multiplication

Learning to Think Mathematically About Multiplication

A Resource for Teachers, A Tool for Young Children

Jeffrey Frykholm, Ph.D.

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Learning to Think Mathematically About Multiplication

A Resource for Teachers, A Tool for Young Children

Authored by
Jeffrey Frykholm, Ph.D.

This book is designed to help students develop a rich understanding of multiplication and division through a variety of problem contexts, models, and methods that elicit multiplicative thinking. Elementary level math textbooks have historically presented only one construct for multiplication: repeated addition. In truth, daily life presents us with various contexts that are multiplicative in nature that do not present themselves as repeated addition. This book engages those different contexts and suggests appropriate strategies and models, such as the area model and the ratio table, that resonate with children's intuitions as they engage multiplication concepts.

These models are offered as alternative strategies to the traditional multi-digit multiplication algorithm. While it is efficient, is not inherently intuitive to young learners. Students equipped with a wealth of multiplication and division strategies can call up those that best suit the problem contexts they may be facing.

The book also explores the times table, useful both for strengthening students' recall of important mathematical facts and helping them see the number patterns that become helpful in solving more complex problems. Emphasis is not on memorizing procedures inherent in various computational algorithms but on developing students' understanding about mathematical models and recognizing when they fit the problem at hand.

About the Author

Dr. Jeffrey Frykholm has had a long career in mathematics education as a teacher in the public school context, as well as a professor of mathematics education at three universities across the United States. Dr. Frykholm has spent over two decades of his career teaching young children, working with beginning teachers in preservice teacher preparation courses, providing professional development support for practicing teachers, and working to improve mathematics education policy and practices across the globe (in the U.S., Africa, South America, Central America, and the Caribbean).

Dr. Frykholm has authored over 30 articles in various math and science education journals for both practicing teachers, and educational researchers. He has been a part of research teams that have won in excess of six million dollars in grant funding to support research in mathematics education. He also has extensive experience in curriculum development, serving on the NCTM Navigations series writing team, and having authored two highly regarded curriculum programs: An integrated math and science, K-4 program entitled Earth Systems Connections (funded by NASA in 2005), and an innovative middle grades program entitled, Inside Math (Cambium Learning, 2009). This book, *Learning to Think Mathematically about Multiplication,* is part of his Learning to Think Mathematically series of textbooks for teachers. Other books in this series include: *Learning to Think Mathematically with the Rekenrek; Learning to Think Mathematically with the Number Line;* and *Learning to Think Mathematically with The Ratio Table.*

Dr. Frykholm was a recipient of the highly prestigious National Academy of Education Spencer Foundation Fellowship, as well as a Fulbright Fellowship in Santiago, Chile to teach and research in mathematics education.

More recently, Dr. Frykholm authored Grades 1 and 2 of *Bridges in Mathematics* Second Edition, published by The Math Learning Center.

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Learning to think Mathematically: An Introduction

The *Learning to Think Mathematically* Series

One driving goal for K-8 mathematics education is to help children develop a rich understanding of numbers – their meanings, their relationships to one another, and how we operate with them. In recent years, there has been growing interest in ***mathematical models*** as a means to help children develop such number sense. These models (e.g., the number line, the rekenrek/number rack, the ratio table, the area/array model of multiplication, etc.) are instrumental in helping children develop structures – or *ways of seeing* – mathematical concepts.

This textbook series has been designed to introduce some of these models to teachers – perhaps for the first time, perhaps as a refresher – and to help teachers develop the expertise to implement these models effectively with children. While the approaches shared in these books are unique, they are also easily connected to more traditional strategies for teaching mathematics and for developing number sense. Toward that end, we hope they will be helpful resources for your teaching. In short, these books are designed with the hope that they will support teachers' content knowledge and pedagogical expertise toward the goal of providing a meaningful and powerful mathematics education for all children.

How to Use this Book

This is not a typical textbook. While it does contain a number of activities for students, the intent of the book is to provide teachers with a wide variety of ideas and examples that might be used to further their ability and interest in approaching the topics of multiplication and division from a conceptual point of view. The book contains ideas about how to teach multiplication through the use of mathematical models like the area model and the ratio table. Each chapter has a blend of teaching ideas, mathematical ideas, examples, and specific problems for children to engage as they learn about the nature of multiplication, as well as these models for multiplication.

We hope that teachers will apply their own expertise and craft knowledge to these explanations and activities to make them relevant, appropriate (and better!) in the context of their own classrooms. In many cases, a lesson may be extended to a higher grade level, or perhaps modified for use with students who may need additional support. Ideas toward those pedagogical adaptations are provided throughout.

Book Chapters and Content

This book is divided into five chapters. The first chapter, *The Nature of Multiplication and Division*, explores various contexts that are multiplicative in nature. While the idea of "repeated addition" is certainly a significant part of multiplicative reasoning, there are other equally important ways of thinking about multiplication. Contexts that promote these different ways of thinking about multiplication are presented in Chapter One.

The second chapter, *The Times Table*, encourages students to discover and appreciate the many patterns that exist in the times table. When students are given the opportunity to investigate the times table deeply, they will discover interesting patterns and number relationships that ultimately help them develop intuitive strategies and conceptual understanding to help master the multiplication facts. For example… every odd number is surrounded by even numbers… the product of two odd numbers is always odd… the product of two even numbers is always even… the product of an even and an odd number is always even… diagonals in the times table increase and decrease in regular increments… there is a line of reflection from the top left to bottom right corner of the times table… etc. There are many number relationships in the times table, and if given the chance, students will make many discoveries about multiplication and division on their own. These findings are important for the development of their confidence and mastery of the basic facts, a topic that is also addressed in the second chapter.

The third chapter of the book, *The Area Model of Multiplication*, explores the area model as a viable method not only to conceptualize multiplicative contexts, but also to find solutions to multiplication problems. Area representations for multiplication are common in geometry, but are rarely used to help students learn how to multiply. Hence, we lose a valuable opportunity to make mathematical connections between, in this case, geometric reasoning and arithmetic. Resting heavily on the important mathematical skill of decomposing numbers, the Area Model recognizes the connection between multiplication as an operation, and area models as representations of multiplication. Moreover, the area model allows student with strong spatial reasoning skills to visualize the product of two numbers as an area. The intent of this chapter is to present students with problems that will help them develop facility with this representational model for multiplication computation, as well as to use that model to better understand what multiplication really is.

The fourth chapter of the book, *The Ratio Table as a Model for Multiplication*, illustrates how children may complete two and three-digit multiplication problems with the ratio table. (There is an entire book in the Thinking Mathematically series devoted to the ratio table: *Learning to Think Mathematically with the Ratio Table*.) Building on the mental math strategies developed more fully in the Ratio Table book noted above, students develop powerful techniques with the ratio table that they may choose as an alternative to the traditional multiplication algorithm. The benefit of the ratio table as a computational tool is its transparency, as well as its fundamental link to the very nature of multiplication. Multiplication is often described to young learners as repeated addition. Yet, this simple message is often clouded when students learn the traditional multiplication algorithm. A young learner would be hard pressed to recognize the link between the traditional algorithm and "repeated addition" as they split numbers, "put down the zero", "carry", and follow other steps in the standard algorithm that, in truth, hide the very simple multiplicative principle of repeated addition. In contrast, the ratio table builds fundamentally on the idea of "groups of…" and "repeated addition." With time and practice, students develop remarkably efficient and effective problem solving strategies to multiply two and three-digit numbers with both accuracy, and with conceptual understanding.

The final chapter of the book, *The Traditional Multiplication Algorithm*, is an important chapter. We must recognize the value of the traditional multiplication model – it has been taught almost exclusively in American schools for over a century. It is ubiquitous in elementary text books, and certainly is one of the most well-recognized and commonly used methods in all of arithmetic. And yet… research has indicated that the traditional algorithm is difficult for children to understand from a conceptual point of view. With practice, children memorize the steps of the traditional model, Without conceptual understanding, however, they are often unable to determine if they have used the algorithm correctly, or whether or not they have obtained a reasonable answer for the problem context. Hence, while we certainly should continue to teach the traditional model, it may not be the multiplication model of choice for many students if they are given the chance to learn other methods of multiplication in the same depth as we typically teach the traditional method. This chapter elaborates the traditional algorithm, drawing comparisons to other methods of multiplication when relevant.

Chapter 1: The Nature of Multiplication and Division

The primary goal for this chapter is to provide students with a conceptual introduction to multiplication (and by extension, division). In order for students to appropriate any method for multi-digit multiplication or division – and to understand what they are doing through the method – they must develop some basic understanding of the nature of multiplication. This would include, for example, *what multiplication is*, how various multiplication contexts can be represented with different models, and how these representations and subsequent models can lead to elegant solution strategies and, ultimately, answers to multiplication problems.

Four primary kinds of multiplication problems are highlighted in this introduction. These include:

- Multiplication as "*repeated addition*" (e.g., 3 groups of 4)
- Comparison problems (e.g., "I have 4 times as many as you have.")
- Area representations
- Combinations

Brief conceptual explanations of these problem types are included below, with the intent that they are presented to students as well. Subsequently, an activity designed to be distributed to students is presented which supports understanding of these initial multiplication contexts.

Multiplication as Repeated Addition

The first, and most common, representation of multiplication is often thought of as "repeated addition." Indeed, the first multiplication algorithms were created to help people solve addition problems of this nature. That is, people sought more efficient methods for adding the same number to itself over… and over… and over.

The most common type of multiplication problem in traditional elementary textbooks is of this variety. For example:

Manuel has 4 packs of gum. Each pack of gum has 6 individual sticks. How many individual sticks of gum does Manuel have?

This problem is most often conceptualized, and therefore solved, by repeated addition. The solution may be found by adding the following:

$$6 + 6 + 6 + 6 = 24$$

Again, most of the multiplication problems we have typically presented to young learners in the traditional elementary classroom come in this form. While there is nothing wrong with doing so – indeed, many problem in real life are of this nature – we would be remiss if we did not present other multiplicative contexts such as the following.

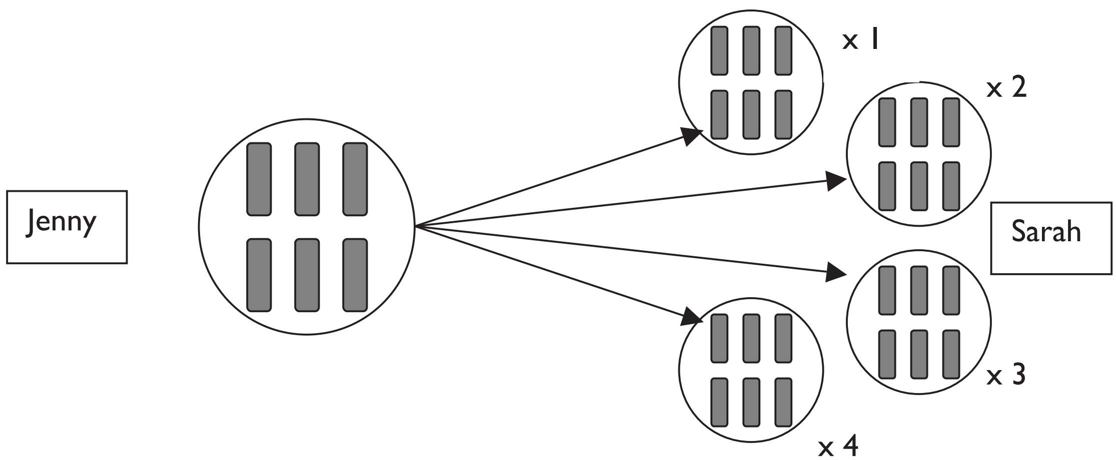
Multiplication as a Comparison

Comparison problems are solved similarly to repeated addition problems: we employ simple addition techniques to resolve the problem. What makes them fundamentally different, however, is that we conceptualize a comparison problem in a different manner. In other words, the comparison problem elicits a different kind of thinking than the typical "repeated addition" problem. Therefore, we must offer students ample opportunities to engage comparison problems, to think about them conceptually, and to consequently solve them with an appropriate tool.

