your knowledge and skill as a constructivist teacher, and work to increase each of your student’s knowledge of constructivism as it applies to him or her.

Activities and Questions for Chapter 4

1. Using introspection and metacognition, work to increase your understanding of your relative rates of learning in different disciplines.

2. As a person progresses in the formal study of mathematics, he or she begins to encounter math books designed for learning math by reading. Take a look at several different math book series used in elementary schools. Compare and contrast these books from the point of view of learning math versus doing/using math to learn.

Chapter 5
Problem Solving

"If I have seen further it is by standing on the shoulders of giants." (Isaac Newton; English mathematician & physicist; Letter to Robert Hooke, February 5, 1675; 1642–1727.)

Problem solving is part of every discipline. This chapter explores the general topic of problem solving. It then looks at roles of ICT in problem solving and the specific topic of problem solving in mathematics.

Problem Solving Writ Large

Earlier parts of this book have talked briefly about the three terms: problem; problem posing; and problem solving. However, the term *problem* has not been carefully defined. A later part of this chapter will provide some definitions. The goal of this section is to broaden your insights into the general idea of gaining an increasing level of expertise in problem solving and the types of problems that people might learn about during their formal education.

Your brain is active all of the time, even when you are asleep. Your brain functions at a subconscious level to direct a wide range of activities that keep your body alive and functioning well. That is, your brain is constantly detecting and solving problems at a subconscious level. However, these are not the types of problems that we have in mind when we explore the development of a school curriculum to help students get better at problem solving.

As you carry on your everyday activities, your five senses input a steady barrage of data into your brain. In very simplified terms, this is what happens with the input data. The input data is temporarily stored at a subconscious level, where one of three things happens. Your brain may pay attention to the data at a subconscious level and process it at a subconscious level. Your brain may bring the data to a conscious level, allowing the brain/mind to then process it at a conscious level. Or, the data may be ignored and quickly forgotten.

In terms of formal school and schooling, we are particularly interested in increasing and improving the brain/mind's capacity to solve the types of problems that come to its conscious attention. You know that it is possible to transform many such problems into problems that can be solved at a subconscious level. For example, consider a basketball player shooting a basket, perhaps while somewhat off balance and moving rapidly. This requires a large amount of mental processing to control a number of muscles, and it must happen very rapidly. There is not enough time for careful, conscious processing of the situation and consciously sending out appropriate signals to the appropriate muscles.

The procedural part of your brain can learn a large number of procedures—that is, it can gain automaticity in the subconscious solving of a wide range of problems. Such procedures are integral components of sports, driving a car, riding a bicycle, playing a musical instrument, fast keyboarding, reading, and doing arithmetic. However, to build and maintain a high level of such procedural knowledge and skill takes a lot of time and continued practice.

If you are a sports fan, you know that Michael Johnson was one of the greatest professional basketball players of all time. For a while during his professional basketball career, he quit basketball and attempted to become a professional baseball player. He never was good enough to make it into the major leagues, and he returned to basketball and continued his successful career in that sport. This example suggests the difficulty in building and maintaining a very high level of procedural knowledge and skill in two different areas, even if they are moderately closely related.

There has been quite a bit of research on how long it takes a person to reach their potential in a discipline. For example, how many hours of practice does it take for a person to get about as good as they are capable of being in a sport, in playing chess, in playing a musical instrument, in solving math problems, and so on? Answers vary with the discipline, but tend to be a minimum of 10 to 12 years, and often longer.

Thus, for example, suppose that you have the genetic disposition to be a world-class chess player. Once you are old enough to learn and understand the rudiments of the game, you can figure on at least 12 years of full time effort—full time meaning perhaps 50 to 60 hours a week—to come close to reaching your potential.

However, suppose your goal is to be as good a research scientist as you can be. Evidence suggests that most such researchers do their best work before they are thirty years old. You might think of these 30 years as being divided into 12 years of childhood, 12 years of concentrated study focusing in a single discipline, and 6 years of highly productive work doing the job of being a researcher. This is not to say that researchers do not continue to do good work after they are 30 years old. Also, some disciplines such as philosophy require many years of study, reflection, and growth in wisdom. Philosophers tend to reach their peak much later in life than scientists.

Suppose that your goal in life is to be as good a teacher as you can be, you are a freshman just starting college, and genetically you have what it takes to become a good teacher. Prior to entering college, you have learned quite a bit about teaching by observing your teachers—by having been taught. You will take four or five years of college, learning content and pedagogy. Your will then move into a teaching job. By the time you complete your first six or seven years of teaching—assuming you have been working really hard for the past dozen years, you will be getting close to being as good as you can be.

BUT, this assertion is misleading. In some ways, being a good teacher is closely related to being a good philosopher. Thus, as you gain more experience, broader knowledge and skills, and increased wisdom, you will continue to improve as a teacher. Also, you can broaden those aspects of the discipline of teaching in which you are achieving a high level of expertise. For example, you might begin learning special education, and add this knowledge and skill to your repertoire. You might decide to increase your knowledge and skill in working with disadvantaged students. To summarize, as a teacher, you are in a career when you can steadily increase your depth and breadth of teaching-related expertise throughout your career.

Computers and Problem Solving

A microcomputer contains one or more processing units and a variety of storage devices. A processing unit is designed to rapidly and accurately carry out a step-by-step set of directions. The set of directions—called a computer program—is stored in a computer memory device. Thus, a computer can be improved by increasing the speed and number of processing units, increasing the speed and capacity of its storage units, and developing more and better computer programs.

Computers are useful aids to problem solving in every discipline. One of the reasons for this is that "building on the previous work of yourself and others" is one of the most important ideas in problem solving. The Web is a huge and steadily growing library. It provides easy access to a lot of the previous work collected knowledge of the human race. For example, this collected knowledge includes information about how to alphabetize a list of words or names. It tells us how to "look up" a word in an alphabetized dictionary or a name in a alphabetized telephone book.

But, a computer can do much more. A computer can store a detailed step-by-step set of directions on how to solve a wide range of math, business, science, and other categories of problems. Thus, many problems can be solved by merely using a computer to retrieve ("look up") the right program and then telling the computer to apply the program to your specific problem.

In summary, each discipline includes a large and steadily growing collection of types of problems that a computer can solve.

Chunks and Chunking

What is a star athlete doing during 10 or more years of hard practice? In essence, this person is "chunking" the physical and mental procedures needed to be a good athlete in a specific sport. The learning process builds meaningful chunks of knowledge and skill that can be accessed and used with little or no conscious effort.

What about the chess player spending 10 or more years to achieve a high level of expertise? Research suggests that this person is internalizing perhaps 50,000 chess chunks—board positions that can be accessed and used very rapidly with little or no conscious effort. A quick glance as a chessboard displaying a game in progress identifies the pertinent meaningful chunks and suggests where to focus conscious attention in deciding on an appropriate move.

The same ideas hold in any area in which a person is seeking to gain a high level of physical and/or mental expertise. The years of study and practice build up chunks of knowledge and skill that can be acted upon at a subconscious level and that can be used by short-term memory as a person consciously thinks about a problem to be solved or a task to be accomplished. Some of the time is used to memorize frequently occurring problems and exactly what to do when faced by one of these problems.

To summarize, increasing expertise in a discipline requires gaining an appropriate combination of:

1. Procedures in procedural memory that can be used rapidly and accurately with little or no conscious thought.
2. Rote memory information in declarative memory that can be retrieved in a timely fashion to solve frequently occurring problems or pieces of problems via rote memory approach.
3. Chunks of meaningful information in declarative memory that can be quickly retrieved and brought into short term (working memory) as chunks used in thinking and problem solving.

4. Learning about oneself as a learner, as a person gaining increasing skill in problem solving, and how to get better at each.
5. Gaining increased knowledge and skill at making effective use of 1-4.

People differ in their relative abilities to accomplish these four learning tasks. Thus, instruction for helping a person gain an increasing level of expertise in a discipline needs to take into consideration such individual differences. A somewhat different way to think about this is that each learner can gradually gain increased expertise in being a learner. A learner can come to better understand his or her strengths and weaknesses in 1-5 in each discipline or learning endeavor that the person is interested in.

Now, consider a child working to gain increasing expertise in mathematics. By the time this child enters kindergarten or the first grade, the child has made a start in each of 1-5. For example, the child has likely memorized into declarative memory counting words one, two, three, four, and so on. The child may well have a procedure in procedural memory that is used to answer questions such "How many pieces of candy are on the table?" The child counts using the counting words, and "knows" without conscious thought that "the answer" is the final counting number that is needed in this counting process.

