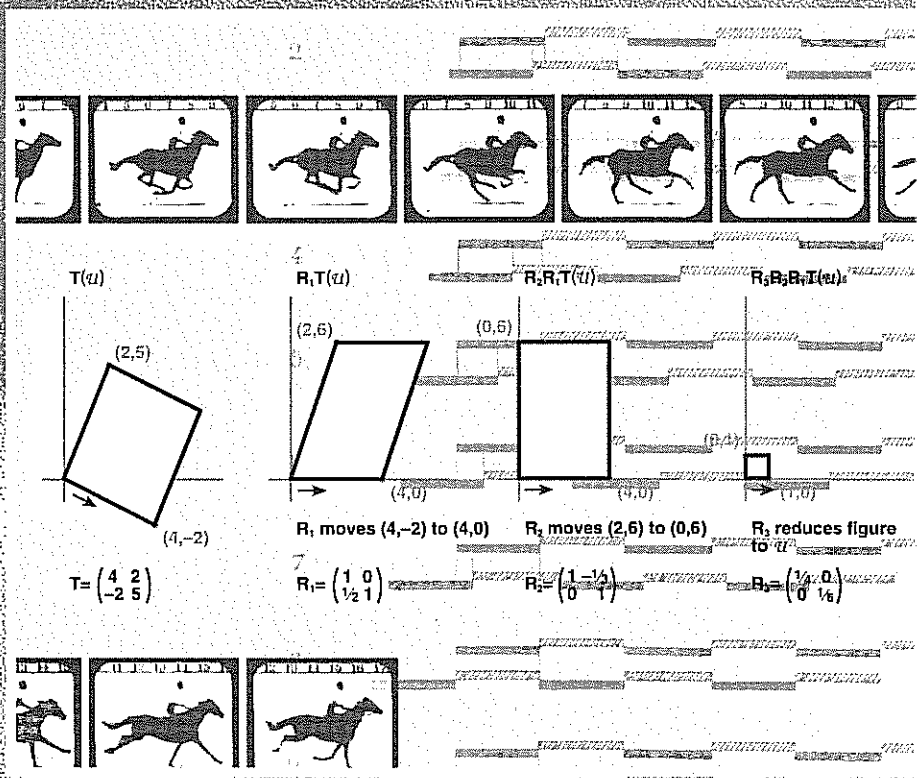
Unit XIV / Math and the Mind's Eye Activities



Looking At Linear Algebra

Eugene Maier

Looking At Linear Algebra

1 Transformations
The effects of transforming the basis vectors of a coordinate grid are depicted.

2 Coordinates and Vector Arithmetic
Coordinates of grids are introduced. Vector arithmetic is developed and used to obtain formulas for linear transformations.

3 Matrices and Determinants
The matrix and the determinant of a linear transformation are defined.

4 Products of Transformations
The *product*, or *composition*, of two transformations is defined and the matrix of the resulting transformation determined.

5 Scaling and Shearing
Scaling and shearing transformations are introduced. Later we shall see that all linear transformations are compositions of these two types of transformations.

6 Inverses
The inverse of a linear transformation is defined and methods of finding inverses are developed.

7 Systems of Equations
Methods of solving systems of two linear equations in two unknowns are developed.

8 Invariance
An *invariant* set of points is defined and examples given. *Eigenvectors* and *eigenvalues* are introduced.

9 Space Transformations
Concepts introduced in previous activities are extended to three-dimensional settings.

Math and the Mind's Eye materials are intended for use in grades 4-9. They are written so teachers can adapt them to fit student backgrounds and grade levels. A single activity can be extended over several days or used in part.

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Transformations

O V E R V I E W

The effects of transforming the basis vectors of a coordinate grid are depicted.

Prerequisite Activity

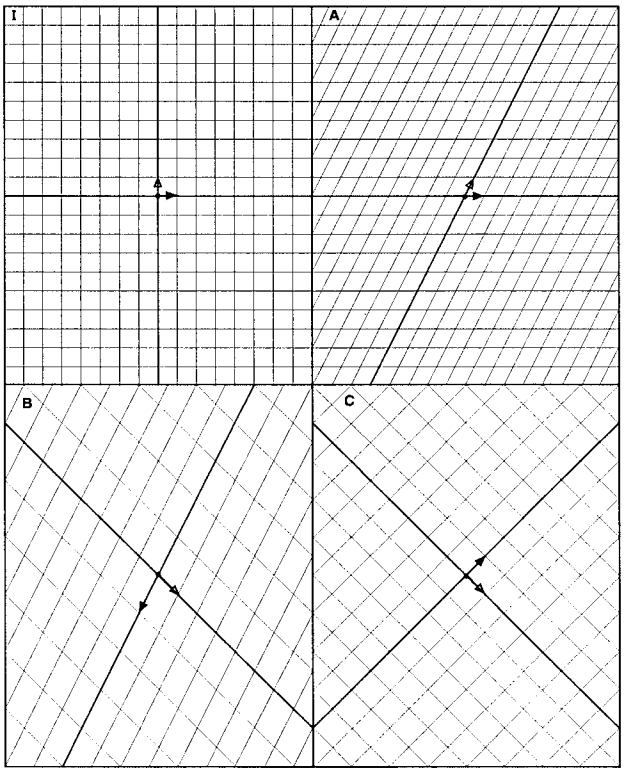
None.

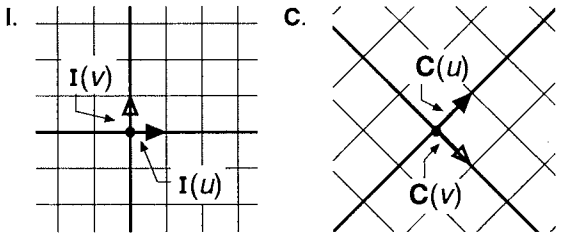
Materials

Multi-grid paper (see Comment 1), a multi-grid overhead transparency, copies of masters as noted.

Actions

1. Distribute multi-grid paper to the students. Discuss the paper with them.





Comments

1. A master for multi-grid paper is attached (Master 1). A sheet of multi-grid paper contains a conventional grid in the upper-left corner, referred to as the *initial grid* and labeled **I**. The remaining grids, **A**, **B**, and **C**, are *transformations* of **I**. These are obtained from **I** by changing the direction and length of the two *basis vectors* of **I**—shown as two arrows, one with a closed arrowhead and the other with an open arrowhead. These vectors emanate from a common point called the *origin* of the grid.

The basis vectors determine the grid. The lines containing the basis vectors are the *axes* of the grid; they intersect at the origin. The remaining grid lines are parallel to the axes, the spacing between them is determined by the length of the basis vectors. The result is a network of congruent parallelograms.

The basis vector of **I** with the closed arrowhead will be denoted by **I**(u) or, simply, u. The basis vector of **I** with the open arrowhead will be denoted by **I**(v) or, simply, v. For the other grids, **A**(u) will designate the basis vector of grid **A** with the closed arrowhead, and so forth. If no grid is designated, it is assumed that the grid is **I**, thus u always stands for **I**(u).

Notice in grid **C**, that not only have the length and direction of both basis vectors of **I** been changed but their *orientation* to one another has also changed: if one moves along **I**(u) towards its tip, **I**(v) is on the left, while if one moves along **C**(u) towards its tip, **C**(v) is on the right.

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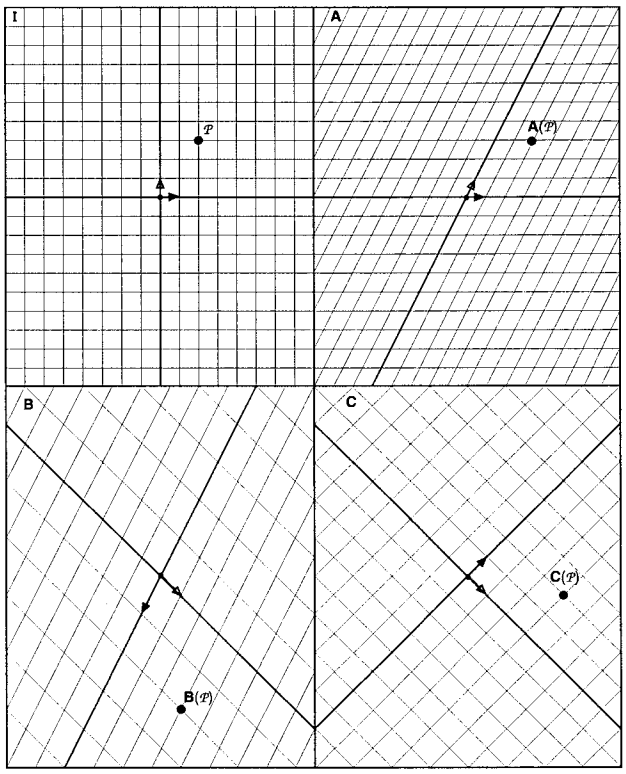
Actions

2. Describe to the students how points are transformed. Have the students mark a point on grid **I** and then transform it to grids **A**, **B**, and **C** or, to say this another way, locate its *image* on grids **A**, **B**, and **C**.

Comments

2. If grid **I** is transformed to another grid, a point $\mathcal{P}$ on **I** is transformed, or changed, to the point on the transformed grid which is in the same relative position to the grid's basis vectors as $\mathcal{P}$ is to the basis vectors of **I**. This point is called the *image* of point $\mathcal{P}$.

For example, if point $\mathcal{P}$ on grid **I** is reached by starting at the origin of **I** and moving 2 lengths of u in the direction it is pointed and then 3 lengths of v in the direction it is pointed, its image on grid **A**, denoted $\mathbf{A}(\mathcal{P})$, is reached by starting at the origin of **A** and moving 2 lengths of $\mathbf{A}(u)$ in the direction it is pointed and then 3 lengths of $\mathbf{A}(v)$ in the direction it is pointed. Master 2 shows $\mathcal{P}$, $\mathbf{A}(\mathcal{P})$, $\mathbf{B}(\mathcal{P})$, and $\mathbf{C}(\mathcal{P})$.



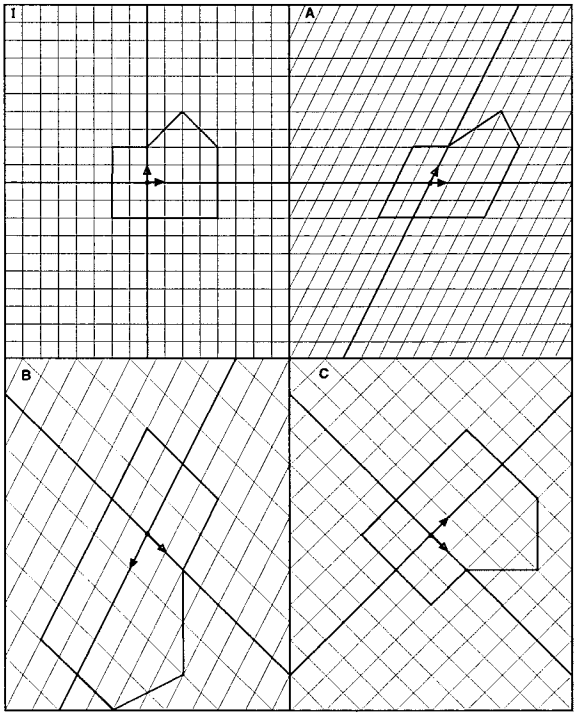
Because the size and direction of the grids vary, the images of some points of grid **I** will not lie on the given portions of grids **A**, **B**, or **C**.