The "comparison" problem often includes language such as, "… times as many as…" For example,

Jenny's found six shells at the beach. Sarah found 4 times as many. How shells did Sarah find?

This problem can also be solved by adding, but we *imagine* the problem in a different way than the traditional "repeated addition" problem. This envisioning includes the notion of a *comparison*. In this case… we compare Jenny's six shells with Sarah's collection – 4 times as many.



Multiplication as Area

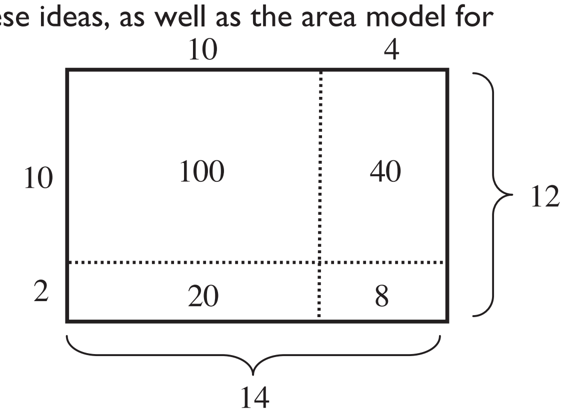
One of the first "formulas" that young children learn in the math classroom is that the area of a rectangle may be found by multiplying the length of the rectangle by its width. We can use this common understanding to provide children with a viable method for multiplication. The product of two numbers can be shown as a visual representation, in the form of a rectangle. For example, the answer to 14 x 12 may be found through the illustration shown below – a rectangle with sides of lengths 14 and 12. This method is extremely effective so long as students understand two primary mathematical ideas: 1) numbers (just like physical areas) can be decomposed into the sum of smaller numbers (or the sum of smaller areas); and 2) the distributive property can be used to break down one large multiplication problem into several smaller ones. With minimal practice, students grasp these ideas, as well as the area model for multiplication, with confidence.

<u>Example**:** Solve 14 x 12 with the Area Model</u>

Find the sum of the individual areas:

100 + 40 + 20 + 8 = **168**

14 x 12

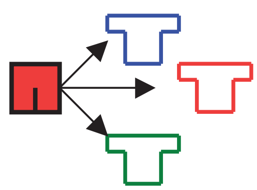


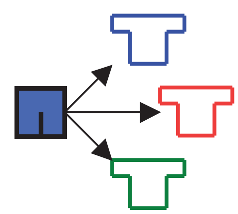
Multiplication as a Combination

Combination problems are probably the least often represented multiplication problems in the elementary curriculum. Yet, they do offer children a unique way to conceptualize multiplication that will be visited later in their mathematics careers as they explore combinations and permutations. For younger children, however, this idea of different "combinations" can be a helpful tool for representing multiplication, particularly when multiplying more than two numbers together.

The "combination" form of multiplication often requires students to imagine the total number of distinct combinations that can be made with two or more unique items. For example:

Nikki has 2 shorts, and 3 shirts. How many outfits can she wear?





Solution: How many distinct outfits? 3 with red shorts, 3 with blue shorts.

Red-Blue	Blue-Blue
Red-Red	Blue-Red
Red- Green	Blue-Green

The problems on *Student Activity Sheet #1* will encourage students to begin thinking about multiplication in these four different forms. As you begin to engage students in divergent thinking about multiplication, be sure they are comfortable with these four problem types. Help your students make connections between verbal descriptions of the problems with their visual representations. Cues in the language of the problem should help students correctly match the description with the model. These pictorial models themselves are suggestive of various ways in which these problems can be solved.

Student Activity Sheet 1

NAME: ______________________________

1. Match the problem statements below with the diagram that best fits the solution.

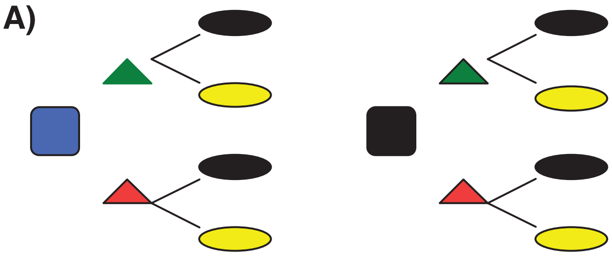
Statement 1: I rode my bike 10 miles an hour for 6 hours. How many miles?

Statement 2: I have 3 fish tanks, each holding 4 fish. How many fish?

Statement 3: The table is 4 feet long, and three feet wide. How many tiles do we need to cover the table?

Statement 4: The soccer team has 2 jerseys, 2 shorts, and 2 pairs of socks. How many uniform combinations?

A)



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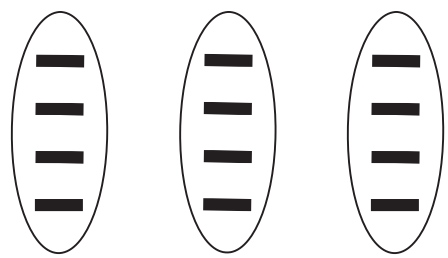

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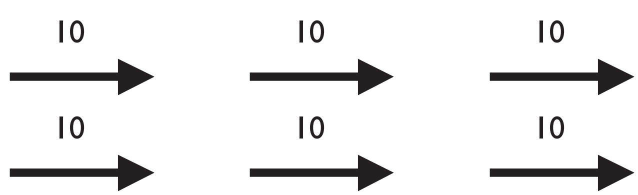
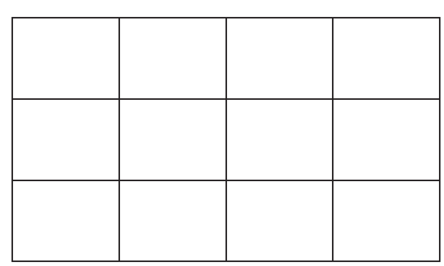


D)



Multiplication and Division: Related Operations

As students begin to develop conceptual understanding of multiplication – what it *is*, and how we use the operation to solve problems – it is important to help children see the close connection between multiplication and division. This relationship is waiting to be explored, and we must take advantage of readily available opportunities to encourage students to see the connections both conceptually, as well as operationally. The activities that appear on *Student Activity Sheet #2* will help young learners recognize the inverse relationship shared by multiplication and division.

The key point to illustrate at this juncture in their learning about these operations is that in both division and multiplication, we are always seeking to discover the unique relationship that exists between three numbers. For example, consider the various ways we might explore the following fact: 3 x 4 = 12.

In one case, we may wish to find the answer (the *unknown result*) of the product:

$$3 \times 4 = \boxed{???}$$

In a second case, we may know the result, but not the starting value of the relationship:

$$\boxed{???} \times 4 = 12.$$

In this form, we are likely to view the problem as one of division, and thereby seek to solve it with a division process as well. Of course, it might just as easily be solved by multiplication, a strategy many students will take if appropriately prepared to engage problem contexts intuitively. The idea of seeking the relationship between three numbers is significant: students should be fluent in expressing these number relationships in both multiplication and division contexts.

The activity sheet asks students to develop this relational view of multiplication and division by asking them to change a multiplication problem to one of division, and vice versa. As students engage these problems, emphasize that the three numbers in each problem are in relationship with one another, and that the relationship can be expressed either by division, or by multiplication.

Student Activity Sheet 2

NAME: ______________________________

<u>Introduction:</u> It can be helpful to think about multiplication and division at the same time. In other words, any multiplication problem can also be thought of as a division problem. It just depends on how you look at it. For example…

I rode my bike 10 miles every hour, for 3 hours. How many miles did I ride?

Could be changed to a division problem in this way…

I rode my bike for 3 hours. I traveled a total of 30 miles. How fast (miles per hour) was I riding?

-----

1. Change the two following multiplication problems into division problems by reworking the statement. (There is more than one correct answer!)

a) *I have 3 fish tanks, each holding 4 fish. How many fish?*

<u>Rewrite as a division problem</u>:

____

b) *The table is 4 feet long, and 3 feet wide. How many square tiles (each tile is !"!) do we need to cover the table?*

<u>Rewrite as a division problem</u>:

____

2. Remember the four different kinds of multiplication problems from an earlier activity? Come up with your own multiplication problem for each kind below.

a) Repeated addition

b) Comparison

c) Area

d) Combination

Chapter 2: The Times Table and Basic Facts

Before moving into the various models for multi-digit multiplication that comprise the bulk of this book, we would be remiss if we were not first thoughtful about the necessary building blocks for multi-digit multiplication. Specifically, in order to multiply larger numbers accurately and with meaning, students must first have command of the single digit multiplication facts.

There are various perspectives on what this "*command of the facts*" means. As adults, we can probably remember our first explorations with multiplication facts, which likely included an emphasis on memorization, flash cards, and instant recall. For many years, the informal standard has been that students should be able to recall any single-digit multiplication fact accurately within three seconds. One might ask the question, however, "What makes three seconds the magic length of time to recall a number fact?" One might also ask if it is indeed necessary to have every multiplication fact *memorized*. If a student is able to *reason* her way to the solution of 6 x 8 quickly and accurately (as opposed to recalling the fact from rote memory), might we agree that she is proficient with that fact?

Imagine, for example, this sort of thinking:

> **Q:** What is 6 x 8?
> **A:** Ok.. 6 groups of 8. Well, I know that 5 groups of 8 is 40. So, one more group of 8 must be 48. So, 6 groups of 8, or 6 x 8, equals 48.
>
> **Q:** What is 5 x 6?
> **A:** Ok.. 5 groups of 6 would be half as much as 10 groups of 6. I know that 10 times 6 is 60. Half of 60 is 30. So… 5 groups of 6 is equal to 30.

The confident mathematical thinker – one with good number sense, with an understanding of what it means to multiply, with the wherewithal to link one fact to another – could complete either of these particular chains of reasoning in 3-4 seconds. One might argue that this child, in fact, has a richer "command" of 6 x 8 than a child who is able to recite the answer from memory, but does not fundamentally know what this number fact means, or how it could be used to determine a different fact.

It is not our intent in this book to take up the argument of whether number facts should be memorized or derived. The point is that before children can multiply large numbers, they must be confident multiplying single digits. The purpose of this chapter is to introduce a decidedly *conceptual* approach to the teaching and learning of the times table. We invite you to use as many of these ideas and strategies as you find useful in the context of your classroom. Several student activity sheets are included throughout this chapter that we have found to be helpful for young learners who are still wrestling to master the multiplication facts.

Patterns in the Times Table

Learning the times table is one of the most memorable aspects of the elementary math experience. Of course… sometimes the memory is not always positive! Based perhaps on the expectation that children leaving the elementary classroom should have a firm command on their multiplication facts, we often jump right into the teaching of the times table without pausing to appreciate the richness of this collection of math facts as they appear in order, in the times table. In truth, there are many patterns that exist in the times table that can help students not only develop a more nuanced sense of multiplication, but also master the multiplication facts much more easily than they might otherwise. *Appendix A* contains a completed multiplication fact table that you may use for various activities that follow in this chapter.