Here is something to think about. Can you explain why the counting procedure produces the correct answer? Do you think that a typical five year old can explain why this procedure works? Quite a bit of math education consists of learning procedures and gaining both speed and accuracy in carrying out the procedures. This task is quite a bit different than learning why a proceduer works—that is, understanding a procedure.

Many math educators have thought about how to design a math curriculum to effectively increase the math expertise of students being taught using the curriculum. But, remember that there are many goals in math education and many different aspects of math in which a person can gain increasing levels of expertise. Math problem solving, for example, is but one aspect of math. Being a world-class expert in the history of math is a worthy goal for some people, and it has little to do with getting better at solving math problems.

Finally, think a little about how ICT affects 1-5 in the list given above. Should the existence of calculators and computers lead to changes in the amount of time that a student spends in each of the four topic areas? For example, consider memorization of how to solve frequently occurring problems. When playing a game of chess during a chess tournament, a player is not allowed to look up the opening move sequences that have been carefully analyzed and stored in books. Contrast this with the real world situation of a person on the job. Certainly there are job situations in which a person does not have the time to look up math information in a book—immediate, off the top of one's head actions are required. However, there are many situations in which there is time to "look it up" in a book.

Computerizable Chunks

Computers have been developed to be especially good at dealing with the types of computational and symbol manipulation chunks that are important parts of math. Many of the problems in other disciplines lend themselves to the use of math in representing and helping to solve the problems. Now you can see why it is necessary to reconsider the design of the math curriculum in schools. Much of the curriculum that is currently in place was designed before computers and powerful calculators (and the Web) became so readily available.

We can build machines that are faster, more accurate, etc. that a person's physical capabilities. Similarly, we have comuters. People can can write computer programs that can remember and accurately carry out certain types of procedures (chunks) that are important in solving intellectual problems.

If an educational goal is for a human aided by available technology to solve a problem or accomplish a task, we need to educate the person to make effective use of the available technology and effective use of the human talents.

What is a Problem?

Up to this point in the book we have repeatedly mentioned the idea of problem solving, but we have not actually defined the term *problem.* People use the term *problem* to encompass a wide range of situations. For example, suppose that you go into a doctor's office and the admitting nurse asks you, "What is you problem?" Most likely you would not present the nurse with a word problem from a math book! Suppose you are talking to a homeless and destitute person on the street and you ask this person, "What is your problem?" Here, you are probably expecting an answer that helps explain why the person is homeless and destitute.

As you undoubtedly know, there are many possible definitions of *problem.* A problem is something that needs to be solved or resolved. Here is a dictionary definition:

> prob·lem n
>
> 1. a difficult situation, matter, or person
> 2. a question or puzzle that needs to be solved
> 3. a statement or proposition requiring an algebraic, geometric, or other mathematical solution
>
> Encarta® World English Dictionary © 1999 Microsoft Corporation. All rights reserved. Developed for Microsoft by Bloomsbury Publishing Plc.

These definitions are helpful, but they lack precision. Here is a definition that I have found useful in my teaching of preservice and inservice teachers at all grade levels and in a variety of subject areas (Moursund, 2003):

You (personally) have a problem if the following four conditions are satisfied:

1. You have a clearly defined given initial situation.
2. You have a clearly defined goal (a desired end situation). Some writers talk about having multiple goals in a problem. However, such a multiple goal situation can be broken down into a number of single goal problems.
3. You have a clearly defined set of resources that may be applicable in helping you move from the given initial situation to the desired goal situation. There may be specified limitations on resources, such as rules, regulations, and guidelines for what you are allowed to do in attempting to solve a particular problem.
4. You have some ownership—you are committed to using some of your own resources, such as your knowledge, skills, time, and energy, to achieve the desired final goal.

These four components of a well-defined (clearly-defined) problem are summarized by the four words: givens, goal, resources, and ownership. If one or more of these components are missing, you have an ill-defined problem situation rather than a well-defined problem. An important aspect of problem solving is realizing when you are dealing with an ill-defined problem situation and working to transform it into a well-defined problem. There is nothing in the definition that says that a particular well-defined problem is solvable. Moreover, there is nothing in the definition that you personally have the knowledge, skills, perseverance, and so on to solve a particular well-defined problem, even if the problem is particularly important to you.

Most of my students do not have trouble understanding parts (1) and (2) of the definition. However, many find that part (3) is confusing. Resources do not tell you how to solve a problem. Resources merely tell you what you are allowed to do and/or use in solving the problem. For example, you want to create a nationwide ad campaign to increase a product's sales by at least 20% . The campaign is to be completed in three months, and it is not to exceed $40,000 in cost. Three months is a time resource and $40,000 is a money resource. You can use the resources in solving the problem, but the resources do not tell you how to solve the problem. Indeed, the problem might not be solvable. (Imagine an automobile manufacturer trying to produce a 20% increase in sales in three months, for $40,000!)

Many writers do not include part (4) in their definition. From my point of view, problems do not exist in the abstract. They exist only when there is ownership. The owner might be a person, a group of people such as the students in a class, or it might be an organization or a country. A person may have ownership "assigned" by his/her supervisor in a company. That is, the company or the supervisor has ownership, and assigns it to an employee or group of employees.

The idea of ownership can be confusing. In this chapter we are focusing on you, personally, having a problem (you, personally, have ownership). That is quite a bit different than saying that our educational system has a problem, our country has a problem, or each academic discipline addresses a certain category of problems that helps to define the discipline.

The idea of ownership is particularly important in teaching. If a student creates or helps create the problems to be solved, there is increased chance that the student will have ownership. Such ownership contributes to intrinsic motivation—a willingness to commit one's time and energies to solving the problem. All teachers know that intrinsic motivation is a powerful aid to student learning and success.

The type of ownership that comes from a student developing a problem that he/she really wants to solve is quite a bit different from the type of ownership that often occurs in school settings. When faced by a problem presented/assigned by the teacher or the textbook, a student may well translate this into, "My problem is to do the assignment and get a good grade. I have little interest in the problem presented by the teacher or the textbook." A skilled teacher will help students to develop projects that contain challenging problems that the students really care about.

Relatively few math educators seen concerned about a student having ownership. As an elementary school educator, you might think about this from the point of view of teaching reading and the point of view of teaching math. When you are helping your students learn to read, you know that it is very helpful to find books that students are interested in—books that students are likely to find intrinsically interesting. Probably you experience a great deal of pleasure when a student selects and reads a book, driven by personal interest and the "fun" of reading the book. Contrast that with math education!

There is nothing in the definition of problem that suggests how difficult or challenging a particular problem might be for you. Perhaps you and a friend are faced by the same problem. The problem might be very easy for you to solve and very difficult for your friend to solve, or vice versa. Through education and experience, a problem that was difficult for you to solve may become quite easy for you to solve. Indeed, it may become so easy and routine that you no longer consider it to be a problem.

What is a Math Problem?

Earlier parts of this book stress that each discipline includes a focus on problem solving. Thus, as might be expected, each discipline has its own definition of what constitutes a problem and what it means to solve or resolve a problem. The 4-part definition given in the previous section tends to be useful over a wide range of disciplines, including mathematics. However, mathematicians tend to argue among themselves on an appropriate answer to, "What is a math problem?"

The following is quoted from Schoenfeld (1992):

> According to the *Mathematics report card,* …[math] lessons are generically of the type Burkhardt (1988) calls the "exposition, examples, exercises" mode. Much the same is true of lessons that are *supposedly* about problem solving. In virtually all mainstream texts, "problem solving" is a separate activity and highlighted as such. Problem solving is usually included in the texts in one of two ways. First, there may be occasional "problem solving" problems sprinkled through the text (and delineated as such), as rewards or recreations. The implicit message contained in this format is "You may take a breather from the real business of doing mathematics, and enjoy yourself for a while." Second, many texts contain "problem solving" sections in which students are given drill-and-practice on simple versions of the strategies discussed in the previous section. In generic textbook fashion, students are shown a strategy (say "finding patterns" by trying values of n = 1,2,3,4 in sequence and guessing the result in general), given practice exercises using the strategy, given homework using the strategy, and tested on the strategy. Note that when the strategies are taught this way, they are no longer *heuristics* in Pólya's sense; they are mere algorithms. **Problem solving, in the spirit of Pólya, is learning to grapple with new and unfamiliar tasks, when the relevant solution methods (even if only partly mastered) are not known. When students are drilled in solution procedures as described here, they are not developing the broad set of skills Pólya and other mathematicians who cherish mathematical thinking have in mind.** [Bold added for emphasis.]