Actions

3. Distribute a copy of Master 3 to the students. Ask them to draw the image of the figure shown on grid **I** on each of the grids **A**, **B**, and **C**.

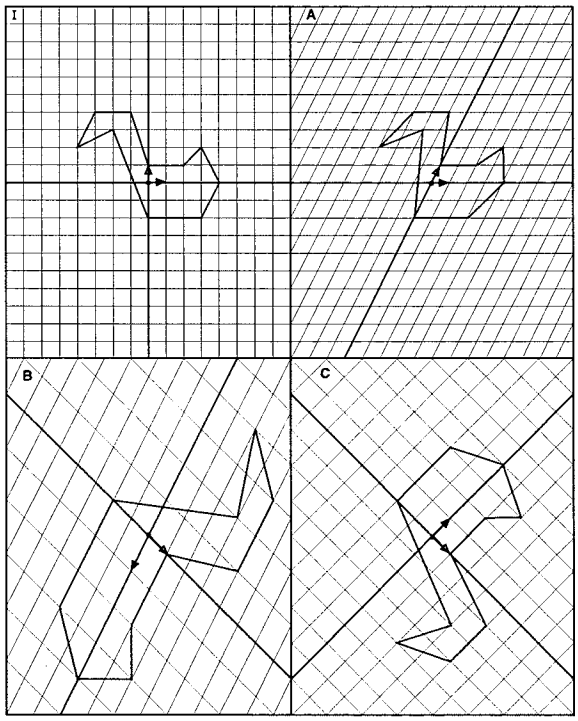
Comments

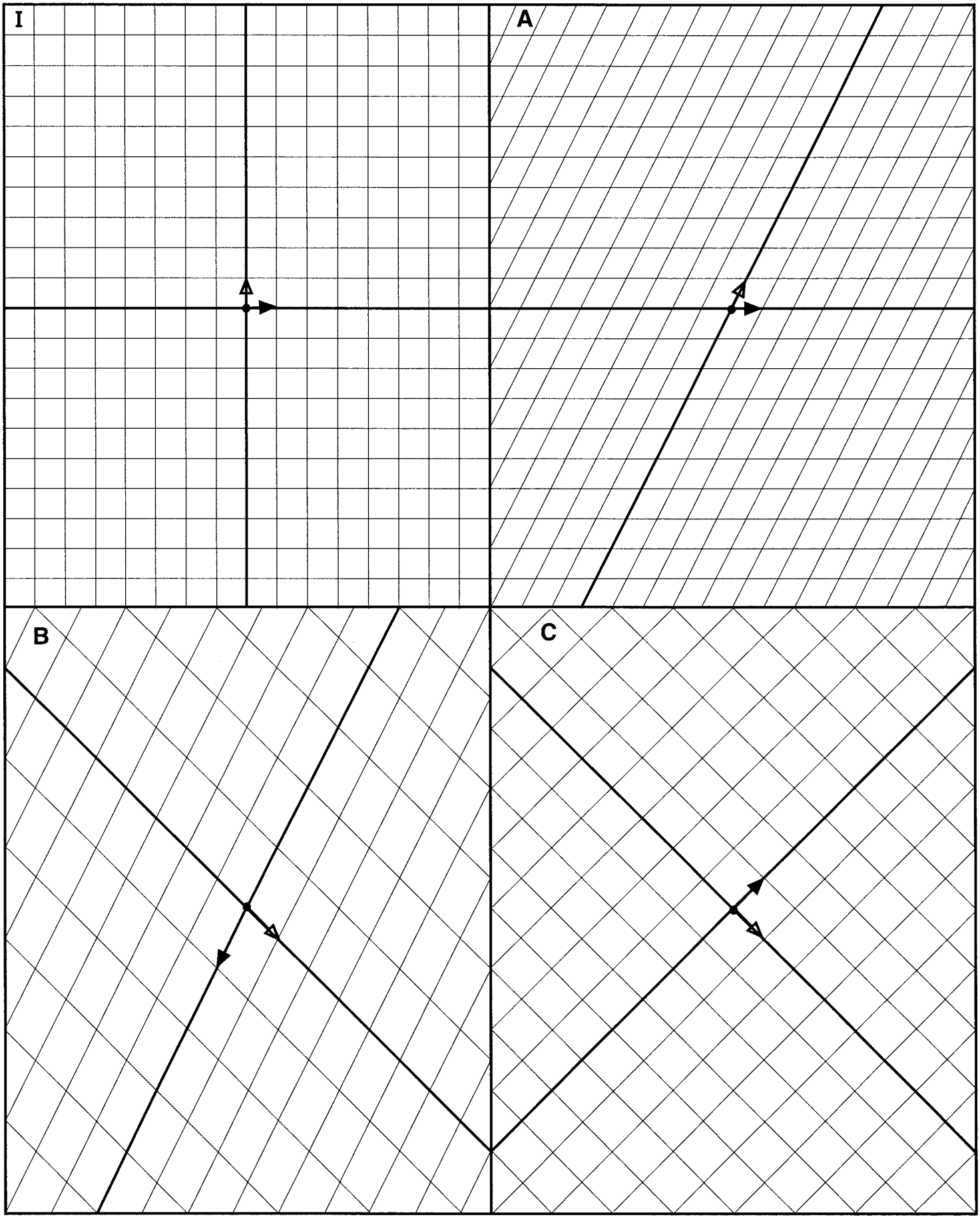
3. Master 3 has a "house" drawn on grid **I**. Its images in the other grids are shown below.



4. Ask the students to each draw a figure of their choosing on grid **I** and draw its image on grids **A**, **B**, and **C**.

4. This can be given as an assignment. The students may be interested in displaying their work. Shown below is a "duck" and its images.





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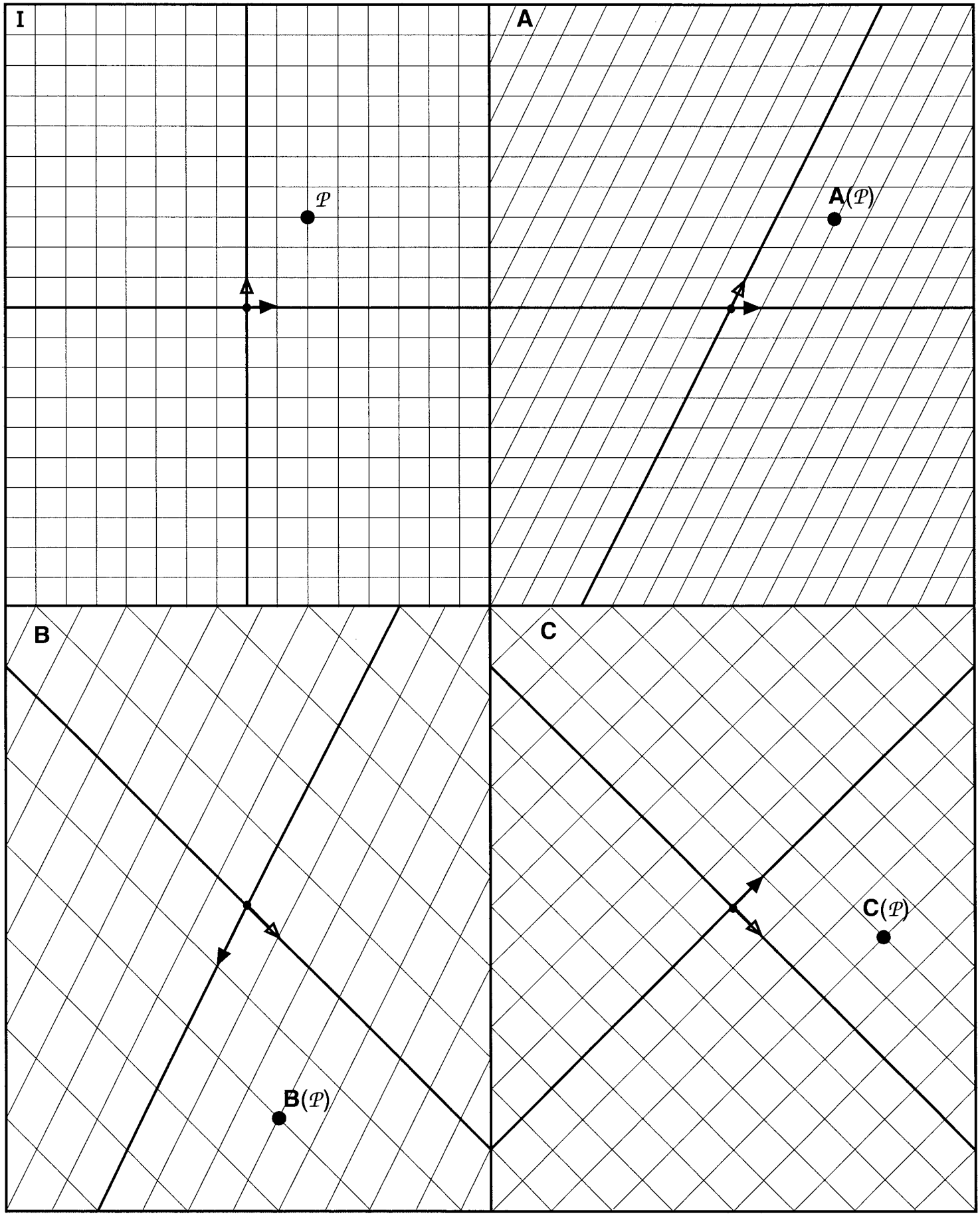