We begin this exploration of the times table – and the many facts it contains – with a search for meaningful patterns that exist in the table itself. If students are given opportunity to investigate the times table and to discover the many interesting patterns that exist within it, there is a much greater chance that they will be able to develop intuitive strategies that will help them master the multiplication facts.

Inquisitive learners will enjoy the pursuit of unique patterns in the table. Given time and perhaps some basic directions, they will discover, for example, that every odd number is surrounded by even numbers… that the product of two odd numbers is always odd… the product of two even numbers is always even… the product of an even and an odd number is always even… diagonals in the times table increase and decrease in regular increments… there is a line of reflection from the top left to bottom right corner of the times table… etc. There are many number relationships like these in the times table. If given the chance, students will make many discoveries about multiplication and division on their own. These findings are important for the development of their confidence and mastery of the basic facts.

On *Student Activity Sheet #3*, students are given a completed times table, and asked to look for patterns. It will be helpful to give the students an introduction to the table as some students may be seeing it for the first time. Others will have varying levels of experience with the table and the facts contained therein. Be sure to offer an appropriate level of introduction which should include a basic explanation of what the table is, how each cell is determined, etc. To prepare students for the subsequent activity, you might highlight a pattern or two with the class, and then as a whole group solicit suggestions for other obvious patterns the children might see.

The activity sheet is somewhat self-guided, and is designed both to be helpful in a large group setting, or as an individual exploration as might best fit the needs of your classroom. Allow students to linger on this problem for at least 15 minutes. They will make many interesting discoveries! Students will notice that some rows or columns increment by certain amounts, or end in a particular number, or have a special relationship with adjacent cells. Some students will notice patterns with odd and even numbers. Some will notice interesting patterns on the diagonals. As students reveal these "discoveries," push them to explain *why* that particular phenomenon occurs.

Student Activity Sheet 3

NAME: ______________________________

Directions: Does this table of multiplication facts look familiar? Maybe you have used a table like this one to help learn your times facts. That is a lot to remember! Or… is it?

There are many interesting patterns in the table that will help you learn and recall the number facts. Take a few minutes to explore the table. Circle any patterns you see, and then describe them below. Two examples have been given for you.

X	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

1) The second row increases by 2 all the way across: 2, 4, 6, 8, …

2) Start with the 6 on the first row (just below the yellow row). Go down one, and over one to the left. Keep doing that. The numbers are like a mirror: 6, 10, 12, 12, 10, 6.

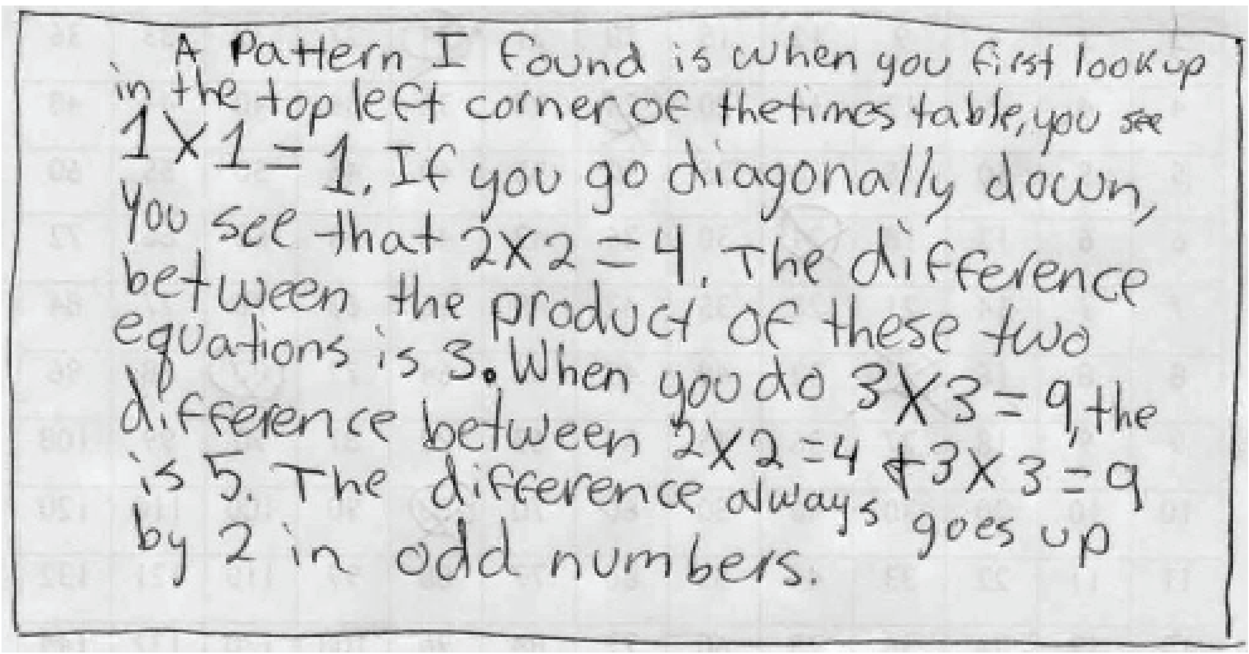
3)

Teaching Tip: Examining the thinking of students

You may wish to have students write about their understandings of various patterns in the times table. The following two examples are reflective of qualitatively different thinking strategies and sophistication that are common among students. By asking students to describe their thinking in writing, you are encouraging powerful metacognitive skills that will serve them well throughout their mathematics education.

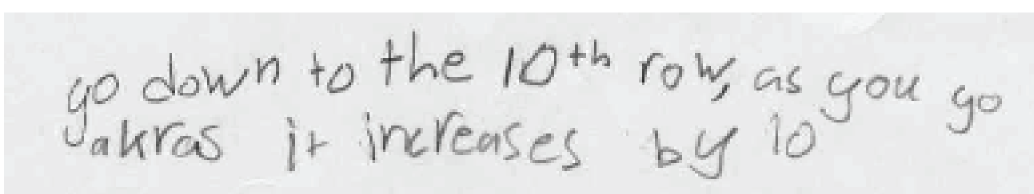
Samples of Student Thinking: Patterns in the Times Table

Sixth Grade Thinking



A Pattern I found is when you first look up in the top left corner of the times table, you see 1X1=1. If you go diagonally down, you see that 2X2=4. The difference between the product of these two equations is 3. When you do 3X3=9, the difference between 2X2=4 + 3X3=9 is 5. The difference always goes up by 2 in odd numbers.

Fourth Grade Thinking



go down to the 10th row, as you go akros it increases by 10

More about Multiplicative Patterns

In the following exercise (*Activity Sheet 4*), students are asked to shade in all multiples of either 4, or 8. While they may choose between the two options, doing both will of course benefit their understanding.

As students complete this task, they may recognize that the multiples of 4 and the 8 overlap – that is, every multiple of 8 (e.g., 8, 16, 24, 32, etc.) is also a multiple of 4, although the reverse is not true). If students do not recognize this on their own, have them partner with a peer who chose to shade the opposite set of multiples (or have the students shade multiples of both 4 and 8).

In fact, it is well worth the time to duplicate the times table provided in the activity, and have them find patterns in the multiples of other numbers as well. Be sure to help them recognize patterns that occur horizontally (jumping by an interval), as well as the patterns that occur up and down the vertical columns as well. The time invested in uncovering patterns with this exercise will pay dividends later for all students, particularly those students that may continue to struggle with the multiplication facts. In fact, it is a good exercise for students to shade the multiples of every number between 2-10. The more they can recognize and predict patterns of multiples, the more they will be able to draw on their intuitions as they learn and use the multiplication facts.

Activity Sheet 5 presents some challenges for students who may be ready to think more deeply about the times table, and multiplication in general. Depending on the grade and level of your students, these problems may not be accessible to every child. They examine a few intriguing mathematical concepts that are embedded in the times table. While these patterns may not be particularly practical or helpful toward the mastery of individual number facts, they nevertheless illustrate interesting nuances in the table that are likely lost on most individuals that casually peruse the times table, or focus exclusively on memorization. Encourage students (as appropriate) to grapple with the questions that exist on *Activity Sheet 5*. Particularly talented mathematical thinkers will find these interesting contexts to be naturally intriguing.

Student Activity Sheet 4: Patterns

NAME: ______________________________

Let's look at some patterns. You get an interesting design when you shade in all ***multiples*** of 6 (all the numbers you list if you started counting by sixes, like… 6, 12, 18, 24… and so on).

X	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

What patterns exist if you shade by 4's? What about shading by 8's? Choose one of these multiples (4 or 8), and shade in the table below.

Your Choice: Shade in the Multiples of _____

X	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Student Activity Sheet 5: Challenge

NAME: ______________________________

Use the times table on the second page to answer the following questions.

1) Did you know…that every odd number is surrounded by even numbers? Check it out for yourself, and then write an explanation for why that is the case.

2) Did you know…that if you start in any box in the top row and go down and to the left diagonally, it will look like a mirror? That is, you will see a group of the same numbers, repeated in order, "reflected" over the diagonal that starts in the upper left corner, and goes to the lower right corner of the table. Please explain why that pattern exists.

3) Did you know…that if you choose any four boxes that make up the corners of a rectangle, and then multiply the ***opposite* corners** together, you will get the same answer? (See the shaded box in the table below.) Find another example that works, and then explain ***why*** the process will always work.

Example: Explore the shaded box. Why does 10 x 24 = 20 x 12 ?

4) Did you know …that this table can be used to multiply even bigger numbers?

- What is 2 x 23? (Check out the **striped box outlined in black:** 46!!)
- How about 4 x 12? (check out the **striped box outlined in Black** → 48!)
- Here are two tough ones. Can you explain the answer to 10 x 78? Or, 8 x 56? (These cells are highlighted in the table.)

Table for Student Activity Sheet 5

X	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	**10**	15	**20**	25	30	35	40	45	50	55	60
6	6	**12**	18	**24**	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Teacher Tips, Activity Sheet 5

These problems points specifically to several unique characteristics of the times table. Explanations for each specific problem appear below.

1) Every odd number is surrounded by even numbers… Why? This phenomenon can be traced to something that students may have noticed in their informal examinations of the times table. Specifically, when multiplying, three options exist: 1) an odd times an odd; 2) an even times and even; or 3) an odd times and even. The only case in which the product of two numbers is odd is when the two numbers being multiplied are themselves odd. This goes back to the first conceptualization of multiplication – repeated addition. The only way to get an odd number is to add an even with an odd. Hence, when we multiply an odd number an odd number of times – say 3 x 5 for example – what we are really doing is repeatedly adding 5, in this case three times. Any number plus itself is even (e.g., 5 + 5 = 10). Hence, if we continue, adding an odd (5) to an even (10) results in an odd answer (10 + 5 = 15). So… back to the original question: Why is every odd number surrounded by even numbers? Because the only way to get an odd is to multiply an odd by an odd. No matter which adjacent cell one selects will be the product of an odd times an even, or an even times an even. To illustrate this phenomenon, select one or two odd entries in the table, and look at the combinations of numbers that lead to the products of the adjacent cells.