George Polya was one of the best mathematicians of the 20th century and he wrote extensively about problem solving. *The Goals of Mathematical Education* (Polya, 1969) is a talk that he gave to a group of elementary school teachers.

> To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. For the higher aims about which I am now talking are some general tactics of problems—to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can

> be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school. But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems.

The essence of math problem from Polya and Schoenfeld's point of view is a "new and unfamiliar tasks, when the relevant solution methods (even if only partly mastered) are not known." These two math educators, and many others, feel that the typical activities and exercises that students spend the majority of their math education time on are not problems.

George Polya and the Six-step Strategy

Polya (1957) contains a general strategy for attempting to solve any math problem. Here I have reworded his strategy so that it is applicable to a wide range of problems in a wide range of disciplines—not just in math. This six-step strategy can be called the Polya Strategy or the Six Step strategy. Note that there is no guarantee that use of the Six Step strategy will lead to success in solving a particular problem. You may lack the knowledge, skills, time, and other resources needed to solve a particular problem, or the problem might not be solvable.

1. Understand the problem. Among other things, this includes working toward having a well-defined (clearly defined) problem. You need an initial understanding of the Givens, Resources, and Goal. This requires knowledge of the domain(s) of the problem, which could well be interdisciplinary. You need to make a personal commitment (Ownership) to solving the problem.

2. Determine a plan of action. This is a thinking activity. What strategies will you apply? What resources will you use, how will you use them, in what order will you use them? Are the resources adequate to the task?

3. Think carefully about possible consequences of carrying out your plan of action. Place major emphasis on trying to anticipate undesirable outcomes. What new problems will be created? You may decide to stop working on the problem or return to step 1 as a consequence of this thinking.

4. Carry out your plan of action. Do so in a thoughtful manner. This thinking may lead you to the conclusion that you need to return to one of the earlier steps. Note that this reflective thinking leads to increased expertise.

5. Check to see if the desired goal has been achieved by carrying out your plan of action. Then do one of the following:

   A. If the problem has been solved, go to step 6.

   B. If the problem has not been solved and you are willing to devote more time and energy to it, make use of the knowledge and experience you have gained as you return to step 1 or step 2.

   C. Make a decision to stop working on the problem. This might be a temporary or a permanent decision. Keep in mind that the problem you are working on may not be solvable, or it may be beyond your current capabilities and resources.

6. Do a careful analysis of the steps you have carried out and the results you have achieved to see if you have created new, additional problems that need to be addressed. Reflect on what you have learned by solving the problem. Think about how your increased knowledge and skills can be used in other problem-solving situations. (Work to increase your reflective intelligence!)

Many people have found that this six-step strategy for problem solving is worth memorizing. As a teacher, you might decide that one of your goals in teaching problem solving is to have all your students memorize this strategy and practice it so that it becomes second nature. Help you students to make this strategy part of their repertoire of high-road strategies. Students will need to practice it in many different domains in order to help increase transfer of learning. This will help to increase your students' expertise in solving problems.

Many of the steps in this six-step strategy require careful thinking. However, there are a steadily growing number of situations in which much of the work of step 4 can be carried out by a computer. This idea will be discussed later in this document. The person who is skilled at using a computer for this purpose may gain a significant advantage in problem solving, as compared to a person who lacks computer knowledge and skill.

A Math-Modeling Strategy

The following diagram is useful in discussing problem solving in math (especially at the precollege level) and roles of computers in math problem solving. A similar diagram is useful in other disciplines.

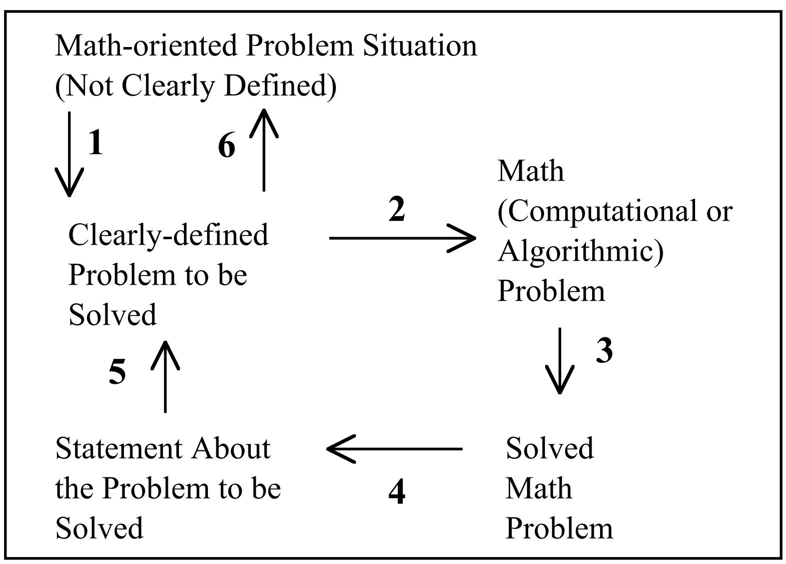

Figure 5.1. Math problem solving.

The six steps illustrated are 1) Problem posing; 2) Mathematical modeling; 3) Using a computational or algorithmic procedure to solve a computational or algorithmic math problem; 4) Mathematical "unmodeling"; 5) Thinking about the results to see if the Clearly-defined Problem has been solved; and 6) Thinking about whether the original Problem Situation has been resolved. Steps 5 and 6 also involve thinking about related problems and problem situations that one might want to address or that are created by the process or attempting to solve the original Clearly-defined Problem or resolve the original Problem Situation.

In some sense, all of the steps except (3) involve higher-order knowledge and skills. They require a significant level of math maturity. Step (3) lends itself to a rote memory approach. It is highly desirable that students develop speed and accuracy in certain types of math mental operations. However, the human mind is not good at memorizing math procedures and then carrying them out rapidly and accurately with the assistance of pencil and paper. On the other hand, calculators and computers are really good at carrying out math procedures.

PreK-12 teachers who teach math tend to estimate that about 75% of the math education curriculum focuses on (3). This leaves about 25% of the learning time and effort focusing on the remaining five steps. Appropriate use of calculators and computers as tools, and Computer-Assisted Learning, could easily decrease the time spent on (3) to 50% or less of the total math education time. This would allow a doubling of the time devoted to instruction and practice on the higher-order knowledge and skill areas.

Some Additional Problem-Solving Strategies

The previous section contains a general-purpose Six Step strategy that is useful in attaching a wide range of problems. This section provides more detail on strategies.

A strategy can be thought of as a plan, a heuristic, a possible way to approach the solving of some type of problem. For example, perhaps one of the problems that you have to deal with is finding a parking place at work or at school. If so, probably you have developed a strategy—for example, a particular time of day when you look for a parking place or a particular search pattern. Your strategy may not always be successful, but you find it useful.

In earlier chapters we have discussed the idea that each discipline is defined by the types of problems it addresses, the methods it uses, and the results it has achieved. The strategies and methods that one uses to solve math problems are quite different than the strategies and methods that one uses to solve a history problem or a music problem. A person might be very good at solving chess problems and very poor at solving economic or social problems. Every problem-solving domain has a number of domain-specific strategies. Research suggests:

1. There are relatively few strategies that are powerful and applicable across all domains. Because each subject matter (each domain) has its own set of domain-specific problem-solving strategies, one needs to know a great deal about a particular domain and its problem-solving strategies to be good at solving problems within that domain.
2. The typical person has few explicit domain-specific strategies in any particular domain. This suggests that if we help a person gain a few more domain-specific strategies, it might make a significant difference in overall problem-solving performance in that domain. It also suggests the value of helping students to learn strategies that cut across many different domains and teaching for higher-road transfer of learning of these strategies.

The next few sub sections give examples of rather general-purpose strategies that cut across many domains. Each might be taught in a math unit of study or course, and each is applicable in many different domains.

Top-Down Strategy

The idea of breaking big problems into smaller problems is called the top-down strategy. The idea is that it may be far easier to deal with a number of small problems than it is to deal with one large problem. The top-down strategy is frequently used in solving math problems. For example, suppose that you are given the dimensions of each room in a house, and the goal is to find the total square footage of the house. The problem is easily broken into one of finding the square footage of each room (a collection of smaller problems) and then adding the results.

The top-down strategy is quite useful in writing. The task of writing a long document may be approached by developing an outline, and then writing small pieces that fill in details on the outline. The smaller problems of writing individual paragraphs are less complex than the overall problem of writing a long document.