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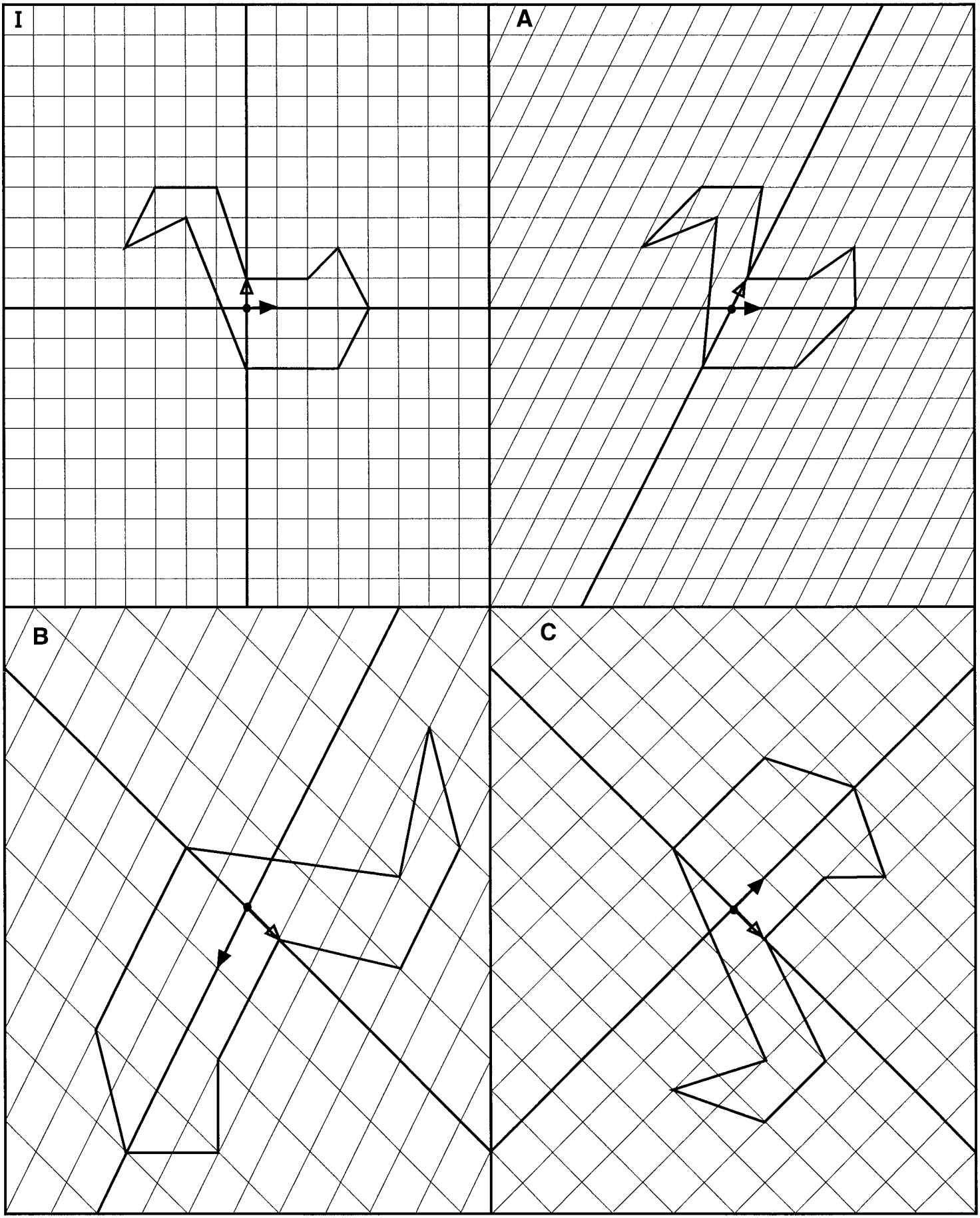

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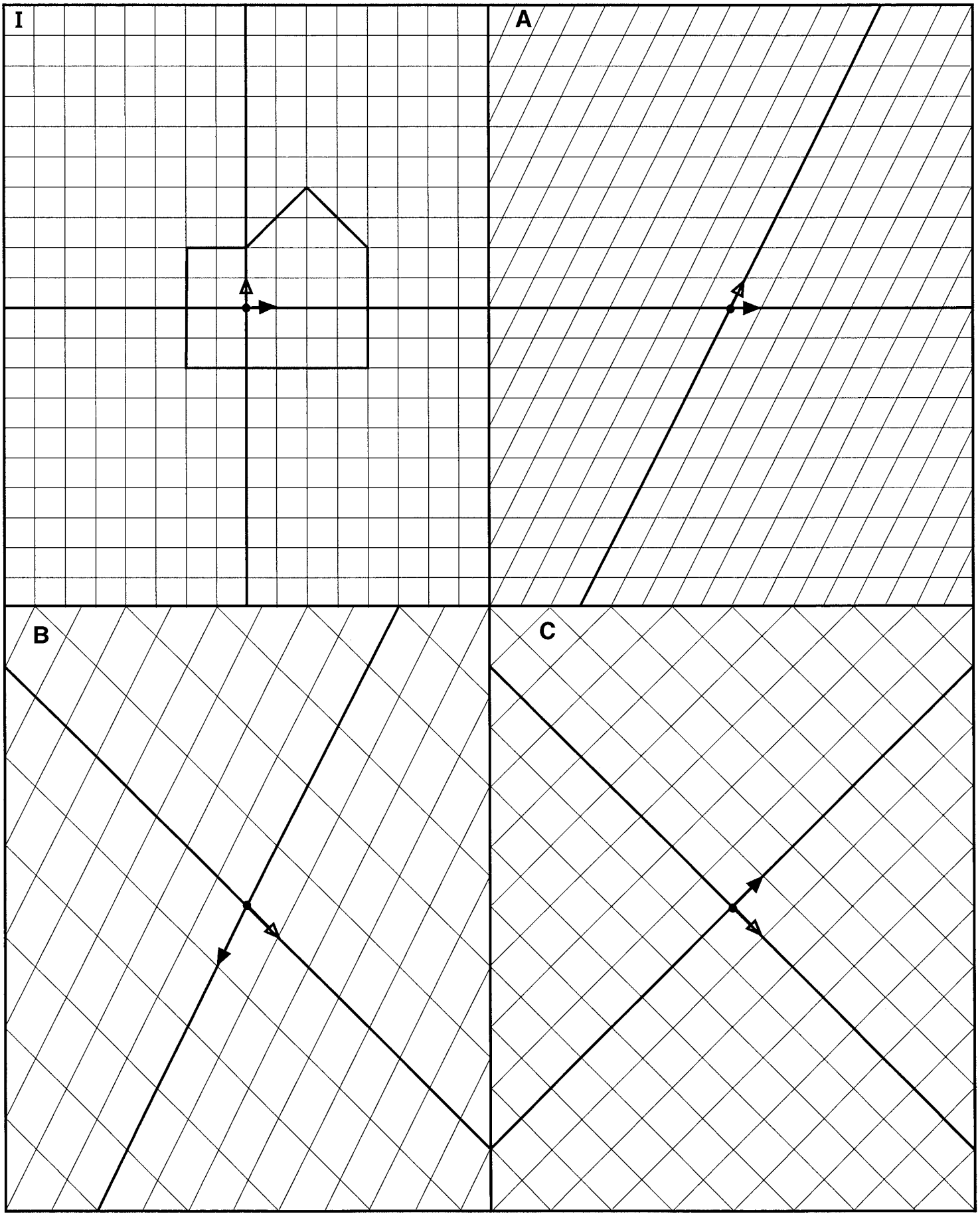

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Coordinates and Vector Arithmetic

O V E R V I E W

Coordinates of grids are introduced. Vector arithmetic is developed and used to obtain formulas for linear transformations.

Prerequisite Activity

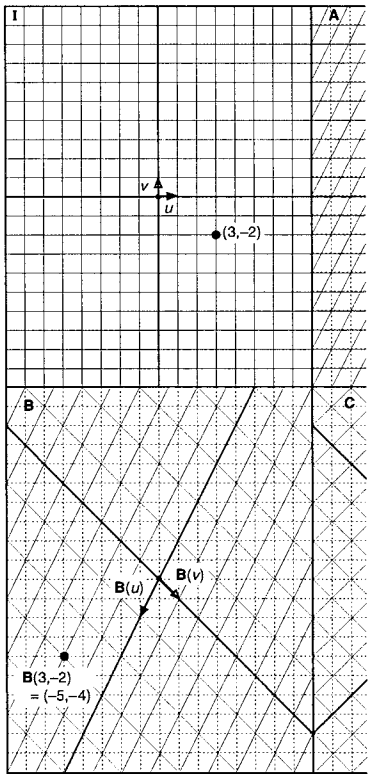
Unit XIV, Activity 1, *Transformations.*

Materials

Multi-grid-**I** paper (see Comment 1), centimeter grid paper (optional; see Comment 2).

Actions

1. Show the students a copy of multi-grid-**I** paper. Describe to the students how points on the various grids are designated.



Comments

1. A master for multi-grid-**I** paper is attached. Multi-grid-**I** paper is multi-grid paper for which the initial grid **I**—with the exception of its basis vectors—is indicated by dotted lines on the **A**, **B**, and **C** grids. The origin of a dotted **I** grid coincides with that of the other grids.

The location of a point on a grid is given by first indicting the grid and then a pair of numbers, called *coordinates*. These numbers indicate how the point is reached by starting at the origin. The first number tells how far one moves in the direction (if the number is positive), or opposite direction (if the number is negative) of the close-tipped basis vector. The second number tells this for the open-tipped basis vector.

If no grid is mentioned, the **I** grid is assumed. For example, (3, –2) means **I**(3, –2). The point (3,–2) is reached by starting at the origin of an **I** grid, moving 3 lengths of u in the direction its arrow is pointing, and then moving 2 lengths of v in the opposite of the direction its arrow is pointing. **B**(3, –2) is reached by starting at the origin of a **B** grid, moving 3 lengths of **B**(u) in the direction its arrow is pointing and then moving 2 lengths of **B**(v) in the opposite of the direction its arrow is pointing.

Points on a grid can be designated either by reference to that grid or to the dotted **I** grid, e.g., **B**(3, –2) and (–5, –4) locate the same point, that is, **B**(3, –2) = (–5, –4). This can be thought of in the following way: if the origin is kept fixed and grid **I** is stretched

Continued next page.