2) The diagonal line from upper left to lower right is the "squares" line. This is a great opportunity to introduce some important mathematical concepts. First, the "squares" diagonal is a line of reflection, which points to the notion of symmetry. The portion of the table to the left of the "squares" line is identical to the portion of the table to the right. The reason for this is that multiplication is ***commutative***. This is a significant concept for students to embrace. 4 x 3, (4 groups of 3) is the same quantity as 3 x 4 (three groups of four). To begin this discussion, ask students to locate a particular cell – say the cell containing the answer to 8 x 4. Then, ask them to find the cell containing the answer to 4 x 8. Do this with several other number pairs as well. They soon will see the relationship – that these pairs of numbers are on the same diagonal, perpendicular to the squares line.

3) This interesting relationship intrigues many students. It can quite easily be explained when students look at the smaller products that comprise the larger problem. Take the given example: the rectangle outlined by the corners 10, 20, 24, and 12. The problem suggests that for any rectangle, the products of the opposite corners will be equal. So, in this case, 10 x 24 = 20 x 12. Help the students explore where each of these particular "corners" came from: 10 is found by multiplying 2 x 5; 24 is found by multiplying 6 x 4; 20 is found by multiplying 5 x 4; 12 is found by multiplying 2 x 6. When you combine these products according to the definition of the rule (the products of the opposite corners are equal to each other), you get: 2 x 5 x 6 x 4 = 5 x 4 x 2 x 6 – the same product of numbers!

4) The number relationships explored in this problem are quite interesting. The times table can be used as a calculator for larger computations assuming that students are comfortable with base-10 principles. The first example highlighted asks students to

consider the product of 2 x 23. This problem is solved by using the distributive property. That is, 2 x 23 is the same as (2 x 20) + (2 x 3). In the table, we first look at the cell containing the product of 2 x 2, which is really 2 x 20, or 40 (4 groups of 10). The second cell we consider the product of 2 x 3 = 6 (ones). So, our answer is 40 +3 = 43.

The same logic can be used anywhere in the table. For example, consider the example of 10 x 78 (highlighted in green). When broken down with the distributive property, this can be expressed as (10 x 70) + (10 x 8). When expanded, we can find each of these products in the table. 10 x 7(0) = 70 (groups of 10). Likewise, 10 x 8 = 80 (ones). Hence, the product is 70 tens plus 80 ones: 780.

Teaching The Multiplication Facts

The times table can be very helpful in learning the multiplication facts. For many years, teachers and text books were inclined to begin the process of memorizing the times facts by starting with the 1's, and gradually progressing toward 10, or beyond. Knowing much more now about how children think, and how they use intuition to solve problems, it is far more effective to consider the big picture – the whole table – when teaching children the multiplication facts. Children probably know more than they think they do. For starters, one doesn't have to memorize every box – only ***half*** of the table! The right half of the table (shaded) is identical to the left half of the table. The diagonal line down the middle (called the "doubles" line) acts as a mirror, separating two identical sets of multiplication facts.

Duplicate facts, to the right of the Doubles Line

The "Doubles" Line

X	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

There are some other helpful facts that students probably know already as well. As an exercise with students, shade in the boxes that correspond to the following number facts:

The ones facts… (one times any number is the number itself)
The twos facts… (doubles… 2x2, 2x3, 2x4, 2x5, etc.)
The fives facts… (five, ten, fifteen, twenty…)
The tens facts… (ten, twenty, thirty, forty…)

In the table below, building on what the students already know, the following facts have been shaded:

The 1's... The 2's... The 5's... The 10's...

The Times Table… what we already know!

X	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Now… if a student knows these facts, she can probably figure out the related "next door neighbor" facts. That is, if you know that 5 groups of 5 is 25, then… **6** groups of 5 (6x5) would be five more than 25 … or, 30. As you are encouraging students to think in this way, return to the times table and shade in the following boxes:

- **The twos (+1)…** for example, if 2 groups of 6 = 12, then 3 groups of 6 would be… 2 x 6 = 12… *plus* one more 6 →18!

- **The fives (+1 and -1)…** for example, if 5 groups of 8 = 40, then 4 groups of 8 would be 5 x 8 = 40 … *minus* one group of 8 → 32!

- **The 10's (+1 and -1)** … for example, if 10 x 6 = 60, then 11 x 6 would be 10 x 6 = 60… *plus* one more 6 → 66!

Shade in these "next door neighbor" facts with the class. When students begin to view the number facts in this way – building on what they already know, and using their mathematical intuitions to construct one fact from another known relationship – the times table quickly becomes a manageable task.

A Question for Students: Are there other number facts that you can quickly solve based on what you already know? Shade them in!

Teacher notes: Summarizing the teaching of the multiplication facts

After the informal explorations with the times table completed previously, students who may not be comfortable with all of their multiplication facts can turn to the table for that purpose – to guide their use of intuition and informal explorations to help them recall the multiplication facts.

Before progressing, the question must be asked: What does it mean to have mastery of the times table? Of course, instant recall of the multiplication facts is desirable. With time and proper exposure, most students develop such facility with the times table. But, how soon should we expect students to have instant recall of the times facts? There is notable research to suggest that over-emphasizing recall of the facts found in the times tables at an early age can actually inhibit students' understanding of the mathematics of multiplication. So, then, what is mastery?

A good rule of thumb is that if students are able to discern a correct answer to a single digit multiplication problem within 3-5 seconds, that is probably quick enough for the moment (assuming that greater efficiency naturally comes with time). Truly, there are few situations in life in which it is crucial that young learners be able to recall multiplication facts in less than 3 seconds. Taking the pressure off students in this regard will lead them to greater confidence in their intuitive ability to discern these facts, as well as their overall efficacy as learners and doers of mathematics. The 60-second multiplication facts tests of the past have long been known to leave indelible and negative marks on students – and for what reason? If students can solve single digit multiplication problems in a few seconds using any number of intuitive strategies, they will be ahead in the long run when compared to students who are forced to simply memorize 100 number facts. So… the moral of the story here is *to relax*. Let your students use informal strategies to solve multiplication problems. With repeated practice and exposure, they will become more efficient –without the scars that many adults carry around today because of the way in which they were intimidated by the number facts. The strategies outlined previously in this section of the book will help students toward this end.

Chapter 3: The Area Model of Multiplication

Overview

The area model of multiplication is often thought of as the most conceptually comprehensible multiplication strategy for young learners. In addition to the fact that this strategy builds on the learner's spatial reasoning (a preferred learning style for more than half of all young children), this strategy also clearly isolates the partial products of the multiplication problem, an essential component of learning any multi-digit multiplication strategy. Three examples are provided below to illustrate the learning path that is elaborated in this chapter.

<u>Example 1: Single digit multiplication with the area model</u>

If teachers introduce the area model with small numbers (e.g., 3 x 4), students can literally count tiles or cells, further solidifying their conceptual understanding of ***area*** (i.e., "covering" a given space) as representative construct for multiplication and division. For example:

Question: What is the answer to 3 times 4?

Response: It might be helpful for learners to think of this problem in terms of rows and columns. That is, we might recommend children to restate the problem in these words:

"How many tiles are there in 3 rows, where each row has 4 tiles?"

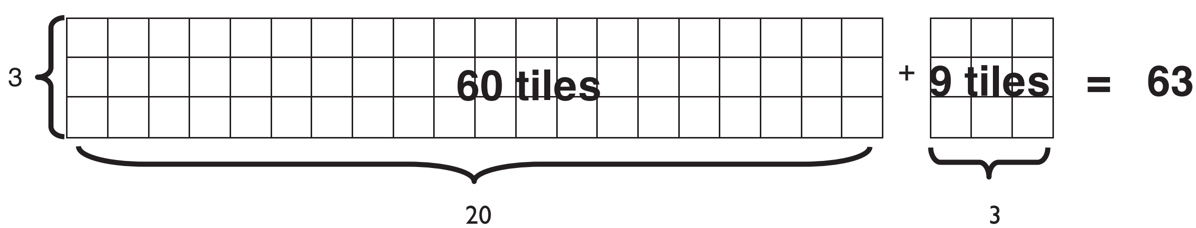
1	2	3	4
5	6	7	8
9	10	11	12

<u>Example 2: Single x Double-digit multiplication with the area model</u>

With practice, students can apply an area model strategy with larger numbers. The ability to decompose numbers will be crucial to the successful implementation of this strategy. *Decomposition* is the ability for the learner "see inside" a number, and to think of it in terms of its component parts. That is, the number 23 might be thought of as one group of 20, plus 3 more. The number 18 might be thought of as one group of 10, and 8 ones. This ability to decompose is essential for successful use of the area model with multi-digit numbers. For example:

Question: What is the answer to 23 times 3?

Response: I need 3 rows of 23. To make it easier, I can think of 23 as one group of 20, plus 3 more. Therefore, I need three rows of 20, plus 3 rows of 3.

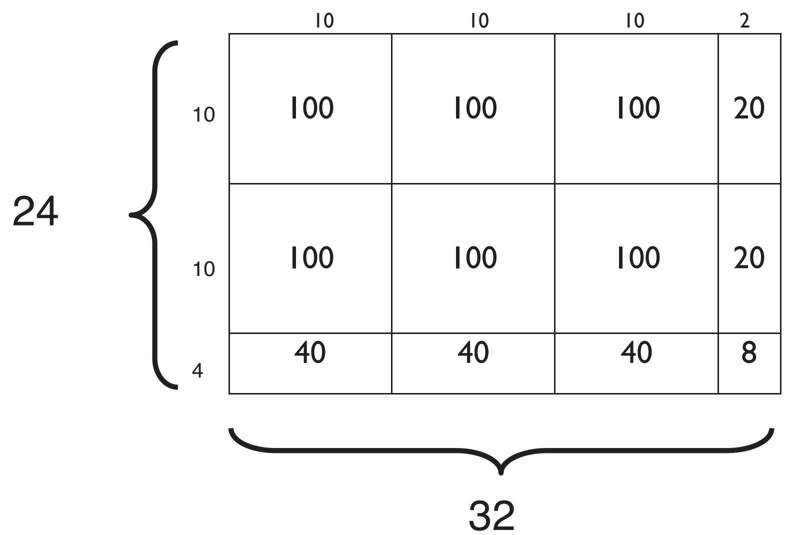


<u>Example 3: Multi-Digit multiplication</u>

As students become comfortable with the use of partial products and various representations that make use of them, they will be able to tackle larger, multi-digit problems with confidence such as the example provided below. A significant portion of this chapter is devoted to this pursuit. It is crucial to note that, whether using area models, ratio tables, or the traditional multiplication algorithm, the degree to which students can see the partial products inherent in the problem will likely determine their success with the strategies.

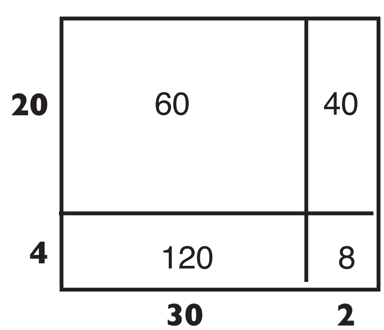
Question: What is 24 x 32?

Response: I can think of 24 as 20 (10 + 10), plus 4 more. Likewise, I can think of 32 as 30 (10 +10 + 10), plus 2 more. Arranging these component parts in a rectangle, I get the following:



Area = (100 + 100 + 100 + 100 + 100 + 100) + (40 + 40 + 40) + (20 + 20) + (8)
= (600) + (120) + (40) + (8)
= **768**

This chapter outlines a progression of activities that will build students' confidence with the area model as illustrated in the previous examples. If students are given multiple opportunities to practice this method, they will quickly gravitate toward nuanced (and rapid) application of the area model, even with larger numbers. A student who is comfortable with this method will, for example, solve the previous problem with a diagram like the following, indicating a sophisticated understanding of decomposition, and indeed, multiplication itself.