Don't Reinvent the Wheel (Ask an Expert) Strategy

Library research is a type of "ask an expert" strategy. A large library contains the accumulated expertise of thousands of experts. The Web is a rapidly expanding global library. It is not easy to become skilled at searching the Web. For example, are you skilled in using the Web to find information that will help you in dealing with Language Arts problems, Math problems, Science problems, Social Science problems, personal problems, health problems, entertainment problems, and so on? Each domain presents its own information retrieval challenges.

An alternative "ask an expert" approach is to actually ask a human expert. Many people make their livings by being consultants. They consider themselves to be experts within their own specific domains, and they get paid for answering questions and solving problems within their areas of expertise.

From the point of view of a young student, a teacher (indeed, perhaps any adult) is an expert. You want to help you students to ask clear, well-defined questions. But, you also want them to learn to find answers using resources such as each other, libraries, and the web.

Scientific Method Strategy

The various fields of science share a common strategy called Scientific Method. It consists of posing and testing hypotheses. This is a type of problem posing and problem solving. Scientists work to carefully define a problem or problem area that they are exploring. They want to be able to communicate the problem to others, both now and in the future. They want to do work that others can build upon. Well done scientific research (that is, well done problem solving in science) contributes to the accumulated knowledge in the field.

Trail and Error and Exhaustive Search Strategies

Trial and error (guess and check) is a widely used strategy. It is particularly useful when one obtains information by doing a trial that helps make a better guess for the next trial. For example, suppose you want to look in a dictionary to find the spelling of a word you believe begins with "tr." Perhaps you open the dictionary approximately in the middle. You note that the words you are looking at begin with "mo." A little thinking leads you to opening the right part of the dictionary about in the middle. You then see you have words beginning with "sh." This process continues until you are within a few pages of the "tr" words, and then you switch strategies to paging through the dictionary, one page at a time.

The "page through the dictionary one page at a time" is an exhaustive search strategy. You could have used it to begin with, starting at the first page of the dictionary. That is a very slow strategy to use for finding a word in a dictionary.

An ICT system might be a billion times as fast as a person at doing guess and check or exhaustive search in certain types of problems. Thus, guess and check, and exhaustive search, are both quite important strategies for the computer-aided solving of certain types of problems.

Concluding Remarks

The essence of learning math is learning to solve math problems. Over the centuries, many aids have been developed to help humans solve math problems. Pencil and paper (chalk and a chalkboard) are very powerful aids to solving math problems. A math library is a powerful aid.

Now we have computers that are powerful aids that are growing increasingly more powerful. Our math education system is struggling with how to appropriately design curriculum content, instructional processes, and assessment that adequately integrate the capabilities of the combined power of human and computer brains.

Recommendations Emerging from Chapter 5

5.1 Help your students learn about problem solving across the curriculum and roles of math as an aid to problem solving across the curriculum.

5.2 Help your students learn the capabilities and limitations of their human brains and of aids to their human brains (such as computers) in problem solving.

5.3 Help your students to understand more about the long and hard path to achieving a high level of expertise in an area.

Activities and Questions for Chapter 5

1. Compare and contrast the ideas of reading across the curriculum, mathing (using math) across the curriculum, and problem solving across the curriculum.

2. Reflect on the similarities and differences between memorizing math procedures with little or no understanding, and using computers to carry out math procedures instead of carrying them out by hand.

3. Students studying math tend to develop the idea that every math problem has “an answer” and their goal is to find “the solution.” Why do you suppose that this is the case? Can you give some examples of math problems that do not have a solution? Can you give some examples of math problems that have multiple solutions?

4. Make a list of strategies that are useful both in solving math problems and that transfer to other disciplines.

Chapter 6
Research, and Closure

April 2012 note from David Moursund: Chapters 0-5 of this manuscript were distributed to students and used in a two-week component of a Math Methods for Elementary Teachers Course in 1994. Chapter 6 was in rough draft form at that time and was not distributed. The Chapter 6 given here has been reconstructed from the rough draft notes.

> Information on its own is not enough to produce actionable knowledge. … People learn in response to need. When people cannot see the need for what's being taught, they ignore it, reject it, or fail to assimilate it in any meaningful way. Conversely, when they have a need, then, if the resources for learning are available, people learn effectively and quickly. (John Seely Brown & Paul Duguid, *The Social Life of Information*. Harvard Business School Press, 2000)

This chapter consists of three parts. The first part presents a little bit of the research literature on improving math education. The second part discusses how Information and Communication Technology (ICT) will change math education. The third part is a few general predictions of where I believe math education is headed.

Research

There is a lot of research literature on math education. This research has helped guide the development of a wide range of curriculum materials. It also helped to guide the NCTM in its development of the NCTM Math Standards.

1999 Article Titled "Parrot Math"

O'Brien, Thomas C. (February 1999). Parrot math. *Phi Delta Kappan.* Accessed 3/31/2012: http://www.professortobbs.com/articles/PDK-Parrot%20Math.htm.

Quoting from this document:

> **A SMALL but vociferous group** of very well-organized critics is espousing a return to "parrot math." These critics believe that mathematics education in elementary schools should be confined largely to arithmetic and that mathematics should be taught by the force-feeding of inert facts and procedures shorn of any real-life context. They have no tolerance for children's invented strategies or original thinking, and they leave no room for children's use of estimation or calculators.
>
> The critics claim that their approach is the only correct approach. Although some of their most vocal leaders have no apparent expertise in mathematics and no

> experience teaching mathematics at any level, they say that anyone who criticizes them is not a mathematician or doesn't understand how students learn mathematics. Their understanding of how children learn mathematics gives short shrift to the notion that knowledge is a personally constructed network of ideas, information, images, and relationships that tends toward coherence, stability, economy, and generalizability.
>
> They criticize new approaches to the teaching of math—approaches that can be summarized by saying that math should make sense to children and that children should be thinkers rather than storage bins for thinking done by others. They also argue that constructivism is a fad—this despite 80 years of empirical research, replicated worldwide, on the construction and growth of children's thinking about essential mathematical and scientific ideas, such as number, space, logic, causality, classification, and contradiction. The main findings of this body of research—that the development of knowledge comes from an interaction between knower and known, that children's thinking is very different from adults' thinking, and that social interaction is a major cause of intellectual growth—are foreign to them.
>
> In the field of children's learning of arithmetic, there is significant research to show that the force-feeding of computational procedures is harmful. But the critics continue to insist that arithmetic— and knowledge in general—is inert stuff to be transmitted and stored.

Here is another part of O’Brien’s paper:

> The back-to-basics approach to learning has been dominant in U.S. math classrooms throughout this century. As early as the 1930s the math education researcher William Brownell saw parrot math as dominant and criticized it heartily, pleading for children to be allowed to find meaning in math.
>
> But the view of math as isolated bits of information to be transmitted to passive receptors continues to be dominant in America's schools. In January 1998 I contacted some 20 colleagues around the country -- education and publishing experts in daily contact with entire states and even regions. I asked them to tell me the proportion of activity-based, constructivist-minded elementary school classrooms in their areas. Many respondents replied, "There are none," and the highest figure cited was 20%. Reporting on the analyses of the video component of the Third International Mathematics and Science Study (TIMSS), James Stigler and James Hiebert state that U.S. eighth-graders spend almost all their time practicing routine procedures transmitted by the teacher, while in Japan students are asked to think. And to judge from what classroom teachers remember, not much has changed in decades. (See "What Teachers Recall," which accompanies this article.)
>
> We cannot go back to basics as the critics demand. We've been there all along. And the fact is that the back-to-basics approach, not the activity-based approach, has failed us. Let's take a look at some of the evidence.

- In the first International Study of Achievement in Mathematics, published in 1967, American 13-year-olds finished next to last among 10 major industrial nations.
- In the Second International Mathematics Study, conducted during the 1981-82 school year, American eighth-graders ranked 10th of 20 national groups in arithmetic, 12th in algebra, 16th in geometry, and 18th in measurement. In all subtests, America's overall scores were at or below the median for the entire group.
- In a comparison of 14 national groups of 9-year-olds reported in 1992, American children came in next to last, besting only Slovenia. In the same research, American 13-year-olds tied with Spain for next-to-last place and bested only Jordan.
- TIMSS is the largest, most comprehensive, and most rigorous international comparison of education ever undertaken. During the 1995 school year, the study tested the math and science knowledge of half a million students from 41 nations. In mathematics, U.S. eighth-graders posted scores in the middle of the pack, slightly below the international average.

2002 US Department of Education Report

In 2002 the United States Department of Education sponsored a meeting to discuss educational research. Here is the reference:

> Ed.gov (2002). Scientifically Based Research. US Department of Education. Retrieved 3/31/2012 from http://www2.ed.gov/nclb/methods/whatworks/research/page_pg4.html. The transcript of this meeting is available in PDF and Microsoft Word formats.