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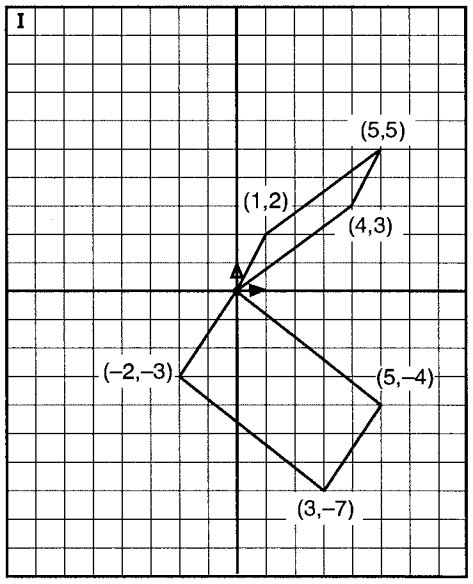
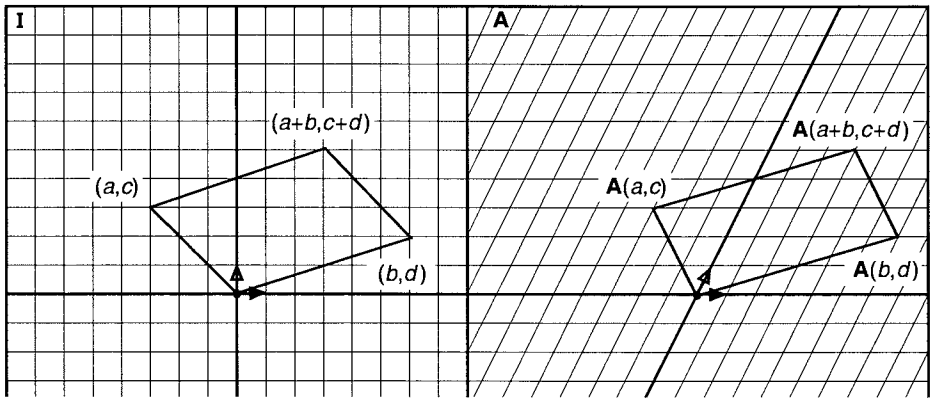
Actions

2. Distribute grid paper to the students. Ask them to find the coordinates of the fourth vertex of a parallelogram for which three successive vertices have coordinates (1, 2), (0, 0), and (4, 3), respectively. Ask them to find the coordinates of the fourth vertex if the coordinates of the three successive vertices are (5, –4), (0, 0), and (–2, –3); if they are (a, c), (0, 0), and (b, d). Discuss their findings.





Comments

1. *Continued.* and turned until it coincides with grid **B**—that is, until u becomes $\mathbf{B}(u)$ and v becomes $\mathbf{B}(v)$ — the point originally located at (3, –2) on the **I** grid would now be located at (–5, –4) on the **I** grid. We say (3, –2) has been *transformed* by **B** to (–5, –4), or that (–5, –4) is the *image* of (3, –2) under *transformation* **B**.

Note that (0, 0) = **A**(0, 0) = **B**(0, 0) = **C**(0, 0).

2. The result of this action motivates the definition of *vector addition* in Action 3. You may wish to provide the students with centimeter grid paper; or they can draw sketches on the **I** grid of a sheet of multi-grid paper.

For the three given situations, the fourth vertex is, respectively, (5, 5), (3, –7), and $(a + b, c + d)$. (See the sketches.) Note that the coordinates of the fourth vertex are the sums of the coordinates of the vertices on either side of the origin.

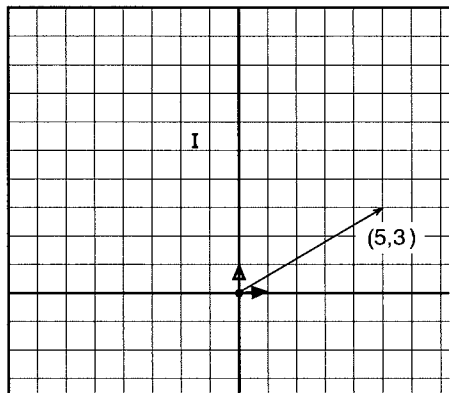
The above results are stated in terms of grid **I** coordinates. Similar results hold for the other grid systems. For example, if $\mathbf{A}(a, c)$, $\mathbf{A}(0, 0)$, and $\mathbf{A}(b, d)$ are three successive vertices of a parallelogram in the grid **A** system, the fourth vertex is $\mathbf{A}(a + b, c + d)$.

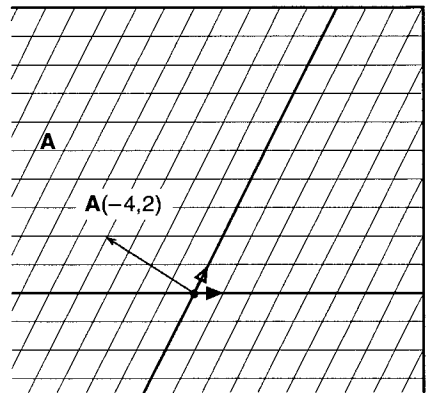
Actions

3. Introduce the term *vector*. Define *vector addition*. Describe its geometric meaning.

Comments

3. Algebraically, we will not distinguish between a vector and a set of coordinates. In two dimensions, a *vector*, like a set of coordinates, is an ordered pair of numbers. Thus, we may refer to expressions like (5, 3), **A**(–4, 2), **B**(x, y), etc., as either coordinates or vectors. Diagrammatically, we will picture a vector as an arrow originating at the origin and terminating at the point with the given coordinates.

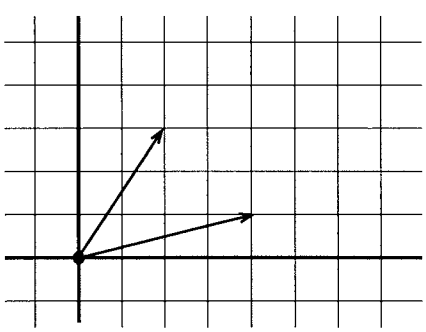


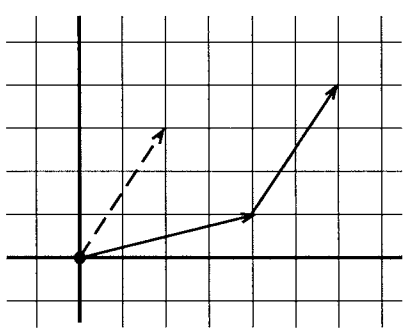


The results of Action 2, prompts the following definition of *vector addition*:

$$(a, c) + (b, d) = (a + b, c + d)$$

Note that the plus sign is used in the above definition to designate two different operations. On the left of the equal sign, it designates vector addition; on the right, within the parentheses, it designates the addition of real numbers. We could have used a different symbol, say ⊕, to designate vector addition. However, it should be clear from the context which addition is meant.



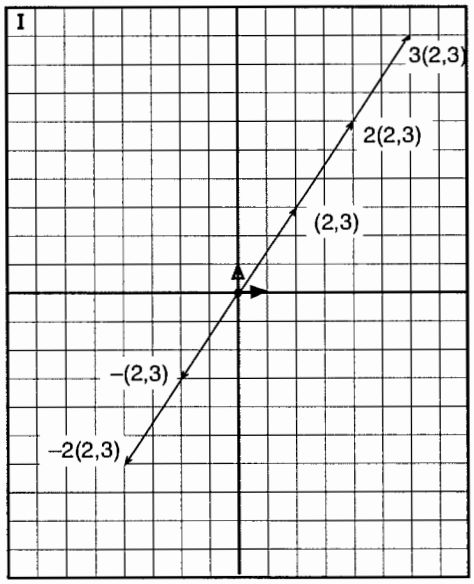


Vector addition is defined so that the sum of two vectors is the diagonal of a parallelogram for which the two vectors are adjacent sides. Graphically, one can find the fourth vertex of a parallelogram which has adjacent sides lying along two vectors by translating one of the vectors so that it originates at the end of the other vector (see the sketch). The end of the translated vector is the fourth vertex of the parallelogram.

The sum of two vectors is similarly defined in the other grid systems, e.g., **C**(2, 3) + **C**(4, –5) = **C**(6, –2).

Actions

4. Define *scalar multiple* and describe its geometric meaning.



Scalar multiples of (2,3)

5. Discuss the properties of vector addition and scalar multiplication, and how the coordinates of a point can be expressed as a *linear combination* of basis vectors.

Comments

4. Akin to ordinary arithmetic, we write the sum of a vector and itself as twice that vector, that is $2(a, c)$ means $(a, c) + (a, c)$. Thus, $2(a, c) = (a, c) + (a, c) = (a + a, c + c) = (2a, 2c)$.

Similarly, $3(a, c) = (3a, 3c)$, and so forth. We extend these results for positive integers by defining the *scalar multiple* $k(a, c)$, for any real number k, to be the vector (ka, kc).

The set of all scalar multiples of the point (a, c)—that is, the set of all points of the form $k(a, c)$, where k is a real number—is the line which passes through the origin and (a, c). In particular, the so-called x-axis is the set of all scalar multiples of $(1, 0)$; the y-axis is the set of all scalar multiples of $(0, 1)$.

The representation of a line as the set of scalar multiples of a point is not unique, for example, the set of all scalar multiples of $(6, 8)$ is the same set of points as the set of all scalar multiples of $(9, 12)$, e.g., $3(6, 8) = 2(9, 12)$.

Scalar multiplication is defined similarly in other grid systems, e.g., in the grid **A** system, $k\mathbf{A}(a, c) = \mathbf{A}(ka, kc)$.

5. It follows from the definition of vector addition and the properties of real numbers that vector addition is associative and commutative. Also, from the definitions of vector addition and scalar multiplication, and the properties of the real numbers, we have the following relationships:

$$\begin{gathered} k(a, c) + m(a, c) = \\ (ka, kc) + (ma, mc) = \\ (ka + ma, kc + mc) = \\ ((k + m)a, (k + m)c) = \\ (k + m)(a, c). \end{gathered}$$

Similar relationships hold in the other grid systems, e. g.,

$$\begin{gathered} (k + m)\mathbf{A}(a, c) = \\ k\mathbf{A}(a, c) + m\mathbf{A}(a, c). \end{gathered}$$

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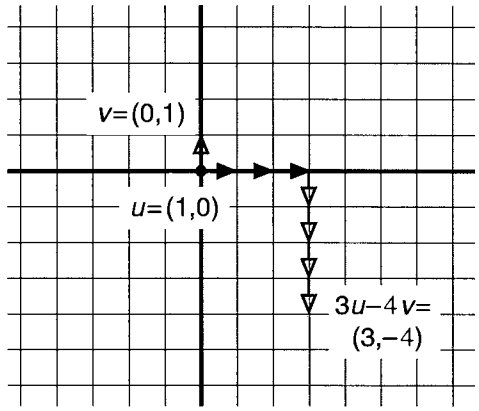


5. *Continued.* A *linear combination* of two vectors is the sum of scalar multiples of the two vectors. Since, in grid **I**, $(a, b) = a(1, 0) + b(0, 1) = au + bv$, every point (a, b) in grid **I** can be written as a *linear combination* of the basis vectors, $u = (1, 0)$ and $v = (0, 1)$ of grid **I**. We say u and v *span* the grid.

For example, $(3, -4) = 3(1, 0) + (-4)(0, 1) = 3u + (-4)v$ or, as we will normally write it, $3u - 4v$. Geometrically, this tells us that $(3, -4)$ can be reached by starting at the origin and laying out in succession 3 copies of u followed by 4 copies of v with its direction reversed. (See the figure.)