The Area Model: Beginning Steps

At the conclusion of this chapter on the Area Model, students should be able to:

- Recognize the connection between multiplication as an operation, and area models as representations of multiplication;
- Use the area model to visualize products of two numbers;
- Use the area model to help understand both decomposition of numbers, as well as the distributive property of multiplication.

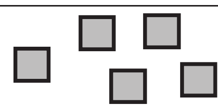
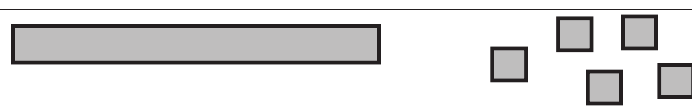
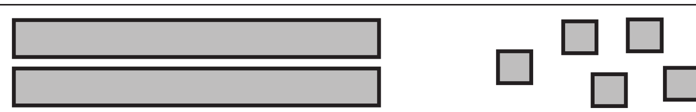
Fundamental to understanding the area model as a computational tool is that, 1) numbers can be decomposed into the sum of smaller parts; and 2) the distributive property can be used to break down one large multiplication problem into several smaller ones. Begin to teach the area model with these two ideas in mind.

Decomposition and Partial Products

To begin this chapter, students must engage the idea that multi-digit numbers can be represented as the sum of a set of smaller parts. This ideas is known as “number decomposition.”

Students must be encouraged to think of 15 as the sum of 10 + 5. Or, 36 might be represented as the sum of 10 + 10 + 10 + 5 + 1. Once this has been established, the next question to consider is whether or not this process of decomposing numbers can actually assist us with other operations and actions on numbers. In the case of multiplication, it turns out that it can.

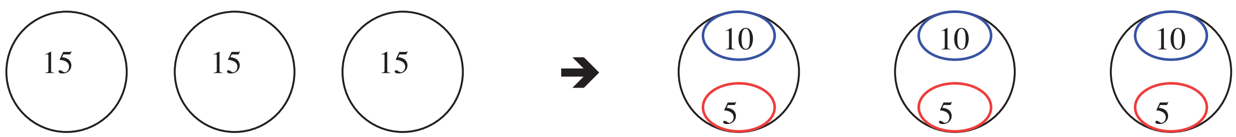
<u>Begin by using base-10 blocks to represent a series of numbers</u>. Engage students in a discussion of the following (and additional) representations. They should be encouraged to represent numbers, decomposed into component parts, with their own base-10 blocks.

Represent 5 with base-10 blocks	
Represent 15 with base-10 blocks	
Represent 25 with base-10 blocks	
Represent 33 with base-10 blocks	
Represent 18 with base-10 blocks	

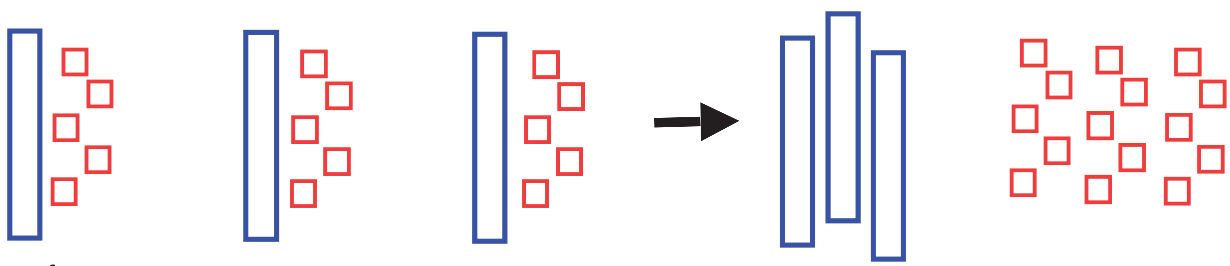
What makes decomposition of numbers important is the way that it can be combined with the distributive property in order to flexibly and powerfully compute with large numbers. Fundamental to the area model is the understanding that 3 x 15, for example, can be thought of in the following way:

3 x 15 ➔ 3 groups of 15 ➔ 15 + 15 + 15 = 45

Building on the previous representations with the base-10 blocks (or other manipulatives), it would be relatively straight forward to rewrite this problem (3 piles of 15 counters) by rearranging the 15 counters strategically. For example, we could separate each pile of 15 into two piles of 10 and 5.



Illustrate this problem with base-10 blocks.



Therefore…

3 x (10 + 5) ➔ 3 groups of 10 and 3 groups of 5 ➔ 10 + 10 + 10 + 5 + 5 + 5 = 45

This concept is fundamentally important to the area model of multiplication. Instead of using piles of counters, however, we can represent the quantity of each number to be multiplied as the dimensions of a rectangle. From this point, students are led to the idea that the whole area of the rectangle is simply the sum of the areas of smaller regions within the rectangle. For example:

10
10
10

+

1	**2**	**3**	**4**	**5**
1	**2**	**3**	**4**	**5**
1	**2**	**3**	**4**	**5**

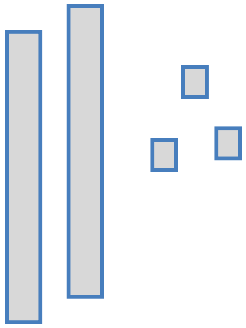
3 x 15 ➔ (10 + 10 + 10) + (5 + 5 + 5) ➔ 30 + 15 = 45

The problems on Activity Sheet 6 will help student develop understanding of these important prerequisite concepts for the area model of multiplication.

Student Activity Sheet 6: Decomposition

NAME: ______________________________

Use Base-10 Blocks to help you answer the following questions. Show each number with as few blocks as possible. Draw in your answer with a pencil.

23 (Example) 	**12**
42	**14**
18	**52**
34	**122**

Beginning with the Area Model

The intent of the problems found in Activity Sheet 7 is help students not only develop facility with a representational model for multiplication computation, but also to use that model to better understand what multiplication really is.

To set the stage for the problems in Activity Sheet 7, begin by motivating the idea that multiplication can be thought of as finding the "area" of a rectangular space. Using tiles on the floor, measurements of a rectangular rug, tiles on the ceiling, an arrangement of blocks, etc., model the connection between area and multiplication.

<u>Teaching Example 1</u>: What is the area of a rug that is 2 feet wide, and 5 feet long?

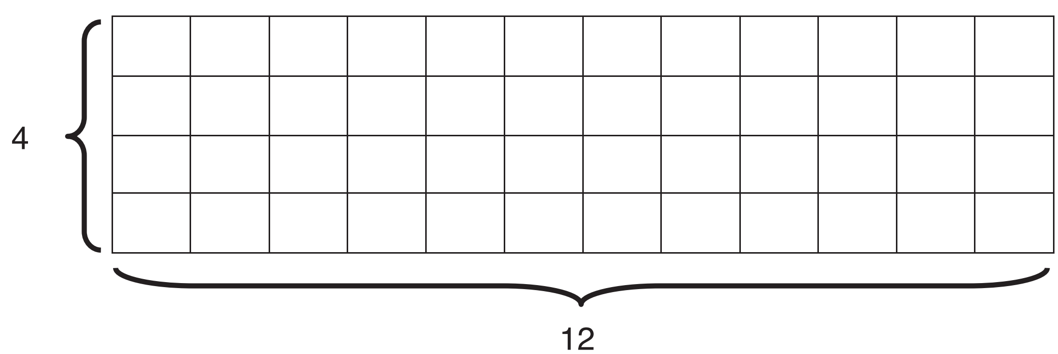
Model the problem with tiles or blocks, and have students count the number of squares required to cover the rug.

1	2	3	4	5
6	7	8	9	10

<u>Teaching Example 2</u>: What is the area of a rug that is 4 feet wide, and 3 feet long?

1	2	3
4	5	6
7	8	9
10	11	12

<u>Teaching Example 3</u>: What is the area of a rug that is 4 feet wide, and 12 feet long? Do we really want to count each square? What if the dimensions were 24 feet wide, by 48 feet long? Is there a better strategy to find the answer than counting each square?

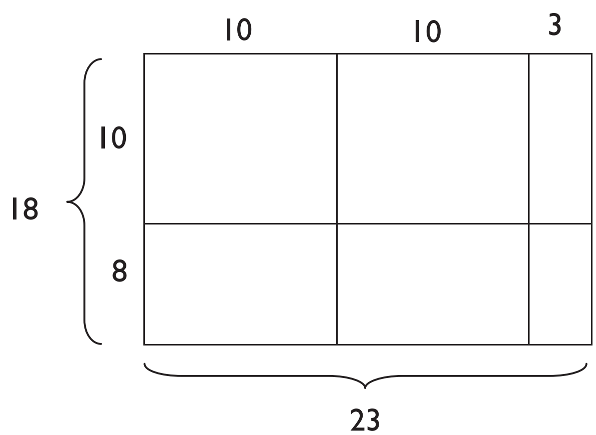


Solicit and record students' suggestions about how to more efficiently find the number of tiles without counting each one.

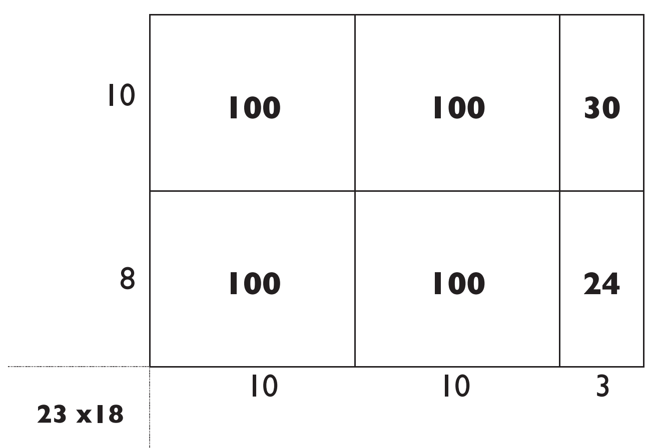
A Teaching Example

Pose the following question to students: How can we use the area of a rectangle to find the answer to 23 x 18? Proceed to explain the following steps to students.

We can use some basic number principles to make a helpful diagram to solve this program by computing its area. We know, for example, that 23 can be expressed as 23 = 10 + 10 + 3. We also know that 18 can be expressed as: 10 + 8. By making a rectangle with these dimensions in mind, we can begin to make progress toward finding a solution – the area of a rug that is 23 feet long, and 18 feet wide.



With this new drawing, we can now compute our original problem (23 x 18) using the smaller rectangles. That is, we can compute the area in each of the smaller squares quite easily, and then add the totals together. For example, this **first square** has dimensions of 10 x 10, which means it has an area of 100. The same can be done for each of the other squares.



So… when we add all of the squares together, we get:

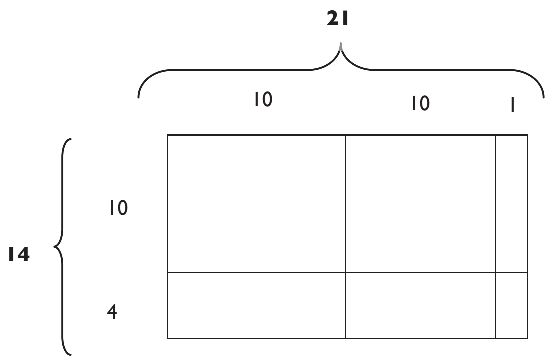
23 x 18 → 100 + 100 + 100 + 100 + 30 + 24 = **454**

Student Activity Sheet 7: Area Model

NAME: ______________________________

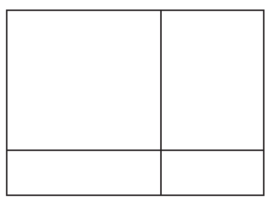
Use the Area Model to complete the following multiplication problems.