Participants in the meeting discussed educaitonal research in general, and in some specifi areas of educaiotn. Dr. Russell Gersten discussed the research in math education. The following is quoted from his presentation:

> MS. NEUMAN: The first presentation is by Russ Gersten. I have read so much of his work over the years. He's at the University of Oregon. He's done a lot of work on reading comprehension, teacher knowledge, and today what he's going to be talking about is the scientific based evidence and what that means for math education and achievement.
>
> MR. RUSSELL GERSTEN: **This is actually an easy topic to be brief on because there isn't a lot of scientific research in math. There's some. There's some promising directions, but it is a somewhat depressing topic.”** [Bold added for emphasis.]
>
> There are two things going on. One, in elementary education there is no question that most teachers, even most parents, -- the reading is the big emphasis there compared to math. But it's not that simple. For other reasons, the math community of math educators at least for forty-plus years has looked at their role as reform, as change, as re-conceptualizing.

Therefore, there hasn't been this steady tradition. There are a few exceptions of really systematically using the methods that Valerie and others talked about earlier to build a knowledge base, but rather to study using the more qualitative methods: teachers understandings, kids understandings.

So, this is something that can change. There have always been little glimmerings of change. There's a slight increase in the amount, but overall the math education community has been quite resistant to that, where let's say in the reading field there have always been at least two schools of thought, one in the experimental group.

…

We found four categories. Notice the small number of studies we found on this. Now, we limited ourselves to low achieving students. These were students whose documentation was well below grade level, at least below the 35^{th} percentile on some standardized measure.

But some of the things that worked, and again we don't have a lot of replications, but they were pretty decent studies, is that when kids and/or their teachers get ongoing information, every two weeks, every four weeks, of where they are in math in terms of either the state standards or some framework, it invariably enhances performance.

This sounds kind of a little boring, it's not as romantic, there's so much of romantic work done in math. But the idea of having a system to know where kids are and what they really know, rather than saying this kid is struggling, this kid is struggling with fractions, manipulating fractions, more than one, with dividing fractions, with a sense of place value once you get into the hundreds. That information can be critical for low achieving kids, can be a life or death issue.

The second group we found, there was only six studies, is peer assisted learning. It's usually tutoring. This is something that could revolutionize practice. Invariably, when kids are partnered up, and it seems to be better if they're heterogeneous pairs, there's one stronger student and one weaker student and they switch off, achievement in math is always improved.

So, peers can be excellent tutors. I'm not talking here about cooperative groups of four, five, six kids. It's [groups of] two.

The nation must seek to narrow the achievement gaps between white students and students of color, between middle-class students and students living in poverty; gaps that have persisted over the past decade. To address these problems, the federal government and the nation's school systems have made and are continuing to make significant investments in the improvement of mathematics education**. However, the knowledge base on which these efforts are founded has often been weak and speculative**."

…

> With the peer-assisted learning, the six studies consistently showed moderate effects—and I'm not giving the exact numbers, but there's statistical ways to cut across called meta-analyis—and that is an important finding.
>
> When kids saw the data, and it was almost always on the computer, how they were doing, which skills they needed work on, whether they were making progress, these were moderately large, these were pretty large. This was especially true not so much for special education students but for that other that kind of at-risk group who are sometimes in Title I programs who sometimes need tutoring, that giving kids this kind of feedback seems invariably to help.
>
> [We found] a very small number of studies on instruction. We broke them two ways: explicit instruction, that includes both the very, very heavily tightly sequenced work that Carnine and some of his colleagues did in math which has everything sequenced exactly for kids and a beautiful array of examples, and some of these other approaches to teach kids problem solving strategies.
>
> In both cases, and we only have a small set because we're looking kindergarten through eighth grade, but there is some evidence that providing this degree of explicitness to kids, showing them strategies, letting them take over and showing what they know is helpful.
>
> …
>
> Contextualized instruction was our way to fit together very, very, very exciting ideas about the discussion teaching fractions and getting kids immersed in real world problems that involve measuring and fractions and equivalents. And the results? I put a question mark there. When we averaged them together—and again we're only dealing with four studies—it came out about zero.
>
> …
>
> The other thing is we have this concept which is still elusive called "number sense." You'll see it around a lot. Nobody knows exactly what it is. It's sort of a sense of numbers, the way some kids just sort of take to it. You ask them, well, you know, here are six things, we want nine, how many more do you need? They'll just go "three." And, others will just go, "Well, you need some more."
>
> But, it's just basically, the idea of both performing and understanding and doing and strategizing. We have his general notion. It seems a fascinating one. It seems a wonderful spur for a generation of new researches to do the kind of array of scientific methods. So, that's one huge area.

In brief summary, the research on effective math education practices and the implementation of this research lags progress that has been occurring in reading education.

Rand Report

The following report has been broadly cited.

> Ball, Deborah (Committee Chair). (2002). Mathematical Proficiency for All Students Toward a Strategic Research and Development Program in Mathematics Education. Rand Corporation. Retrieved 3/31/2012 from

http://www.rand.org/pubs/monograph_reports/MR1643.html#toc. The individual chapters of this report can be downloaded free fro the Website cited above.

Here are some quotes from this report:

[Summary] A clear need exists for substantial improvement in mathematics proficiency in U.S. schools. While the federal government and the nation's school systems have made significant investments toward improving mathematics education, the knowledge base supporting these efforts has generally been weak. The RAND Mathematics Study Panel was convened as part of a broader effort to inform the U.S. Department of Education's Office of Educational Research and Improvement on ways to improve the quality and usability of education research and development (R&D). The panel has proposed a strategic R&D program supporting the improvement of mathematics proficiency, and equity in proficiency, among U.S. school students. The panel identified three areas for focused R&D-development of teachers' mathematical knowledge used in teaching, teaching and learning of skills needed for mathematical thinking and problem-solving, and teaching and learning of algebra from kindergarten through the 12th grade. The panel also recommends that the initial stages of the program include three key study areas: collecting evidence to support decisions concerning standards of mathematical proficiency, creating analytic descriptions of current instructional practice and curriculum in U.S. classroom, and developing measures of mathematical proficiency.

…

Complicating the process of improving school mathematics are disputes about what content should be taught and how it should be taught. Arguments rage over curriculum materials, instructional approaches, and what aspects of the content to emphasize. Should students be taught the conventional computational algorithms or is there merit in exploring alternative procedures? Should calculators be used in instruction? What degree of fluency is necessary and how much depth of conceptual understanding? What is the most appropriate view of algebra? These questions unhelpfully dichotomize important instructional issues. **The intense debates that filled the past decade, often based more on ideology than on evidence, have hindered improvement.**

…

However, despite more than a century of efforts to improve school mathematics in the United States, investments in research and development have been virtually nonexistent. Recent federal efforts to foster improvement in mathematics education are infrequently based on solid research, and federal funding for mathematics education research and development have been sporadic and uncoordinated. There has never been a long-range programmatic effort to fund research and development in mathematics education, nor has funding been organized to focus on knowledge that would be usable in practice.

…

The limited resources that likely will be available for mathematics education research and development in the near future make it necessary to focus those resources on a limited number of topics. Because students' opportunities to develop mathematical proficiency are shaped within classrooms through their interaction with teachers and with specific content and materials, the proposed program addresses issues directly related to teaching and learning. We have selected three domains in which both proficiency and equity in proficiency present ubstantial challenges, and where past work would afford resources for some immediate progress:

1. Developing teachers' mathematical knowledge in ways that are directly useful for teaching
2. Teaching and learning skills used in mathematical thinking and problem solving
3. Teaching and learning of algebra from kindergarten through the 12th grade (K–12).

Final Remarks

Math is a human endeavor and one of humananities great acbievements. Here is a book that I found to quite insightful.

> Kilpatrick, J., Swafford, J., & Findell, B. (2002). *Adding it up: Helping children learn mathematics.* The National Academies Press. The book is available free on the Web. The book is available free (online) at http://www.nap.edu/books/0309069955/html/index.html.