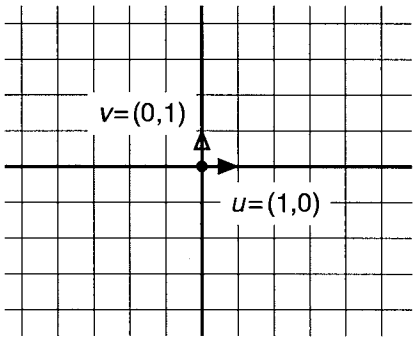
Similar results hold in the other grids, e.g., $\mathbf{C}(a, b) = a\mathbf{C}(1, 0) + b\mathbf{C}(0, 1) = a\mathbf{C}(u) + b\mathbf{C}(v)$.

6. Distribute copies of multi-grid-**I** paper to the students. For each of the grids **I**, **A**, **B**, and **C**, ask the students to write the coordinates of the basis vectors of that grid.

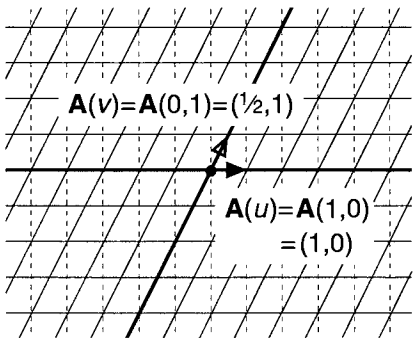
6. Coordinates of a grid's basis vectors can be given either as coordinates in that grid or in grid **I** coordinates. In the following listing, for grids **A**, **B**, and **C**, both are given.

Grid **I**: $u = (1, 0)$, $v = (0, 1)$
Grid **A**: $\mathbf{A}(u) = \mathbf{A}(1, 0) = (1, 0)$,
$\mathbf{A}(v) = \mathbf{A}(0, 1) = (1/2, 1)$
Grid **B**: $\mathbf{B}(u) = \mathbf{B}(1, 0) = (-1, -2)$,
$\mathbf{B}(v) = \mathbf{B}(0, 1) = (1, -1)$
Grid **C**: $\mathbf{C}(u) = \mathbf{C}(1, 0) = (1, 1)$,
$\mathbf{C}(v) = \mathbf{C}(0, 1) = (1, -1)$

I.



A.



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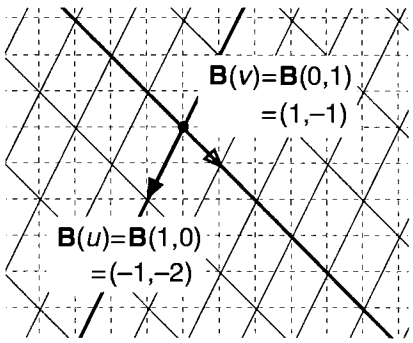
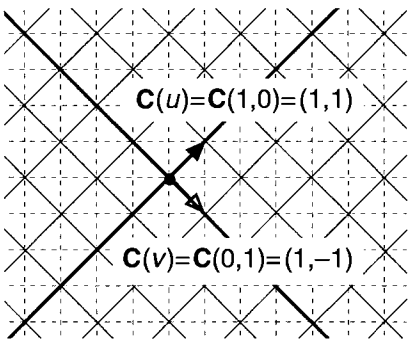


C.



Actions

7. Ask the students to find the grid **I** coordinates of **A**(10, 15), **B**(10, 15), and **C**(10, 15).

8. Ask the students to find the grid **I** coordinates of **A**(x, y), **B**(x, y), and **C**(x, y). Discuss the results.

Comments

7. There are several ways to proceed. One way is to note that **A**(10, 15) = 5**A**(2, 3). From multi-grid-**I** paper, one sees that **A**(2, 3) = ($3^1/_2$, 3), so **A**(10, 15) = 5**A**(2, 3) = 5($3^1/_2$, 3) = ($17^1/_2$, 15).

Alternatively, writing **A**(10, 15) as a linear combination of **A** grid basis vectors and using properties from Comment 5, one has **A**(10, 15) = 10**A**(1, 0) + 15**A**(0, 1) = 10(1, 0) + 15($^1/_2$, 1) = (10, 0) + ($7^1/_2$, 15) = ($17^1/_2$, 15).

Similarly, **B**(10, 15) = (5, –35) and **C**(10, 15) = (25, –5).

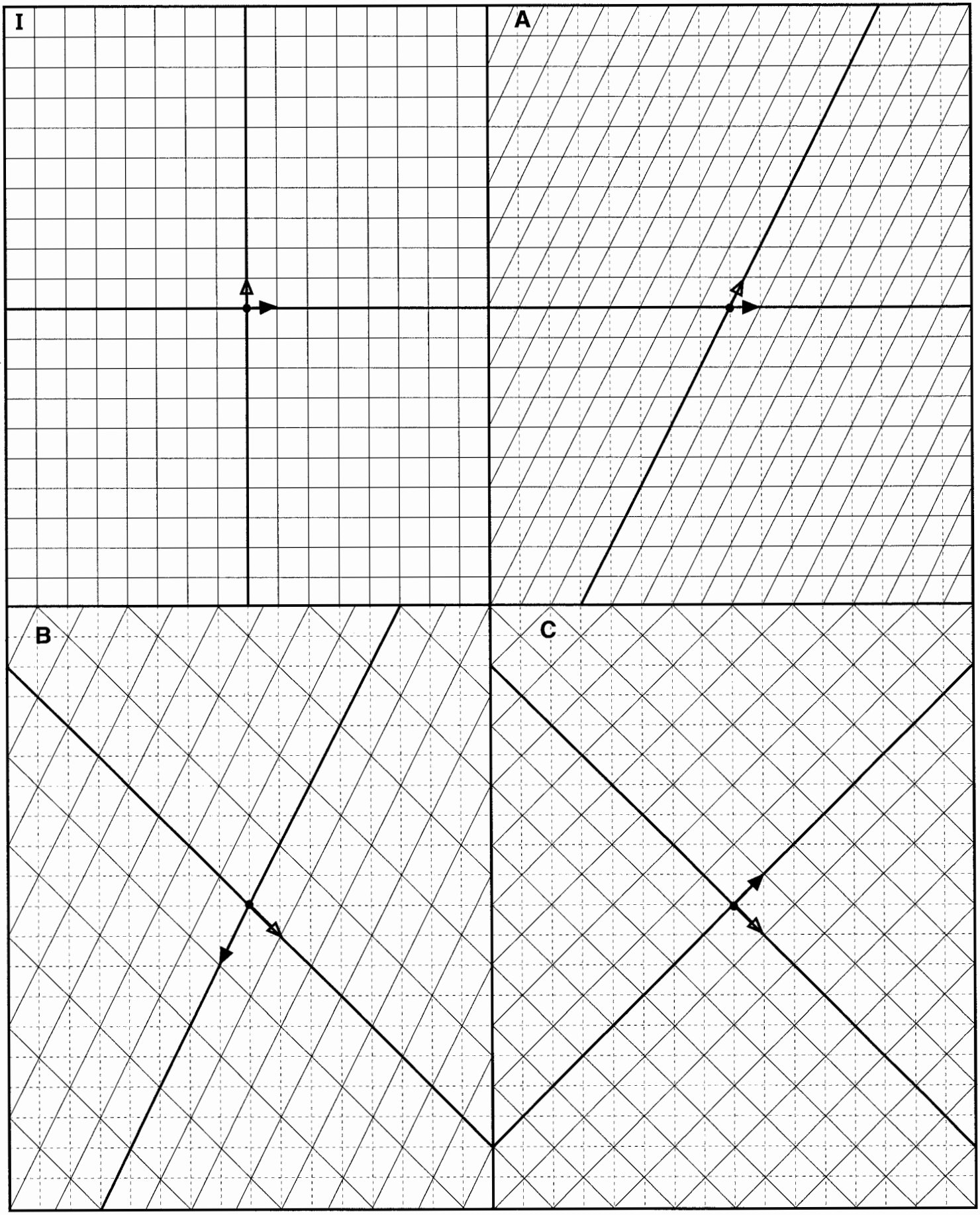
8. Writing the coordinates as linear combinations of basis vectors and using properties from Comment 5, one has:

A(x, y) = x**A**(1, 0) + y**A**(0, 1) = x(1, 0) + y($^1/_2$, 1) = (x + $^1/_2y$, y)

B(x, y) = x**B**(1, 0) + y**B**(0, 1) = x(–1, –2) + y(1, –1) = (–x + y, –2x – y)

C(x, y) = x**C**(1, 0) + y**C**(0, 1) = x(1, 1) + y(1, –1) = (x + y, x – y)

These equations provide formulas for determining how the grid **I** coordinates of a point are changed when the gird **I** system is changed, or transformed, to the grid **A**, **B**, and **C** systems respectively. For example, **B**(7, –3) = (–7 + (–3), –2(7) – (–3)) = (–10, –11). Thus, **B** transforms the point (7, –3) to (–10, –11).



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Matrices and Determinants

O V E R V I E W

The matrix and the determinant of a linear transformation are defined.

Prerequisite Activity

Unit XIV, Activity 2, *Coordinates and Vector Arithmetic.*

Materials

Multi-grid-**I** paper.

Actions

1. Review the results of Action 8 in Unit XIV, Activity 2. Introduce the term *linear transformation.*

2. Describe how a linear transformation can be represented by a *matrix.* Point out the relationship between the columns of a matrix and basis vectors.

Comments

1. In Action 8 of the previous activity, the **I** grid coordinates of the images of (x, y) under transformations **A**, **B**, and **C** where determined to be $(x + {}^1/_2y, y)$, $(-x + y, -2x - y)$, and $(x + y, x - y)$, respectively. These coordinates are *linear combinations* of x and y, that is expressions of the form $ax + cy$, where a and c are real numbers. A *linear transformation* is a transformation for which the image of (x, y) is a point whose **I** grid coordinates are linear combinations of x and y.

2. The equations obtained in Action 8 of the previous activity can be written with coordinates specified by vertical, rather than horizontal, arrays. For example, $\mathbf{B}(x, y) = (-x + y, -2x - y)$ can be written $\mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + y \\ -2x - y \end{pmatrix}$. One advantage of vertical notation is that no punctuation marks are necessary to separate coordinates.