1) 14 x 21



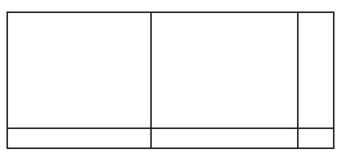
ANSWER: __________________

2) 12 x 15



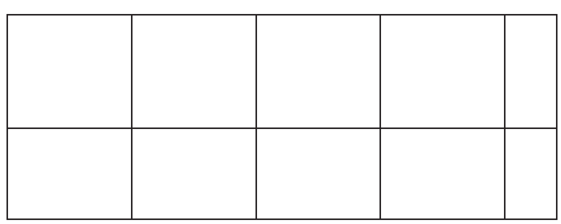
ANSWER: __________________

3) 11 x 22



ANSWER: __________________

4) 17 x 42



ANSWER: ______________

5) 13 x 12

NAME: ______________________________

Student Activity Sheet 7: Area Model
Continued

6) 18 x 34

ANSWER: ______________

7) 26 x 24

ANSWER: ______________

8) 12 x 300

	100	100	100
10			
2			

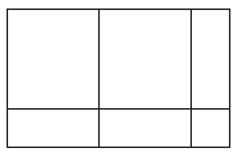
ANSWER: ______________

9) 26 x 24

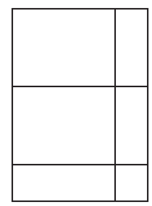
10) 15 x 211 (Challenge!)

Teacher notes for Activity Sheet 7

The progression of problems on Activity Sheet 7 is intended to guide students through gradually more sophisticated problems that require subtle nuances in the use of the area model diagrams. The starting diagrams provided in some of the problems are intended to be helpful scaffolds for students to lean on as they engage this method of multiplication on their own. Be aware that some students may see the possibility of setting up *different* rectangular area models that might also be used to solve the problem successfully. For example, an area model for *12 x 23* might be set up in one of two representations:



12 x 23



23 x 12

Teachers might challenge students to use both representations to solve the problem which will not only help students develop confidence with this method, but also more fully come to understand the nature (and definition) of multiplication.

As students work the problems, continue to help them recognize the value in deliberately choosing to decompose numbers so as to later take advantage of multiplication "facts" that may come more easily than others. Multiplying by 10, for example, is one of the first conventions of multiplication that students cling to. Multiplying by 100 is also a powerful strategy that is helpful when computing with larger numbers. On the final challenge problem, for instance, students that are grasping this method may, on their own, be able to diagram the problem by using 1's, 10's, and 100's in the following way:

15 x 211	**100**	**100**	**10**	**1**
10	**1000**	**1000**	100	**10**
5	**500**	**500**	50	**5**

Solution: 15 x 211 = 1000 + 1000 + 500 + 500 + 100 + 50 + 10 + 5
= **3165**

Please note that this diagram is not drawn to scale. Such is the nature of the use of some mathematical models. At first, some students may wish to represent each problem to scale. It may be helpful, in fact, to give students standard graph paper that they can use to model simple problems. (Base-10 blocks may be used with similar effect.) The purpose of using models to represent and solve problems, however, is to provide students with a powerful method that can effectively enable them to represent and illuminate mathematical relationships and meaning, and ultimately find correct answers to problems with a high degree of confidence. In fact, in this specific case of area models for multiplication, the representations of students will tell you a great deal about the depth of their understanding. While at first many students will draw their area models with a great deal of care and precision, with time, those children who begin to grasp the method will spend less time on the diagram itself. This is a great sign that they are understanding the decomposition of numbers, conservation of area, and indeed, the nature of multiplication itself. The model should facilitate the development of more sophisticated understanding of the problem itself, if not multiplication more generally.

For example, consider how three students might solve the following problem: 22 x 25 = ?

Example 1: 22 x 25

	10	**10**	**5**
10	**100**	100	50
10	**100**	100	50
2	**20**	**20**	**10**

Example 2: 22 x 25

	20	**5**
10	200	50
10	200	50
2	**40**	**10**

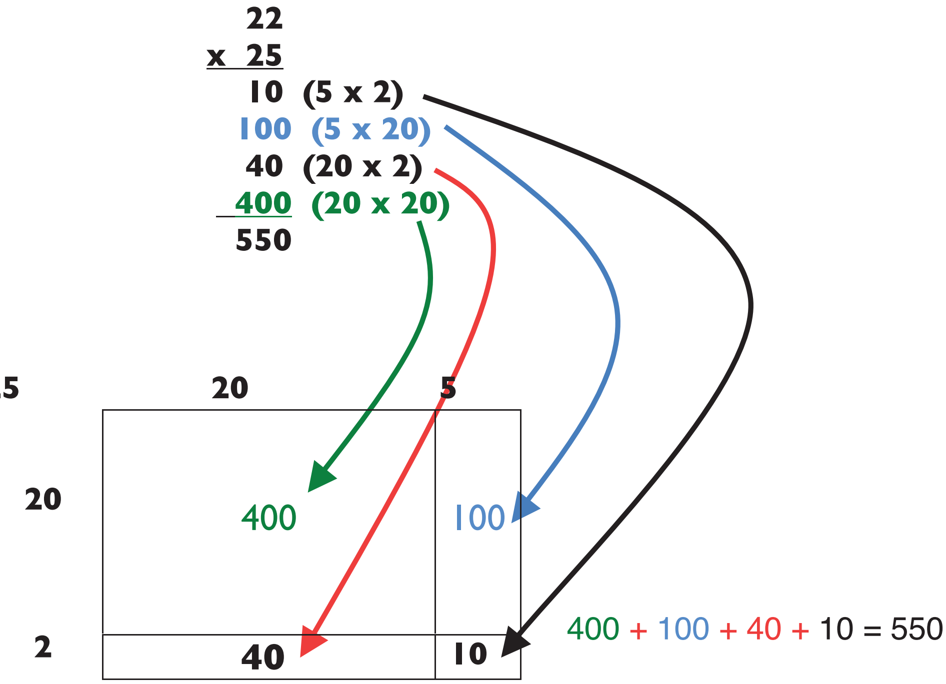
Example 3: 22 x 25

	20	**5**
20	400	100
2	**40**	**10**

Finally, please note the connection between the area models illustrated above, and the traditional multi-digit algorithm that is so commonly taught in the upper elementary grade levels. Each step in the traditional method is shown below. When color coded and compared to the area model, the links between the model and the algorithm become transparent.

Traditional Method: 22 x 25

Area Model: 22 x 25



Up for consideration is the degree to which each model allows not only for the correct solution to be found, but also, and perhaps more importantly, a chance to fundamentally understand *what it means to multiply*. One might argue that the mathematical meaning of multiplication is somewhat veiled in the traditional method, whereas the area model quite clearly illustrates the ways in which the partial products may be combined to determine the final answer of the problem.

In summary, central to the area model are a few key mathematical ideas: 1) the notion of *number decomposition* (e.g., 23 can be thought of as a group of 20, plus 3 more), 2) the distributive property (i.e., 3 x 23 = 3 x 20 + 3 x 3), as well as 3) the more abstract notion of the conservation of area – a rectangle can be broken into smaller parts that, when added together, represent the same area as the original rectangle. With repeated use of the area model – tying these constructs together – students develop a rich understanding of multiplication, both a useful and efficient method to solve multiplication problems, but also more fundamentally, a nuanced understanding of *what it means to multiply*.

Chapter 4: The Ratio Table as a Model for Multiplication

In the ongoing debate in the mathematics education community about the relative merits of teaching for algorithmic proficiency versus teaching for understanding, the ratio table has emerged as a mathematical model that can be viewed as a conduit between these two sides of the debate.

Given its rise to popularity in contemporary math education curriculum, a stand-alone book in the *Thinking Mathematically* series has been devoted entirely to an explanation of the ratio table as a model for multiplication and division. The information shared in this chapter is only a summary of the full range of applications of the ratio table that can be found in the full book (*Learning to Think Mathematically with the Ratio Table*). Please understand that this chapter is a proxy to the power and applicability of the ratio table as a mathematical model that has great potential to influence the thinking and mathematical understanding of children.

What is a ratio table? And, why should we use it?

The ratio table is an excellent computational tool that, when understood well by students, can be used quickly, efficiently, and accurately to multiply and divide, calculate percentages, explore ratios and proportions, etc. It is particularly effective in problem contexts that present themselves in a way that elicits proportional reasoning. For example, consider the following two problems:

> *Example 1*: What is the square footage of a garage that is 15 feet wide and 12 feet long?
>
> *Example 2*: Markers come in packs of 12. The teacher ordered 15 packs. How many markers are there in 15 packs?

These problems are fundamentally the same: 12 x 15 = 180. Yet, if students are encouraged to let the context inform their progress toward a solution, they will approach each problem in very different ways.

Students that have only experienced the traditional algorithm will likely solve both problem contexts in the following way, which is likely familiar to most all of us:

Solution 1: Traditional Model (12 x 15)

$$\begin{array}{r} 15 \\ \underline{\times\ 12} \\ 30 \\ \underline{150} \\ 180 \end{array}$$

Students who are comfortable with the area model as articulated in the previous chapter, however, would likely solve *Example !* (i.e., square foot of a garage) with an area diagram such as the following:

Solution 2: Area Model (12 x 15)

	10	5
10	100	50
2	20	10

Area = 100 + 50 + 20 + 10 = 180.

Example 2 (i.e., packs of markers), in contrast, is stated in a way that elicits proportional reasoning, and therefore is a great candidate for the use of a ratio table as a solution strategy. The ratio table below is a representation of the following thinking as a solution to the problem of 15 packs of 12 markers.

Step 1: "One pack contains 12 markers."
Step 2: "Okay… if one pack contains 12 markers, then *two* packs contain 24."
Step 2: "Likewise… if one pack contains 12 markers, then *ten* packs would contain 120 markers."
Step 3: "Ok… if ten packs contain 120 markers, then 5 packs must have 60 markers."
Step 4: "So… if ten packs contain 120, and if 5 packs contain 60, then 15 packs must contain 180 markers."

Expressed in a table, this thinking might look like the following:

Solution 3: Ratio Table (12 x 15)

Packs	1	2	10	5	15
Markers	12	24	120	60	180

As suggested in this abbreviated chapter on the model, ratio tables like the one above are extremely powerful tools. As you and your students explore this chapter, opportunities will be presented for students to embrace ratio tables not only as a tool for calculations, but also as a *way of thinking* about mathematical relationships. Bear in mind that a full explanation of the nuanced uses of the model articulated below may be found in the ratio table book that is part of this *Thinking Mathematically* series.

Common Ratio Table Strategies

<u>Strategy 1: Multiplication by 10</u>

The number relationship inherent in multiplication by ten is comforting to young learners, and it becomes one of the most important and well-used strategies with ratio tables. In ratio table form, the "tens" strategy looks like the following:

Example 1:

x10

1	10
15	150

x10

Example 2:

x10

3	30
12	120

x10

Students should be encouraged to adopt the following language to accompany the mental strategy:

Example 1: *"If one group has 15, then 10 groups would have 150."*

Example 2: *"If 3 groups have 12, then 30 groups would have 120."*

Strategy 2: Multiplication by Any Number

Related to the 10's multiplication strategy above, students may also be led to recognize that multiplication by any factor is a viable ratio table strategy. For example:

x 3

Boxes	2	6
Nails	50	150

> Student thinking: "I know 2 x 3 = 6. So… I could also multiply 50 by 3: 50 x 3 = 150. That is … if 2 boxes contained 50 nails, then 6 boxes would contain 150 nails."