Quoting from the book:

> Mathematics is one of humanity's great achievements. By enhancing the capabilities of the human mind, mathematics has facilitated the development of science, technology, engineering, business, and government. Mathematics is also an intellectual achievement of great sophistication and beauty that epitomizes the power of deductive reasoning. For people to participate fully in society, they must know basic mathematics. Citizens who cannot reason mathematically are cut off from whole realms of human endeavor. Innumeracy deprives them not only of opportunity but also of competence in everyday tasks.
>
> The mathematics students need to learn today is not the same mathematics that their parents and grandparents needed to learn. When today's students become adults, they will face new demands for mathematical proficiency that school mathematics should attempt to anticipate. Moreover, mathematics is a realm no longer restricted to a select few. All young Americans must learn to slink mathematically, and they must think mathematically to learn. Adding It Up: Helping Children Learn Mathematics is about school mathematics from pre-kindergarten to eighth grade. It addresses the concerns expressed by many Americans, from prominent politicians to the people next door, that too few students in our elementary and middle schools are success- fully acquiring the mathematical knowledge, the skill, and the confidence they need to use the mathematics they have learned. Moreover, certain segments of the U.S.

population are not well represented among those who do succeed in school mathematics.

As with all attempts to improve education, there is a huge problem of translating research theory and the recommendations from the NCTM and others into practice.

Here are a few examples to help illustrate this situation.

1. The NCTM has been recommending use of calculators in the curriculum since 1980. Progress can be seen by the available of classroom sets of simple calculators in elementary schools, the use of scientific and/or graphing calculators in many secondary school math courses, and the acceptance of use of calculators in many state and national testing situations. However, in the elementary school there is still and inordinate amount of time spent teaching and learning paper and pencil arithmetic algorithms. Many elementary teachers and many parents feel that mastery of paper and pencil arithmetic algorithms is an essential component of math education. In talking to such people one frequently hears: "I think it is all right for students to use calculators after them have learned the computational algorithms."
2. The concepts of math development and math maturity are barely mentioned in the various NCTM Standards documents. There is a significant amount of research in this area that is not appropriately and adequately integrated into the Standards documents.
3. The NCTM has long been a strong proponent of teaching higher-order thinking and problem-solving knowledge and skills. While some progress has occurred over the years, the progress has been slow. For example, the math curriculum in the US is often described as being "a mile wide and an inch deep" (TIMSS). A significant percentage of math courses in K-12 education are taught by teachers who lack the mathematical knowledge, teaching skills, and maturity for the task. To a large extent, such teachers tend to focus on lower-order knowledge and skills.
4. Math education in this country does not adequately reflect:
   - Highly interactive intelligent computer-assisted learning.
   - The powerful and still growing capabilities of computers to solve a huge range of math problems. We need a math education system that prepares students to understand and use math in a world in which computers are commonplace.

The Future of Math Education

This sub-section presents some ideas on inventing the future of math education. The future of math education is strongly intertwined with the field of Information and Communication Technology. Here are a few examples/ideas to help lay the groundwork for one possible future.

1. Nowadays, people determine the time of day, as well as the day of the week and the date, using an inexpensive battery-powered electronic digital watch. There is a clear separation of learning the meaning of time, day, and date and determining the time, day, and date.

2. Nowadays, people determine their location on earth using an inexpensive battery-powered electronic global position system (GPS). There is a clear separation between understanding the meaning of location and distance between locations, and determining them by use of a GPS.

3. Nowadays, people who have the need to solve equations and produce graphs of functions or relations make use of computers or "powerful" calculators. There is a clear distinction between understanding the meaning of the results of solving an equation and the value/use of graphs, and the process of solving equations and producing graphs.

4. Over the past two decades, computational mathematics, computational biology, computational chemistry, computation ional physics, etc. have developed to a level that they are now major components of each of their respective disciplines. That is, computerized mathematical modeling has become a routine and very important tool in research and application in math, the sciences, and many other disciplines.

5. The development of computers has fostered the development of a relatively large groups of people who use statistical methods, as contrasted with a relatively small group of people who understand the underlying mathematical theory of the statistical methods. Users of computational statistics do not gain their understanding of statistics by memorizing procedures and developing speed and accuracy at carrying out these procedures by hand or by use of a simple calculator. In addition, most do not gain their understanding of statistics by learning the underlying mathematical theory. Thus, statistics provides an example where (for most users) there is a relatively clear distinction between concepts/uses and the underlying mathematical theory.

6. Consider an architect designing a structure. Nowadays, the design work is done on a computer. Each design can be checked for energy efficiency, meeting the earthquake, wind, fire, etc codes, and so on by the computer. Indeed, artificially intelligent software can make suggestions for improvement in all of these areas. A computer can also develop a virtual "walk through" for the structure. The architect and the computer together are making use of a large amount of mathematics, physics, and other disciplines. But, there is a clear distinction between understanding of the concepts and knowing details of carrying out the procedures by hand and/or aided by tables, calculators, and other simple aids.

7. Consider a person using the Web in buying an airline ticket, making a hotel reservation, and renting a car, all as part of a trip the person intends to make. The concepts of airline ticket, hotel reservation, and rental car are not too difficult to learn. The concept of paying money for these things is part of the larger concept of purchasing goods and services. Most of the underlying details are programmed into the computer systems used to complete these transactions. The computer systems access large databases that are accessed in real time and updated in real time. There is a large amount of use of mathematics in solving these purchasing problems.

This list can easily be extended. The point being made is that increasingly, there is a separation between concepts and procedures. This is true not only in math, but many other disciplines. ICT makes it increasingly possible to separate concepts and procedures.

For many years to come, we can expect continued rapid power of computers, computerized equipment, telecommunications, and artificial intelligence. With appropriate education within the various disciplines, students will gain more and more power in solving problems that can be represented mathematically. Increasingly, the example given above for architecture will prevail. Students will learn how to understand and represent problems on a broad scale. Computers will interface with the human-developed representations (for example, an architectural drawing done on a computer) and carry out a tremendous amount of the underlying and necessary work. Knowledge of what needs to be done and how to do it will be stored in computer programs.

My conclusions from these assertions (predictions) is that math education should place much more emphasis on developing mathematical maturity, on exploration of math as a human endeavor, and in helping students to gain an understand of the discipline of mathematics. Math education should place increased emphasis on posing and representing computational-math-based problems in all disciplines, with students learning how to make use of computers to solve the resulting problems.

I close with three quotes from:

> Conference Board of the Mathematical Sciences (2001). The Mathematical Education of Teachers. Retrieved 3/32/2012 from http://www.cbmsweb.org/MET_Document/index.htm.

> Arithmetic skills, and occasionally a little algebra, were once the mathematics required for almost all jobs outside of engineering and the physical sciences. In recent years, computers and an associated explosion in the use of quantitative methods in business and science have dramatically increased the mathematical skills needed in many jobs. Facility at creating spreadsheets is becoming required in many entry-level positions for high school graduates. Assembly line workers may be expected to learn elements of statistical quality control. The level of mathematical sophistication common in financial analyses today would have been unthinkable a generation ago.
>
> …
>
> Throughout U. S. educational history, teachers have generally provided the style and level of instruction that society expected of them. Until 1900, teachers of mathematics were largely seen as drill masters, training students to accurately perform numerical computations. Beyond the eight primary grades, most teachers had at best a year or two of preparation at a special high school, called a normal school. The introduction of universal high school around 1900 gave rise to secondary level subject specialists, who majored in their subject in teachers' colleges. Teachers for earlier grades also were eventually required to go to college, but their education focused on the psychological and social development of children. It was generally assumed, and is still assumed by some today, that prospective elementary school teachers, and perhaps middle school teachers, learn

all the mathematics they need to teach mathematics well during their own schooling.

…

There are a number of statements in this report about prospective teachers acquiring a "deep understanding" of school mathematics concepts and procedures. The emphasis is on the mathematics that teachers need to know but also there is a recognition that teachers must develop "mathematical knowledge for teaching." This knowledge allows teachers to assess their students' work, recognizing both the sources of student errors and their students' understanding of the mathematics being taught. They also can appreciate and nurture the creative suggestions of talented students. Additionally, these teachers see the links between different mathematical topics and make their students aware of them. Teachers with deep understanding are also more able to excite students about mathematics. Some mathematicians may react skeptically to setting these goals for prospective teachers, because, in their experience, prospective teachers, like many other students in introductory mathematics courses, seem to struggle to gain a minimal understanding of the basic concepts. Indeed, it is only realistic to expect teachers to develop a deep understanding over years of professional study, undertaken alone, with other teachers, and in continuing education classes. However, its foundation—deep understanding of school mathematics—must be laid during preservice education.

Appendix A
Goals of Education in the US

> The principle goal of education in the schools should be creating men and women who are capable of doing new things, not simply repeating what other generations have done; men and women who are creative, inventive and discoverers, who can be critical and verify, and not accept, everything they are offered. (Jean Piaget, 1896–1980)

Chapter 2 contains David Perkin's list of three very general, unifying goals of education:

1. Acquisition and retention of knowledge and skills.
2. Understanding of one's acquired knowledge and skills.
3. Active use of one's acquired knowledge and skills. (Transfer of learning. Ability to apply one's learning to new settings. Ability to analyze and solve novel problems.)