The two expressions in the array $\begin{pmatrix} -x + y \\ -2x - y \end{pmatrix}$ are linear expressions in x and y. If we agree to always write x-terms first, the array can be specified by recording only the coefficients of x and y, thus: $\begin{pmatrix} -1 & 1 \\ -2 & -1 \end{pmatrix}$. Such an array of numbers is called a *matrix.*

We will identify a transformation with its matrix. For example, we will write $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ -2 & -1 \end{pmatrix}$. Hence, we have:

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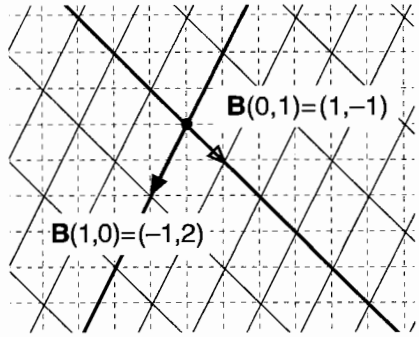
Actions



$$\mathbf{B} = \begin{pmatrix} -1 & 1 \\ -2 & -1 \end{pmatrix}$$

Comments

2. *Continued.*

$$\mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+y \\ -2x-y \end{pmatrix}.$$

For example,

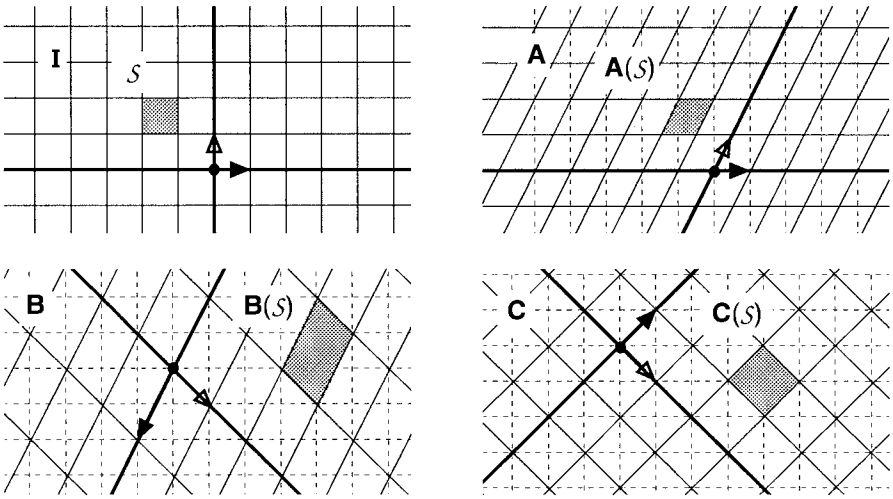
$$\mathbf{B}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} (-1)3+(1)4 \\ (-2)3+(-1)4 \end{pmatrix} = \begin{pmatrix} 1 \\ -10 \end{pmatrix}.$$

Note that $\mathbf{B}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$, which is the first column of matrix **B**, and $\mathbf{B}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which is the second column of matrix **B**. Thus, the columns of matrix **B** are the **I** grid coordinates of the **B** grid basis vectors — the first column is the **I** grid coordinates of the closed-arrow basis vector, the second column is the **I** grid coordinates of the open-arrow basis vector.

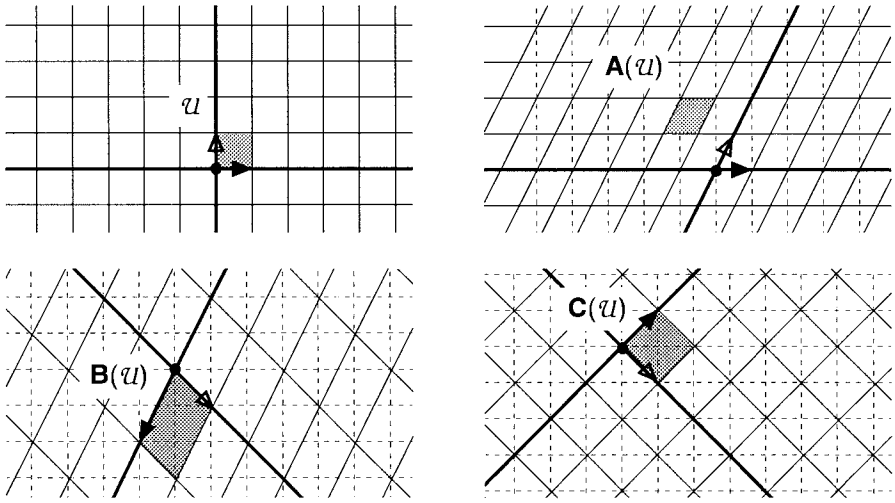
In matrix notation, $\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Note that **I** can also be thought of as the linear transformation in which each point is its own image, that is, $\mathbf{I}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1x+0y \\ 0x+1y \end{pmatrix}$. Thus, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the so-called *identity* matrix.

Actions

3. Distribute multi-grid-**I** paper to the students. Ask them to shade in a single square of the **I** grid. Then have them shade in the image of this square in the **A**, **B**, and **C** grids. Ask them to find the areas of these images, given that the area of one square of the **I** grid is 1. Discuss how transformations **A**, **B**, and **C** affect areas.



4. Define the *determinant* of a linear transformation. Ask the students to find the determinants of transformations **A**, **B**, and **C**.



Comments

3. A student may shade a square whose images don't all lie on the grid paper. If so, ask them to shade another one whose images do.

The sketch shows a single square $\mathcal{S}$ of grid **I** and its images **A**($\mathcal{S}$), **B**($\mathcal{S}$) and **C**($\mathcal{S}$). The areas of these images are respectively 1, 3, and 2. Hence, transformation **A** leaves areas unchanged, **B** triples areas, and **C** doubles them. This can be seen in the images drawn in Activity 1.

4. The numerical value of the *determinant* of a transformation **T**, denoted |**T**|, is the factor by which the area of a figure is changed when it is transformed by **T**. The determinant is positive if it does not change the orientation of the basis vectors, it is negative if it changes their orientation.

If $\mathcal{U}$ is the square of grid **I** which has the basis vectors as edges, the determinant of a transformation can be determined by its effect on $\mathcal{U}$.

The images **A**($\mathcal{U}$), **B**($\mathcal{U}$), and **C**($\mathcal{U}$) are shown below. Notice in grid **I**, $\mathcal{U}$ lies to the left of the closed-tip basis vector, while in grid **C**, **C**($\mathcal{U}$) lies to the right of the closed-tip basis vector. Hence, **C** changes the orientation of the basis vectors, so its determinant is negative. **A** and **B** do not change orientation, so their determinants are positive. From these observations and the previous Action, one has |**A**| = 1, |**B**| = 3 and |**C**| = –2.

Products of Transformations

O V E R V I E W

The *product*, or *composition*, of two transformations is defined and the matrix of the resulting transformation determined.

Prerequisite Activity

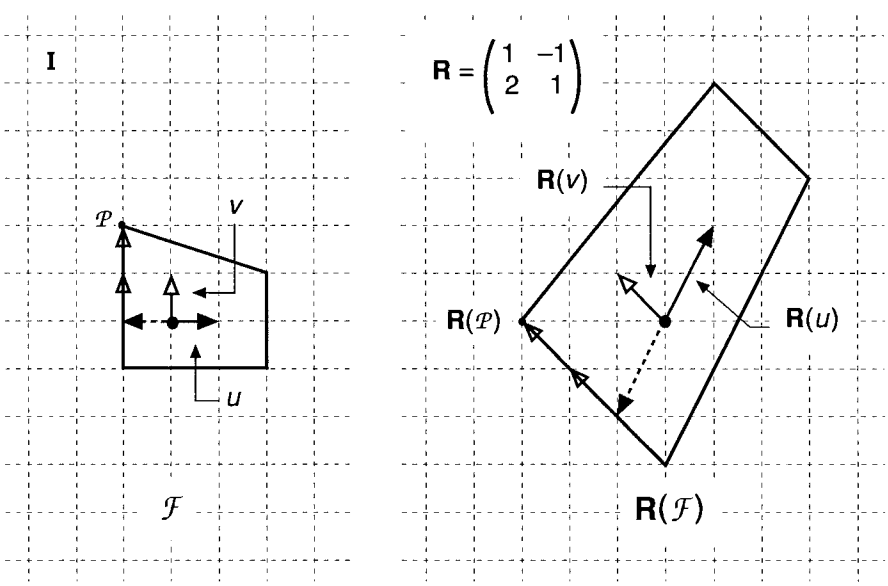
Unit XIV, Activity 3, *Matrices and Determinants.*

Materials

Copies of masters as indicated.

Actions

1. Distribute copies of Master 1 to the students. In the space provided, ask them to draw $\mathbf{R}(\mathcal{F})$, the image of the figure $\mathcal{F}$ under the transformation $\mathbf{R}$. Discuss.



Comments

1. Masters are attached at the end of this activity. Master 2 is a completed version of Master 1. A transparency of it can be used in discussing the results of Action 1. The lower half can be shielded until Action 2 is completed.

One way to proceed is to locate the basis vectors of an **R** grid and then locate the images of points relative to this grid. The **I** grid coordinates of the basis vectors $\mathbf{R}(u)$ and $\mathbf{R}(v)$ of the **R** grid are the first and second columns, respectively, of the matrix form of **R**. (See Comment 2 of the previous activity). Thus $\mathbf{R}(u) = (1, 2)$ and $\mathbf{R}(v) = (-1, 1)$, as shown in the sketch.

As shown, the point $\mathcal{P}$ is a vertex of figure $\mathcal{F}$. It is reached by moving one length of u in the opposite of the direction it is pointed, and then moving two lengths of v in the direction it is pointed. Thus, $\mathbf{R}(\mathcal{P})$ is reached by moving one length of $\mathbf{R}(u)$ in the opposite of the direction it is pointed, and then moving two lengths of $\mathbf{R}(v)$ in the direction it is pointed.

Alternatively, the coordinates of the vertices of $\mathbf{R}(\mathcal{F})$ can be determined from its matrix. For example, point $\mathcal{P}$ has coordinates $(-1, 2)$. So the coordinates of $\mathbf{R}(\mathcal{P})$, in vertical form, are

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1(-1)+(-1)2 \\ 2(-1)+1(2) \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

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Actions

2. For the transformations **S** and **T**, have the students draw **S**($\mathcal{F}$) and **T**($\mathcal{F}$).

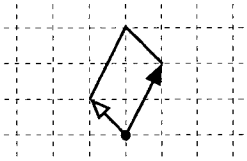
3. Ask the students to find the values of the determinants of **R**, **S**, and **T**.

4. Distribute copies of Master 3 to the students. Have them draw **S**(**R**($\mathcal{F}$)) in the space provided. If **U** is the transformation which directly transforms $\mathcal{F}$ to **S**(**R**($\mathcal{F}$)), ask the students to write the matrix of **U** in the space provided and sketch **U**($\mathcal{F}$).

Comments

2. **S**($\mathcal{F}$) and **T**($\mathcal{F}$) are shown on the bottom half of Master 2.