Quite readily, students realize that multiplication can be a helpful tool in quickly scaling any particular ratio. Moreover, recognize that students may choose various multipliers that they feel comfortable using in this process. Consider the following two examples as illustrations of these important considerations.

Example 2: There are 5 pencils in a box.
How many pencils are in 18 boxes?

x 6 x 3

Boxes	1	6	18
Pencils	5	30	90

To find the solution for this problem, the student first multiplied by 6, followed by multiplication by 3.

Consider an alternative strategy for the same problem:

Example 3: There are 5 pencils in a box.
How many pencils are in 18 boxes?

Boxes	1	9	18
Pencils	5	45	90

The strategy employed by this student was to first multiply by 9, and then multiply by 2.

Strategy 3: Doubling

A particular subset of the multiplication strategy outlined above is the special case of "doubling" a column in a ratio table. The strategy of *doubling* is quite common among children and, once again, is a wonderful mental math strategy that will serve them well throughout their many mathematical explorations in contexts both in and outside of school. Consider the following example:

Doubling Example 1: I get paid $8 per hour.
How much did I earn after working 20 hours?

Hours	1	2	4	5	10	20
Wages	8	16	32	40	80	160

Doubling Strategy

A "doubling" strategy is used repeatedly toward the solution for this problem. Each of the arrows above the table indicates that a doubling calculation between adjacent columns has taken place. On the top row, then, the student has completed the following doubles facts: 1 x 2 = 2; 2 x 2 = 4; 5 x 2 = 10; 10 x 2 = 20. Across the bottom row, we see the following numbers were likewise doubled: 8 to 16; 16 to 32; 40 to 80; and 80 to 160. Doubling is a powerful strategy with wide reaching application and, again, easily motivated and developed through the use of ratio tables.

Strategy 4: Halving

The inverse operation to doubling the entries in one column to another is to "*halve*" a column. Again, this is a powerful mental math strategy that children use intuitively in many contexts. Given that *halving* makes a quantity smaller, it should be noted therefore that this halving strategy is often used to reduce quantities in a ratio table. Hence, as discussed later in the book, halving strategies are instrumental in using the ratio table for division. Consider the following example:

Halving Example 1: 120 new baseballs must be split evenly between 8 teams.
How many new baseballs does each team get?

Halving Strategy

Teams	8	4	2	1
Baseballs	120	60	30	15

A "halving" strategy is used repeatedly toward the solution for this problem. Each of the arrows above the table indicates that a *halving* calculation between adjacent columns has been performed. On the top row, then, the student has completed the following halving calculations: Half of 8 is 4; half of 4 is 2; half of 2 is 1. Likewise, across the bottom row we see the following calculations: half of 120 is 60; half of 60 is 30; half of 30 is 15. Therefore, each team receives 15 baseballs.

Be aware that halving strategies are not exclusive to division problems. Sometimes a student will use a halving strategy as part of a string of calculations toward a desired end. Consider the following example in which the student is trying to determine how many eggs are in 6 cartons:

Cartons	1	10	5	6
Eggs	12	120	60	72

In the context of the solution strategy for this problem, the student has "halved" 10 to get 5, and subsequently taken half of 120 to arrive at 60. The student is trying to arrive at 6 groups of 12 (for a total of 72).

The thinking might go as follows: "I know that one group is 12. I am trying to find the total for 6 groups. Well, 10 groups would be 120. Half of that is 5 groups, or, 60. So, if 5 groups is 60, then one more group, six in all, would be 60 + 12 = 72 eggs." In this example we see how a halving strategy might very well be appropriately applied in a problem that is, by nature, multiplicative.

Strategy 5: Combining Columns (Adding and Subtracting)

The final strategy consists of combining columns – either by addition, or by subtraction. The basic idea that we want children to understand is that we are, in a sense, "combining buckets." For example, imagine the following ratio table that indicates the number of cherries that may be found in various combinations of baskets (15 cherries per basket).

Baskets	1	2	4	7
Cherries	15	30	60	105

We begin with our initial ratio: 1 basket holds 15 cherries. Using our multiplication by two strategy, our next column indicates that 2 baskets would therefore hold 30 cherries. The third column simply doubles the second column: if two baskets hold 30 cherries, 4 baskets would hold 60.

Now, at this point, we are ready to combine columns. Observe the 4th column. We see 7 in the top row, and 105 in the bottom row. How did we arrive at those figures? We did so by combining the previous columns. 1 + 2 + 4 = 7 baskets. Correspondingly, 15 + 30 + 60 = 105 cherries.

$$1 + 2 + 4 = 7$$

\+

=

Baskets	1	2	4	7
Cherries	15	30	60	105

=

As noted previously, one of the positive features of the ratio table is the extent to which it fosters students' mental math ability. There are several key arithmetic strategies that are essential to successful work with the ratio table. These strategies are highlighted below, followed by a series of activities that encourage the development of these mental math skills.

As you teach these methods, it is essential that students become comfortable with each of the following strategies. The ratio table can become an instrumental tool in helping children develop confidence with mental math strategies. This will occur, however, only if students are given ample opportunity to experiment with the ratio table, to "play" with different strategies, and to create their own pathways to solutions.

Though guidance is needed at first, eventually the students should take ownership of their chosen strategies. *Teachers should not force students to use a prescribed set of steps with the ratio table*, even if those steps are more efficient than the path being taken by the child. With time, students gravitate toward efficiency; they derive great satisfaction in determining their own solution strategy, and will naturally seek to complete the table with as few steps as possible. Please bear in mind the following: <u>*Pushing students toward efficiency too early will stunt the development of their native mathematical intuitions and flexibility with mental math strategies.*</u>

Strategy Summary

In the previous pages, we have explored a number of unique strategies:

- Multiplying by 10
- Multiplying by any number
- Doubling
- Halving
- Adding or Subtracting

While students will naturally gravitate to some strategies over others, it is important that they understand how the various strategies work, and how they may be used together to solve a given problem. Teachers should make every attempt to compare and share the various solution strategies used by students to solve the same problem. *Encourage a diversity of strategy use. Be sure to regularly model multiple approaches to a given problem*. This will pay dividends down the road. As students begin to trust their own intuitions and native strategies, they develop an expanded vision of mathematical thinking, of their own efficacy as young mathematicians, of the ratio table itself as a mathematical model for multiplication and division, and powerful mental math strategies.

NAME: ______________________________ **Student Activity Sheet 8: Ratio Table Strategies**

Here are some common strategies for solving problems with a ratio table.

Multiply by 10

1	10
15	150

Multiplying

2	6
50	150

Doubling

4	8
15	30

Halving

20	10
30	15

Adding

1	2	3
25	50	75

Subtracting

1	10	9
12	120	108

Find the missing numbers in the shaded boxes in the ratio tables below. Then write which of the above strategies you used.

1.

1	10	11
12	120	

Strategy:________________________

2.

1	10	5
12	120	

Strategy:________________________

3.

1	10	20
12	120	

Strategy:________________________

4.

2	12
5	

Strategy:________________________

5.

4	40
18	

Strategy:________________________

6.

2	10	12
8	40	

Strategy:________________________

7.

4	8
10	

Strategy:________________________

8.

12	
8	4

Strategy:________________________

NAME: ______________________________

Student Activity Sheet 9: Ratio Table Practice Problems

Directions: Solve each problem with a ratio table.

Example: There are 5 pieces of gum in a pack. How many pieces of gum are in 10 packs?

x 10

Pack	1	10		
Pieces	5	50		

x 10

1) There are 6 chairs per row. How many chairs are there in 10 rows?

Rows	1	10		
Chairs	6			

2) There are 3 eggs per nest. How many eggs are there in 10 nests?

Nests	1			
Eggs	3			

3) There are 4 legs per table. How many legs are there in 10 tables?

Tables				
Legs				

4) Two trays contain 12 ice cubes. How many ice cubes are there in 20 trays?

Trays	2			
Ice Cubes	12			

5) There are 4 balloons on each string. How many balloons are there on 20 strings?

Strings	1					
Balloons	4					

6) There are 12 eggs in every carton. How many eggs are there in 8 cartons?

Cartons	1					
Eggs	12					

7) There are three gallons in each container. How many gallons are in 12 containers?

Containers						
Gallons						

8) 5 students need 20 pencils. How many pencils do 40 students need?

Students						
Pencils						

9) 60 eggs fit into 4 baskets. How many eggs fit into one basket?

Baskets	4					
Eggs	60					

10) You can buy 12 peaches for $4. How much does it cost to buy 6 peaches?

Peaches						
Cost						

11) It takes 1 minute to travel 2 miles on the high speed train. How many miles can you travel in 12 minutes?

Minute	1	2	10	12		
Miles	2	4				

12) It takes 2 minutes to run 1 lap around the track. How long would it take to run 5 laps?

Laps	1	2	3	5		
Minutes	2					

13). A tube contains 3 tennis balls. How many tennis balls are there in 32 tubes?

Tube							
Tennis Balls							

14) Comic books cost $8. How much will it cost to buy 12 comic books?

Comic Books							
Cost							

15) At the market, 2 apples sell for \$3. How much would it cost to buy 12 apples?

16) At the zoo, the seals eat 25 pounds of fish a day. How much would they eat in one week (7 days)?

17) At the same zoo, 25 visitors are allowed in the gates each hour. How many visitors are allowed into the zoo in one day? The zoo is open for 8 hours each day.

18) It costs \$12 per student to get into the movie theater. How much would it cost for a group of 21 students to go to the movies?

19) Every 4 hours, the dentist sees 8 patients. How many patients does the dentist see after 11 hours?

Hours									
Patients Seen									

20) Use a ratio table to solve this problem: What is 6 x 15? (Think… six groups of 15.)

Groups of 15									
Total									

Teacher notes for Activity Sheet 9

The problems on this activity sheet reflect various strategies highlighted in Activity Sheet #8. Problems 1-4, for example, may be solved most efficiently with the "multiply by 10" strategy. Problems 5-8 can be solved by using various multiplication strategies – multiplying by 2, by 4, or by various combinations of multiplication factors. For example, on problem 8, students are asked to determine how many pencils will be needed to supply 40 students, assuming that for every 5 students, 20 pencils are required. Solution strategies will vary. While one student may choose to use a doubling strategy (Solution 1), another student may use a variety of multiplication facts (Solution 2). Both strategies are correct, and teachers should take advantage of the opportunity to compare the thinking of various students as captured in their ratio tables. These are powerful moments in the classroom in which students, when challenged to compare their own thinking to that of their peers, can make leaps in their understanding of the nature of multiplication.

<u>Problem 8</u>: 5 students need 20 pencils. How many pencils do 40 students need?

<u>Solution 1</u>: Doubling

	x 2	x 2	x 2			
Students	5	10	20	40		
Pencils	20	40	80	160		

Doubling strategy use: 5 x 2 = 10; 10 x 2 = 20; 20 x 2 = 40
20 x 2 = 40; 40 x 2 = 80; 80 x 2 = 160

<u>Solution 2</u>: Combination of strategies

	x4	x2				
Students	5	20	40			
Pencils	20	80	160			

Strategy use: Multiply by 4; Double 5 x 4 = 20… and therefore… 20 x 2 = 40
20 x 4 = 80… and therefore… 80 x 2 = 160

Problems 9 and 10 are unique in the sense that they are division problems. One of the great benefits of the ratio table is that if students understand proportional reasoning as developed through the ratio table model, then they are not inclined (nor need) to make a distinction between multiplication and division problems; they may be solved with the same ratio table strategies.