The appendix contains a more detailed list of goals of education. It is a list of goals that many people in American society generally agree upon. Each of the goals is followed by brief comments that about how the goal is being affected by information technology.

The list has been divided into three categories—Conserving Goals, Achieving Goals, and Accountability Goals. In most societies, education has a major goal of conserving and preserving the culture and values of the society. Interestingly, this tends to create some stress between Conserving Goals and Achieving Goals. As students gain increasing knowledge and skills, they sometimes rebel against the conservative nature of schools and their society.

Conserving Goals

G1 **Security:** All students are safe from emotional and physical harm. Both formal and informal educational systems must provide a safe and secure environment designed to promote learning.

Comment: In recent years there has been a great deal of media coverage about potential physical and emotional harm that might occur as students are given access to the Internet and the World Wide Web. Schools are responding by trying to shelter students from Web sites that are deemed to be inappropriate. In addition, students are being asked email and the Web in a responsible manner.

G2 **Values:** All students respect the traditional values of the family, community, state, nation, and world in which they live.

Comment: Not all people are equally appreciative of and supportive of Information and Communication Technology (ICT). Our educational system must allow for such differences in values. In some cases, this means that students must be given options on assignments and on information sources, as well as guidance in selecting options that are supportive of values of their family and culture.

G3 **Environment:** All students value a healthy local and global environment, and they knowingly work to improve the quality of the environment.

Comment: Some of the most successful uses of ICT in schools have centered around environmental projects. Students work on environmental problems in their own communities and/or on a wider scale. For example, students make use of microcomputer-based instrumentation to gather data on water and air quality. Data may be shared from sites throughout the city, state, nation, or world through use of email. It has become common for students to develop hypermedia documents as an aid in disseminating the results of their studies.

Achieving Goals

G4 **Full Potential:** All students are knowingly working toward achieving and increasing their healthful physical, mental, and emotional potentials.

Comment: Notice the emphasis on students "knowingly" working to increase their potentials. The goal is to empower students to empower themselves. Achieving full potential includes learning to make effective use of contemporary tools that are used in the fields where one is developing their potentials.

G5 **Basic Skills:** All students gain a working knowledge of speaking and listening, observing (which includes visual literacy), reading and writing, arithmetic, logic, and storing and retrieving information. All students learn to solve problems, accomplish tasks, deal with novel situations, and carry out other higher-order cognitive activities that make use of these basic skills.

Comment: Many people now argue that ICT is a basic skill. A number of states have set goals for having all of their students to gain basic knowledge and skills in use of a variety of information technology tools.

G6 **General Education:** All students have appreciation for, knowledge about, and understanding of a number of general areas of education, including:

- Artistic, intellectual, scientific, social, and technical accomplishments of humanity.
- Cultures and cultural diversity; religions and religious diversity.
- Governments and governance.
- History and geography.
- Mathematics and science.
- Nature in its diversity and interconnectedness.

Comment: ICT is part of the technical accomplishments of humanity. ICT is now a valuable aid to learning and using one's knowledge in each of the areas listed above.

G7 **Lifelong Learning:** All students learn how to learn. They have the inquiring attitude and self-confidence that allows them to pursue life's options. They have the knowledge and skills needed to deal effectively with change.

Comment: ICT will continue to change quite rapidly. This will present a learning challenge to students of all ages throughout their lifetimes. However, ICT is becoming an increasingly powerful aid to learning for learners of all ages.

G8 **Problem Solving:** All students make use of decision-making and problem-solving skills, including the higher-order skills of analysis, synthesis, and evaluation. All students pose and solve problems, making routine and creative use of their overall knowledge and skills.

Comment: ICT is a powerful aid to problem solving in every academic discipline

G9 **Productive Citizenship:** All students act as informed, productive, and responsible members of organizations to which they give allegiance, and as members of humanity as a whole.

Comment: ICT, including the World Wide Web, is fast becoming a routine component of life in our society.

G10 **Social Skills:** All students interact publicly and privately with peers and adults in a socially acceptable and positive fashion.

Comment: ICT has brought us new forms of communication and social interaction, including desktop conferencing, picture phones, email, and groupware.

G11 **Technology:** All students have appropriate knowledge and skills for using our rapidly changing Information Age technologies as well as relevant technologies developed in earlier ages.

Comment: ICT is both a discipline in its own right and a driving force for change in many different areas of technology, science, and research.

Accountability

G12 **Assessment:** The various components of an educational system that contribute to accomplishing the goals (such as those listed above) are assessed in a timely and appropriate manner to provide formative, summative, and long term impact evaluative data that can be used in maintaining and improving the quality of the educational system.

Comment: Accountability and assessment are thoroughly intertwined. In the past two decades, the issue of authentic assessment has received a lot of attention. As ICT is more thoroughly integrated into curriculum content, the assessment (authentic assessment) of this student learning becomes a new challenge to an educational system. Electronic portfolios are gradually increasing in importance as an aid to authentic assessment.

G13 **Accountability:** All educational systems are accountable to key stakeholder groups, including:

- Students stakeholders
- Parents and other caregivers of the students
- Employees and volunteers in educational systems
- Voters and taxpayers

Appendix B
Goals for ICT in Education

> "If you don't know where you are going, you're likely to end up somewhere else." (Lawrence J. Peter, of "Peter's Principles" fame.)

This appendix provides a brief overview of the field of Information and Communication Technology (ICT) in education. The approach used is via presenting a summary of goals for ICT in education.

The Information Age

Historians have identified four important eras or "ages" in the development of human societies:

- The Hunter-Gatherer Age
- The Agricultural Age
- The Industrial Age
- The Information Age

We live in the Information Age, where computers are used as personal productivity tools and for entertainment. The Information Age has brought us such concepts and tools as telecommunications, the information superhighway, and the Internet. It is characterized by the availability of digitized information disseminated through multimedia such as television, audio and videotape, camcorders, compact discs, and digital tape. The Information Age has seen the development of hypermedia (computer-based, interactive multimedia) and groupware (productivity tools for groups of people working together).

The Information Age has also brought us a new way of "knowing, researching, and using" the various academic disciplines that we study in school. As an example, in 1998 one of the winners of the Nobel Prize in Chemistry was a **Computational Chemist.** The prize was awarded for his work in computer modeling and simulation of chemicals and chemical processes. Nowadays, there are three major categories of scientists within each science discipline: Experimental Scientists, Theoretical Scientists, and Computational Scientists. In math we now have pure mathematicians, applied mathematicians, and computational mathematicians.

The Information Age has shrunk our world and is helping to create a Global Village. It has changed business, industry, government, and education. This transformation has been fueled by rapid progress in computer-related technologies and telecommunications systems that link computers and other machines to each other and to people. The computer's role as a "mind" tool has further fueled change in the Information Age. One person who can use computers effectively can often do the work of several people who don't know how to use computers.

ICT Goals

A variety of people and organizations have recognized the need for and value of having widely agreed upon ICT goals for students, teachers, teacher's assistants, and school administrators.

However, ICT is both a complex and rapidly growing field. Thus, goal setters have been faced by the problem of developing and implementing goals that are appropriate to a rapid pace of change. This has led many people to be rather cautious about formulating and attempting to implement rather precisely defined goals for ICT in education.

A significant part of the challenge of such goal setting is to develop goals that will continue to be appropriate as ICT and ICT in education change quite rapidly. As you read this appendix, examine each goal from the point of view of its potential longevity and flexibility.

The 13 goals given here are slight modifications of goals given in Moursund (1997, Chapter 4). A number of these goals were first published in Moursund and Ricketts (1988).

Student Goals—Functional ICT Literacy

The four goals listed in this section serve to define functional ICT literacy and provide guidelines to K-12 curriculum developers. Notice the combined emphasis on both basic skills and on higher-order, problem-solving skills.

Goal 1: ICT literacy, basic level. All students shall be functionally literate in ICT. A basic level of ICT literacy should be achieved by the end of the eighth grade. It consists of a relatively broad-based, interdisciplinary, general knowledge of ICT applications, capabilities, limitations, how computers work, and societal implications of computers and other information technology. Here are six specific objectives that underlie this information technology literacy goal.

A. General knowledge. Students shall have oral and reading knowledge of computers and other information technology, and their effects on our society. More specifically, each discipline that students study shall include instruction about how electronic aids to information processing and problem solving are affecting that specific discipline.