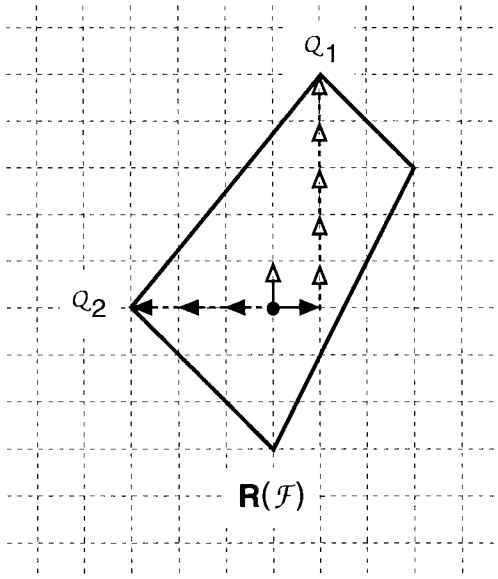
3. The parallelogram determined by the basis vectors of the **R** grid has area 3. (See the sketch below.) **R** does not change the orientation of the basis vectors. Thus, |**R**| = 3.

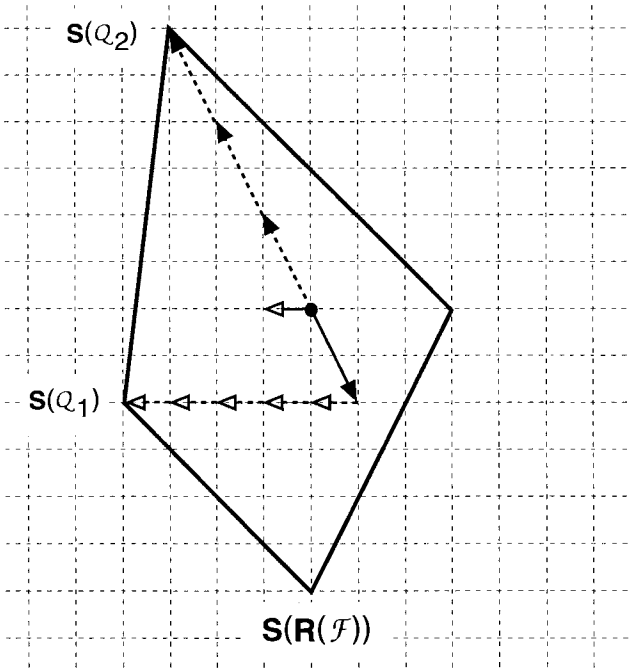


Similarly, |**S**| = –2 and |**T**| = 4. Note that **S** changes the orientation of the basis vectors.

4. Master 4 is a completed version of Master 3.

S(**R**($\mathcal{F}$)) is the **S** image of the figure **R**($\mathcal{F}$). Since the matrix of **S** tells how **S** transforms **I**-grid coordinates of points (rather than **R**-grid coordinates), the points of **S**(**R**($\mathcal{F}$)) are located relative to **S**-grid basis vectors as the corresponding points of **R**($\mathcal{F}$) are located relative to **I**-grid basis vectors. The sketch shows two vertices, Q_1 and Q_2, of **R**($\mathcal{F}$) and their **S** images, **S**(Q_1) and **S**(Q_2).





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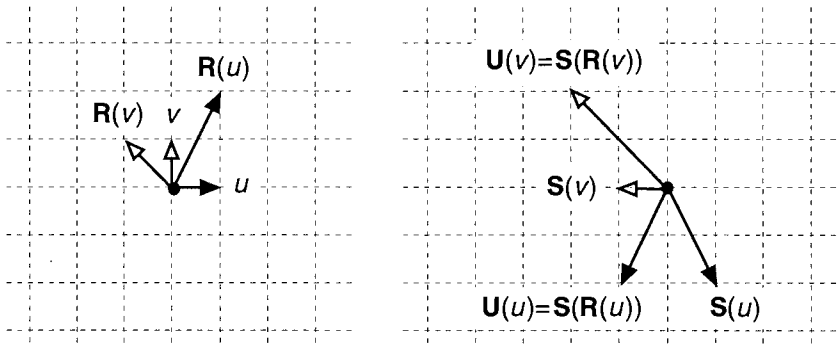
Actions



5. Distribute copies of Master 5 to the students. Have them draw $\mathbf{R}(\mathbf{S}(\mathcal{F}))$ in the space provided. If $\mathbf{V}$ is the transformation which directly transforms $\mathcal{F}$ to $\mathbf{R}(\mathbf{S}(\mathcal{F}))$, ask the students to write the matrix of $\mathbf{V}$ in the space provided and sketch $\mathbf{V}(\mathcal{F})$.

6. If $\mathbf{A}$ and $\mathbf{B}$ are linear transformations, define the *products* $\mathbf{AB}$ and $\mathbf{BA}$. Discuss the properties of transformation products.

Comments

4. *Continued.* The columns of the matrix for $\mathbf{U}$ are $\mathbf{U}(u)$ and $\mathbf{U}(v)$ where $u = (1, 0)$ and $v = (0, 1)$ are the $\mathbf{I}$-grid basis vectors. $\mathbf{U}(u)$ and $\mathbf{U}(v)$ are obtained by first transforming u and v by $\mathbf{R}$ and then transforming the result by $\mathbf{S}$, that is, $\mathbf{U}(u)$ is the $\mathbf{S}$ image of $\mathbf{R}(u)$—the first column of the $\mathbf{R}$ matrix—and $\mathbf{U}(v)$ is the $\mathbf{S}$ image of $\mathbf{R}(v)$—the second column of the $\mathbf{R}$ matrix. From the sketch, one sees that $\mathbf{U}(u) = (-1, -2)$ and $\mathbf{U}(v) = (-2, 2)$. Hence, $\mathbf{U} = \begin{pmatrix} -1 & -2 \\ -2 & 2 \end{pmatrix}$.

$\mathbf{U}(1, 0)$ and $\mathbf{U}(0, 1)$ can also be determined by matrix calculations:

Since $\mathbf{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

$$\mathbf{U}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{S}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+(-1)2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

$\mathbf{U}(0, 1)$ can be determined similarly.

Since $\mathbf{S}(\mathbf{R}(\mathcal{F}))$ and $\mathbf{U}(\mathcal{F})$ are identical, a sketch of $\mathbf{U}(\mathcal{F})$ can be obtained by reproducing $\mathbf{S}(\mathbf{R}(\mathcal{F}))$. Alternately, $\mathbf{U}(\mathcal{F})$ can be obtained by referring to the original figure $\mathcal{F}$ and sketching its image when transformed directly by $\mathbf{U}$.

5. Master 6 shows $\mathbf{R}(\mathbf{S}(\mathcal{F})))$, $\mathbf{V}$ and $\mathbf{V}(\mathcal{F})$. $\mathbf{R}(\mathbf{S}(\mathcal{F}))$ and $\mathbf{V}(\mathcal{F})$ are identical.

6. The *product*, or *composition*, of two transformations is obtained by performing the transformations one after the other. More precisely, if $\mathbf{A}$ and $\mathbf{B}$ are transformations, $\mathbf{AB}$ is the transformation which maps (x, y) to $\mathbf{A}(\mathbf{B}(x, y)$, i. e., $\mathbf{AB}(x, y) = \mathbf{A}(\mathbf{B}(x, y))$. Similarly, $\mathbf{BA}(x, y) = \mathbf{B}(\mathbf{A}(x, y))$.

Continued next page.

Actions

7. Ask the students to explore the relationships among $|\mathbf{R}|$, $|\mathbf{S}|$, $|\mathbf{U}|$, and $|\mathbf{V}|$. Then ask them to describe in general how the determinant of the product of matrices is related to the determinants of the matrices which are the factors of that product.

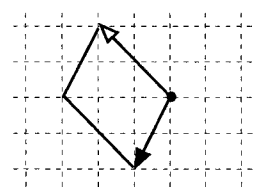
Comments

6. *Continued.* In Actions 4 and 5, $\mathbf{U} = \mathbf{SR}$ and $\mathbf{V} = \mathbf{RS}$. Note that $\mathbf{SR} \neq \mathbf{RS}$, so, in general, the composition of transformations is not commutative—the order in which the transformations are carried out makes a difference. It may be confusing to the students that the order in which a product of transformations **AB** is carried out is the reverse of the order in which the transformations are written. To remind students of this, it may be helpful to read "**AB**" as "**A** preceded by **B**."

Transformation composition is associative. If **A**, **B**, and **C** are transformations, then **(AB)C** means to first carry out **C** and then carry out **AB**, which, in turn, is accomplished by carrying out **B** and then **A**. Thus **(AB)C** is equivalent to carrying out **C**, **B**, and **A**, in that order. On the other hand, **A(BC)** means to first carry out **BC**, that is carry out **C** followed by **B**, and then carry out **A**. Hence, **A(BC)** is also equivalent to carrying out **C**, **B**, and **A** in that order.

The transformation $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ leaves the basis vectors unchanged and hence is an identity for composition, that is, for a linear transformation **A**, $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

7. From Action 3, $|\mathbf{R}| = 3$ and $|\mathbf{S}| = -2$. $|\mathbf{U}|$ can be computed directly by noting, on Master 4, that the parallelogram determined



by the basis vectors of the **U** grid has area 6 and the orientation of these vectors is reversed from that of the **I** grid vectors. Hence, $|\mathbf{U}| = -6$.

On the other hand, **U** is the result of carrying out **R** and then **S**. Since **R** doubles areas and **S** triples them, **U** will increase areas sixfold. Since **R** does not change orientation but **S** does, **U** will change orientation. Thus, $|\mathbf{U}| = -6$. Summarizing, $|\mathbf{SR}| = |\mathbf{U}| = |\mathbf{S}||\mathbf{R}| = -6$. Similarly, $|\mathbf{RS}| = |\mathbf{V}| = |\mathbf{R}||\mathbf{S}| = -6$.

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Actions

8. Define the *inner product* of two vectors. Develop a formula for computing the product of two matrices.

Comments

7. *Continued.* In general, the determinant of a product is the product of the determinants, that is, if **A** and **B** are linear transformations, then $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$. For, if **A** changes areas by a factor of k and **B** changes areas by a factor of m, then **B** followed by **A**, that is **AB**, will change areas by a factor of km. Further if $|\mathbf{A}||\mathbf{B}|$ is negative, just one of **A** and **B** changes orientation, in which case **AB** does also, and hence $|\mathbf{AB}|$ is negative. If, on the other hand, $|\mathbf{A}||\mathbf{B}|$ is positive, than neither or both of **A** and **B** change orientation; in either case, **AB** does not change orientation and hence $|\mathbf{AB}|$ is positive.

8. Given vectors (a, b) and (c, d), their *inner product,* or *dot product*, written $(a, b)\cdot(c, d)$ or $(a, b)\cdot\begin{pmatrix} c \\ d \end{pmatrix}$ is the number $ac + bd$.

Suppose $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and the matrix of the product **AB** is desired.