The remaining problems on the page require students to apply various strategies to resolve the problems. Bear in mind that students will need to practice with ratio tables well beyond the problems that exist on this activity sheet. Indeed, an entire book in this series is devoted to the use of ratio tables as a tool for multiplicative reasoning. Teachers may need to adapt the problems on this activity sheet depending on the level of understanding exhibited by students. In every case, however, the power of the ratio table is that it allows students to apply unique strategies toward the solution of the problem. Teachers should make use of every opportunity that arises to compare strategies among and between students. Observing the thinking patterns of peers who may have solved a problem with a different strategy is a powerful teaching and learning tool.

Chapter 5: The Traditional Multiplication Method

Overview

The intent of a model-based approach to mathematics education is to provide a scaffold for students to approach and solve problems. Effective models typically have a structure that is inherently reflective of the essential features of the mathematics embedded in the problem. In the first chapter of the book, several types of multiplication problems were presented (e.g., multiplication as repeated addition, as combinations, as an area, etc.). These models were explored in subsequent chapters. The intent of a model-based approach is, once again, to represent the mathematics of the problem in a form that reflects the inherent structure of the mathematics at hand, as well as to use that structure as an avenue toward a solution.

As we conclude this book by exploring the traditional method of multiplication, it is important to note that one of the criticisms of the traditional algorithm that has been taught almost exclusively in public schools for decades is that, while efficient, it is not inherently conceptual. There is little in the method itself that would be intuitively obvious to the student. Simply put, steps must be memorized and applied. The algorithm was promoted heavily in school textbooks in an era of American education that modeled itself on the industrial revolution in which efficiency and decontextualization were preferred over conceptual understanding. Hence, while the algorithm is efficient and, when memorized, quite dependable, it provides little scaffolding for students to understand the very discrete steps inherent in the method.

As the most common method for multiplication in the American educational context, however, it is important for teachers and students alike to grapple with the algorithm, hopefully understanding not just the steps themselves, but the mathematics behind each stage of the algorithm. Further, when fully understood, it is possible for students to make connections between the algorithm itself, and other strategies for multi-digit multiplication. The intent of this chapter is to uncover the inner-workings of the traditional method. For example, why do we put a zero down in the right hand column when we compute the second row of computation? Why do we "carry" as we multiply? These and other questions are briefly explored in this section.

Unpacking the Traditional Method of Multiplication

It is important for students to understand that there are several viable ways to multiply. No one method is better than another, as long as the method of choice is both reliable for the student, and is understood well enough such that the student has an idea when the result of a given computation is reasonable (and equally importantly, *un*reasonable). In the steps below, the traditional algorithm is dissected for students. The explanation below might be replicated with students, proceeding from simple problems to those that are more complex. Consider the following problem, solved with the traditional algorithm. What are the steps used in solving this problem, and how might we teach them to students?

Solve: 86 x 24 with the Traditional Model

$$\begin{array}{r} {}^{1}_{2} \\ 86 \\ \times\ 24 \\ \hline 344 \\ 1720 \\ \hline 2064 \end{array}$$

Let's take it one step at a time, and begin with a slightly easier problem: **86 x 4.** The key to understanding the traditional method is to recognize the use of partial products. That is, we can think of 86 x 4 in the following way:

"We need 86 groups of 4. Well… we could start with 80 groups of 4 to begin with, and then we could add in 6 more groups of 4. That gives us a total of 86 groups of 4." In mathematical symbols, we get the following:

$$86 \times 4 = (80 \times 4) + (6 \times 4).$$

The key to understanding the traditional model is to recognize this use of number decomposition (e.g., 86 = 80 + 6), as well as the use of the distributive property. Below, these concepts are reflected in the sequence of computations that comprise the traditional method.

$$\begin{array}{r} {\scriptstyle 2} \\ 86 \\ \times\ \ 4 \\ \hline 344 \end{array}$$

In step #1, we multiply 4 times 86. Only, we do it in two steps. **First,** we multiply the "ones" of the problem: 4 x 6 (6 groups of 4).

4 x 6 = 24, or… 2 tens, and 4 ones. So, we put the "4" in the ones column.

We have not yet recorded the 2 tens below the line, but we keep track of them until we can tally them together with the other tens we get as we continue the problem.

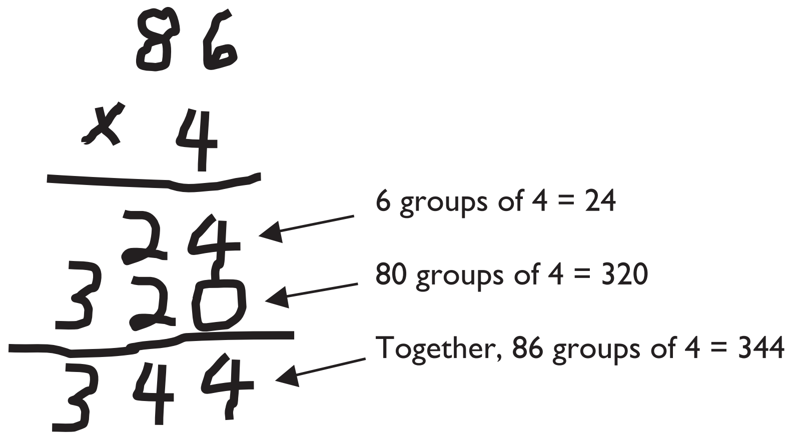
$$\begin{array}{r} {\scriptstyle 2} \\ 86 \\ \times\ \ 4 \\ \hline 344 \end{array}$$

Hundreds Tens Ones

In step #2, we continue toward the goal of finding the answer to 86 groups of 4. So far, we have recorded the total for 6 groups of 4. Now we need to determine the answer to the remaining 80 groups of 4. For this step, we will be counting tens and hundreds.

4 x 8(0) = 32(0). That means 3 hundreds, and 2 tens. Recall that we still had the 2 extra tens from the first step, which are collected in the tens column. In all, we have three hundreds, 4 tens, and 4 ones.

Recall that the traditional method was designed to be as efficient as possible, combining steps when available. One might argue that understanding is lost at the expense of efficiency. If we were to help students see the entire set of calculations for this problem, we might represent the solution to the problem in the following way:



Notice how the traditional algorithm reduces this process by a step as it requires one to "carry the two" (i.e., collect 4 ones below the line, and record the 2 tens above the problem) on the first step rather than to represent the 24 as one number below the line.

We might also consider the way students are taught to "put a zero down under the 4" prior to multiplying 4 x 32. Hidden in this step is that the calculation being made is not really 4 x 8, but rather 4 x 80.

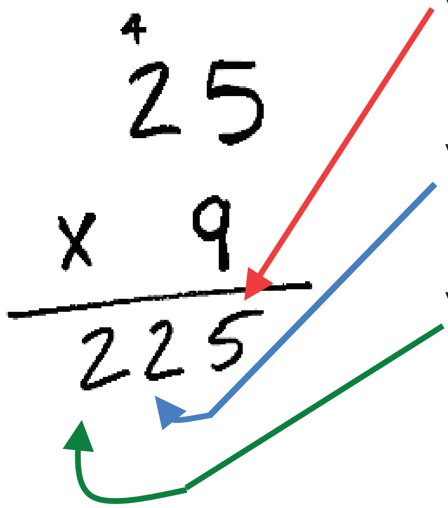
The following activity sheet asks students to provide explanations for various steps in the problem, starting with a single digit multiplier, and then moving toward a double-digit multipliers. It may be necessary to walk through several practice problems prior to assigning the tasks below. Take time to illuminate the meaning behind these steps with students as they develop both facility with, and comprehension of, the traditional multi-digit multiplication algorithm.

Student Activity Sheet 10: Traditional Method

NAME: ______________________________

Directions: Provide explanations for the steps highlighted below.

1. Can you explain each of the steps involved in the strategy shown below?

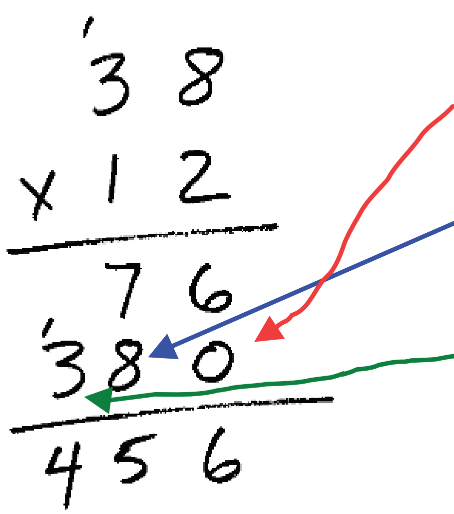


Where did these **5 (ones)** come from?

Where did these **2 (tens)** come from?

Where did these **2 (hundreds)** come from?

2. Can you explain each of these steps in this problem?



Where did the **0 (ones)** come from?

Where did the **8 (tens)** come from?

Where did the **3 (hundreds)** come from?

How did we arrive at a final answer of 456?

Student Activity Sheet 10: Continued

NAME: ________________________________

3. Solve the following problems with the traditional method.

a)

$$\begin{array}{r} 13 \\ \underline{\times\ 22} \end{array}$$

b)

$$\begin{array}{r} 34 \\ \underline{\times\ 25} \end{array}$$

c)

$$\begin{array}{r} 45 \\ \underline{\times\ 32} \end{array}$$

Appendix A
The Multiplication Table

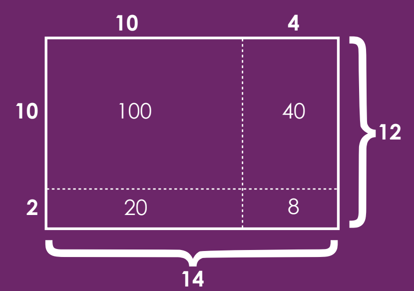
x	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

This book is designed to help students develop a rich understanding of multiplication and division through a variety of problem contexts, models, and methods that elicit multiplicative thinking. Elementary level math textbooks have historically presented only one construct for multiplication: repeated addition. In truth, daily life presents us with various contexts that are multiplicative in nature that do not present themselves as repeated addition. This book engages those different contexts and suggests appropriate strategies and models that resonate with children's intuitions as they engage multiplication concepts. The book also addresses common approaches to multiplication, including a close look at the multiplication facts, as well as the traditional multiplication alogorithms. Students are also led to see connections between multiplication and division.

- Understand the nature of multiplication
- Recognize multiplicative contexts
- Develop fluency with various multiplication models
- Make the connection between multiplication models
- See the connection between multiplication and division

Example Problem

12 x14 with the Area Model



Jeffrey Frykholm, Ph.D.

An award winning author, Dr. Jeffrey Frykholm is a former classroom teacher who now focuses on helping teachers develop pedagogical expertise and content knowledge to enhance mathematics teaching and learning. In his ***Learning to Think Mathematically*** series of textbooks for teachers, he shares his unique approach to mathematics teaching and learning by highlighting ways in which teachers can use mathematical models (e.g., the rekenrek, the ratio table, the number line, the area model) as fundamental tools their classroom instruction. These books are designed to support teachers' content knowledge and pedagogical expertise toward the goal of providing a menaingful and powerful mathematics education for all children.