B. Procedural thinking. Students shall have knowledge of the concept of effective procedure, representation of procedures, roles of procedures in problem solving, and a broad range of examples of the types of procedures that computers can execute.

C. Generic tools. Students shall have basic skills in use of word processing, database, computer graphics, spreadsheet, and other general purpose, multidisciplinary application packages. This also includes basic skills in using menu-driven hypermedia software to create hypermedia materials as an aid to communicating.

D. Telecommunications. Students shall have basic skills in using telecommunications to communicate with people and to make effective use of computerized databases and other sources of information located both locally (for example, in a school library, a school district library, a local community library) and throughout the world. They shall have the knowledge and skills to make effective use of the Internet, the World Wide Web, and email.

E. Hardware. Students shall have basic knowledge of the electronic and other hardware components and how they function sufficient to "dispel the magic." They shall understand the functionality of hardware sufficient to detect and correct common difficulties, such as various components not being plugged in or not receiving power, various components not being connected, and printer out of paper.

F. Computer input. Students shall have basic skills in use of a variety of computer input devices, including keyboard and mouse, scanner, digital still and video camera, touch screen, and probes used to input scientific data. They shall have introductory knowledge of voice input and pen-based systems.

Goal 2: ICT literacy, intermediate level. Deeper knowledge of computers and other information technology as they relate to the specific disciplines and topics one studies in senior high school. Some examples:

A. Skill in creating hypermedia documents. This includes the ability to design effective communications in both print and electronic media, as well as experience in desktop publication and desktop presentation.

B. Skill in use of information technology as an aid to problem solving in the various high school disciplines. A student taking advanced math would use computer modeling. A commercial art student would create and manipulate graphics electronically. Industrial arts classes would work with computer-aided design. Science courses would employ microcomputer-based laboratories and computer simulations.

C. Skill in computer-mediated, collaborative, interdisciplinary problem solving. This includes students gaining the types of communication skills (brainstorming, active listening, consensus-building, etc.) needed for working in a problem-solving environment.

Goal 3: Computer-as-tool in curriculum content. The use of computer applications as a general-purpose aid to problem solving using word processor, database, graphics, spreadsheet, and other general-purpose application packages shall be integrated throughout the curriculum content. The intent here is that students shall receive specific instruction in each of these tools, probably before completing elementary school. Middle school, junior high school, and high school curriculum shall assume a working knowledge of these tools and shall include specific additional instruction in their use. Throughout secondary school and in all higher education, students shall be expected to make regular use of these tools, and teachers shall structure their curriculum and assignments to take advantage of and to add to student knowledge of computer-as-tool.

Goal 4: ICT literacy courses. A high school shall provide both of the following "more advanced" tracks of computer-related coursework.

A. Computer-related coursework preparing a student who will seek employment immediately upon leaving school. For example, a high school business curriculum shall prepare students for entry-level employment in a computerized business office. A graphic arts curriculum should prepare students to be productive in use of a wide range of computer-based graphic arts facilities. Increasingly, some of these courses are part of the Tech Prep (Technical Preparation) program of study in a school.

B. Computer science coursework, including problem solving in a computer programming environment, designed to give students a college-preparation type of solid introduction to the discipline of computer science. These courses or may not be Advance Placement courses.

Student Goals—Independent Lifelong Learning

The three goals listed in this section focus on computer technology as an aid to general learning.

Goal 5: Distance education. Telecommunications and other electronic aids are the foundation for an increasingly sophisticated distance synchronous and asynchronous education system. Education shall use distance education, when it is pedagogically and economically sound, to increase student learning and opportunities for student learning.

Note that in many cases distance education may be combined with computer-assisted learning (CAL, see Goal 6) and carried out through the WWW (see Goal 1D), so that there is not a clear dividing line between these two approaches to education. In both cases students are given an increased range of learning opportunities. The education may take place at a time and place that is convenient for the student, rather than being dictated by the traditional course schedule of a school. The choice and level of topics may be more under student control than in our traditional educational system.

Goal 6: Computer-assisted learning (CAL). Education shall use computer-assisted learning when it is pedagogically and economically sound, to increase student learning and to broaden the range of learning opportunities. CAL includes drill and practice, tutorials, simulations, and microworlds. It also includes computer-managed instruction (see Objective C below). These CAL systems may make use of virtual realities technology.

A. All students shall learn both general ideas of how computers can be used as an aid to learning and specific ideas on how CAL can be useful to them. They shall become experienced users of CAL systems. The intent is to focus on learning to learn, being responsible for one's own learning, and being a lifelong learner. Students have their own learning styles, so different types of CAL will fit different students to greater or lesser degrees.

B. In situations in which CAL is a cost-effective and educationally sound aid to student learning or to overall learning opportunities, it will be an integral component of the educational system. For example, CAL can help some students learn certain types of material significantly faster than can conventional instructional techniques. Such students should have the opportunity to use CAL as an aid to learning. In addition, CAL can be used to provide educational opportunities that might not otherwise be available. A school can expand its curriculum by delivering some courses largely via CAL.

C. Computer-managed instruction (CMI) includes record keeping, diagnostic testing, and prescriptive guides as to what to study and in what order. CMI software is useful to both students and teachers. Students should have the opportunity to track their own progress in school and to see the rationale for the work they are doing. CMI can reduce busywork. When CMI is cost-effective and instructionally sound, staff and students shall have this aid.

Goal 7: Students with special needs. Computer-related technology shall be routinely and readily available to students with special needs when research and practice have demonstrated its effectiveness.

A. Computer-based adaptive technologies shall be made available to students who need such technology for communication with other people and/or for communication with a computer.

B. When CAL has demonstrated effectiveness in helping students with particular special learning needs, it shall be made available to the students.

C. All staff that work with students with special needs shall have the knowledge and experience needed to work with these students who are making use of computer-based adaptive technologies, CAL, and computer tools.

Educational System Goals—Capacity Building

The three goals in this section focus on permanent changes in our educational system that are needed to support achievement of Goals 1-7 listed previously.

Goal 8: Staff development and support. The professional education staff shall have computers to increase their productivity, to make it easier for them to accomplish their duties, and to support their computer-oriented growth. Every school district shall provide for staff development to accomplish Goals 1-7, including time for practice, planning, and peer collaboration. Teacher training institutions shall adequately prepare their teacher education graduates so they can function effectively in a school environment that has Goals 1-7.

This means, for example, that all teachers shall be provided with access to computerized data banks, word processors, presentation graphics software, computerized gradebooks, telecommunications packages, and other application software that teachers have found useful in increasing their productivity and job satisfaction. Computer-based communication is becoming an avenue for teachers to share professional information. Every teacher should have telecommunications and desktop presentation facilities in the classroom. Computer-managed instruction (CMI) can help the teacher by providing diagnostic testing and prescription, access to item data banks, and aids to preparing individual education plans.

Goal 9: Facilities. The school district shall integrate into its ongoing budget adequate resources to provide the hardware, software, curriculum development, curriculum materials, staff development, personnel, and time needed to accomplish the goals listed above.

Goal 10: Long-term commitment. The school district shall institutionalize computers in schools through the establishment of appropriate policies, procedures, and practices. Instructional computing shall be integrated into job descriptions, ongoing budgets, planning, staff development, work assignments, and so on. The school district shall fully accept that "computers are here to stay" as an integral part of an Information Age school system. The community-the entire formal and informal educational system-shall support and work to achieve the goals listed above.

Assessment and Evaluation Goals

The three goals listed in this section focus on doing strategic planning and on obtaining information about the effectiveness of programs for information technology that are implemented by teachers, schools, and school districts.

Goal 11: Strategic plan. Each school and school district shall have a long-range strategic plan for information technology in education. The plans shall include ongoing formative evaluation and yearly updating.

Goal 12: Student assessment. Authentic and performance-based assessment shall be used to assess student learning of information technology. For example, when students are being taught to communicate and to solve problems in an environment that includes routine use of the computer as a tool, they shall be assessed in the same environment.

Goal 13: Formative, summative, and residual impact evaluation. Implementation plans for information technology shall be evaluated on an ongoing basis, using formative, summative, and residual impact evaluation techniques. Formative evaluation provides information for mid-program corrections. It is conducted as programs are being implemented. Summative evaluation provides information about the results of a program after it has been completed, such as a particular staff development program, a particular program of loaning computers to students for use at home, and so on. Residual impact evaluation looks at programs in retrospect, perhaps a year or more after a program has ended. For example, a year after teachers participated in an inservice program designed to help them learn to use some specific pieces of software in their classrooms, are they actually using this software or somewhat similar software?

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