The first column of **AB** is the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ after it has been successively transformed by **B** and then **A.** Since **B** transforms $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} e \\ g \end{pmatrix}$, $\mathbf{AB}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{A}\begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} ae + bg \\ ce + dg \end{pmatrix} = \begin{pmatrix} (a,b)\cdot\begin{pmatrix} e \\ g \end{pmatrix} \\ (c,d)\cdot\begin{pmatrix} e \\ g \end{pmatrix} \end{pmatrix}$. Similarly, the second column of **AB** is $\mathbf{AB}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{A}\begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} (a,b)\cdot\begin{pmatrix} f \\ h \end{pmatrix} \\ (c,d)\cdot\begin{pmatrix} f \\ h \end{pmatrix} \end{pmatrix}$.

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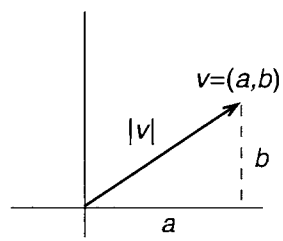
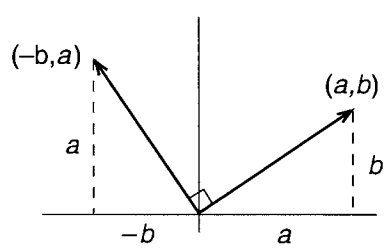
Actions

9. (Optional.) Discuss how inner products are related to vector length and orthogonality.



$vv = a^2 + b^2 = |v|^2$



Comments

8. *Continued.*

Hence, $\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} =$

$$\begin{pmatrix} (a,b)\cdot\begin{pmatrix} e \\ g \end{pmatrix} & (a,b)\cdot\begin{pmatrix} f \\ h \end{pmatrix} \\ (c,d)\cdot\begin{pmatrix} e \\ g \end{pmatrix} & (c,d)\cdot\begin{pmatrix} f \\ h \end{pmatrix} \end{pmatrix}.$$

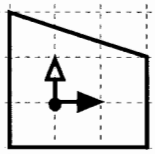
If we view **A** as consisting of two row vectors $r_1 = (a, b)$ and $r_2 = (c, d)$ and **B** as consisting of two column vectors $c_1 = \begin{pmatrix} e \\ g \end{pmatrix}$ and $c_2 = \begin{pmatrix} f \\ h \end{pmatrix}$, then

$$\mathbf{AB} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \end{pmatrix} = \begin{pmatrix} r_1\cdot c_1 & r_1\cdot c_2 \\ r_2\cdot c_1 & r_2\cdot c_2 \end{pmatrix}.$$

9. If vector $v = (a, b)$ and $|v|$ denotes the length of v, then $v\cdot v = a^2 + b^2 = |v|^2$ by the Pythagorean Theorem. Hence, the inner product of a vector with itself is the square of its length.

In the bottom sketch, the non-zero vectors (a, b) and $(b, -a)$ are *orthogonal*, or perpendicular, to one another. If the vector (c, d) is orthogonal to (a, b), then it is a scalar multiple of $(-b, a)$; i. e., there is a k such that $(c, d) = k(-b, a) = (kb, -ka)$). Then, the inner product $(a, b)\cdot(c, d) = (a, b)\cdot(kb, -ka) = akb - bka = 0$. Conversely, suppose $(a, b)\cdot(c, d) = ac + bd = 0$. Solving for c one has $c = {}^{-bd}/_{a}$ (at least one of a and b is not zero, if $a = 0$, solve for d instead of c). Hence, $(c, d) = ({}^{-bd}/_{a}, d) = {}^{d}/_{a}(-b, a)$ and (c, d) is a scalar multiple of $(-b, a)$. Thus, two non-zero vectors are orthogonal if, and only if, their inner product is 0.

$\mathbf{I}$



$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

$\mathbf{R}(\mathcal{F})$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

$\mathbf{S}(\mathcal{F})$

$\mathbf{T} = \begin{pmatrix} -2 & -1 \\ 2 & -1 \end{pmatrix}$

$\mathbf{T}(\mathcal{F})$

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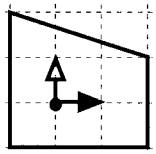
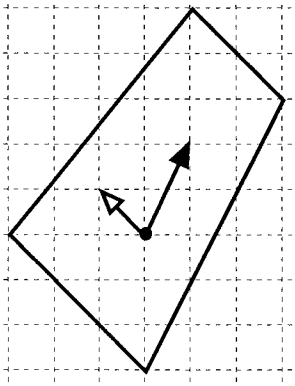
I



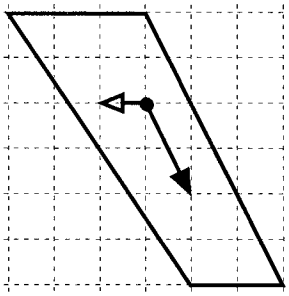
$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$



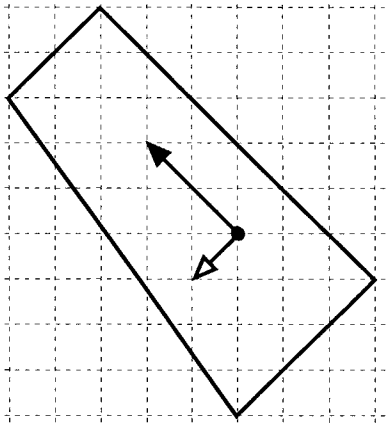
$\mathbf{R}(\mathcal{F})$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$



$\mathbf{S}(\mathcal{F})$

$\mathbf{T} = \begin{pmatrix} -2 & -1 \\ 2 & -1 \end{pmatrix}$



$\mathbf{T}(\mathcal{F})$

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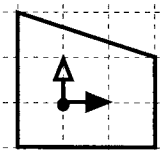
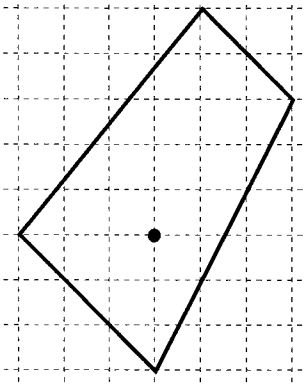


$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$



$\mathbf{R}(\mathcal{F})$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

$\mathbf{S}(\mathbf{R}(\mathcal{F}))$

$\mathbf{U} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

$\mathbf{U}(\mathcal{F})$

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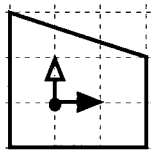

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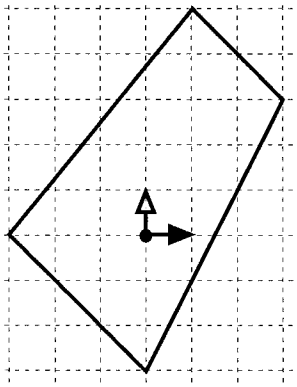
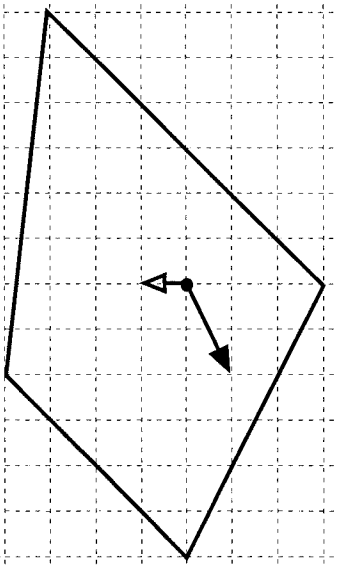
$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$



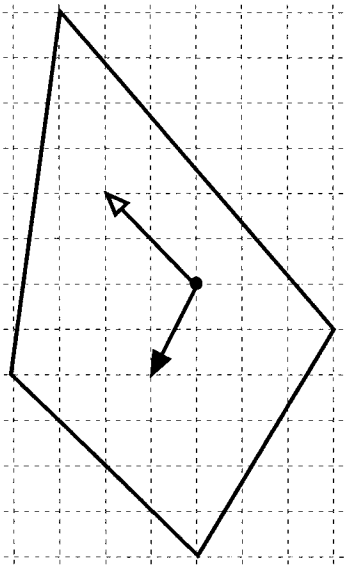
$\mathbf{R}(\mathcal{F})$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$



$\mathbf{S}(\mathbf{R}(\mathcal{F}))$

$\mathbf{U} = \begin{pmatrix} -1 & -2 \\ -2 & 2 \end{pmatrix}$



$\mathbf{U}(\mathcal{F})$

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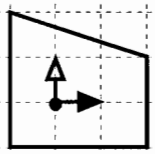
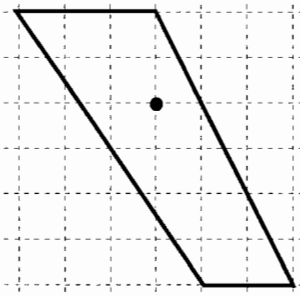


$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

$\mathbf{R}(\mathbf{S}(\mathcal{F}))$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$



$\mathbf{S}(\mathcal{F})$

$\mathbf{V} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

$\mathbf{V}(\mathcal{F})$

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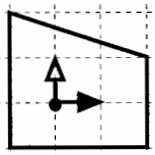
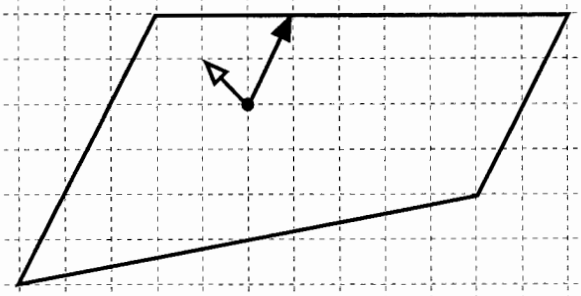


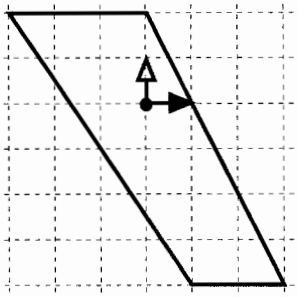
$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$



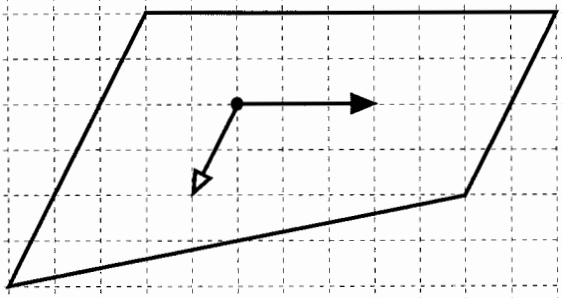
$\mathbf{R}(\mathbf{S}(\mathcal{F}))$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$



$\mathbf{S}(\mathcal{F})$

$\mathbf{V} = \begin{pmatrix} 3 & -1 \\ 0 & -2 \end{pmatrix}$



$\mathbf{V}(\mathcal{F})$

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Scaling and Shearing

O V E R V I E W

Scaling and shearing transformations are introduced. Later we see that all linear transformations are compositions of these two types of transformations.

Prerequisite Activity

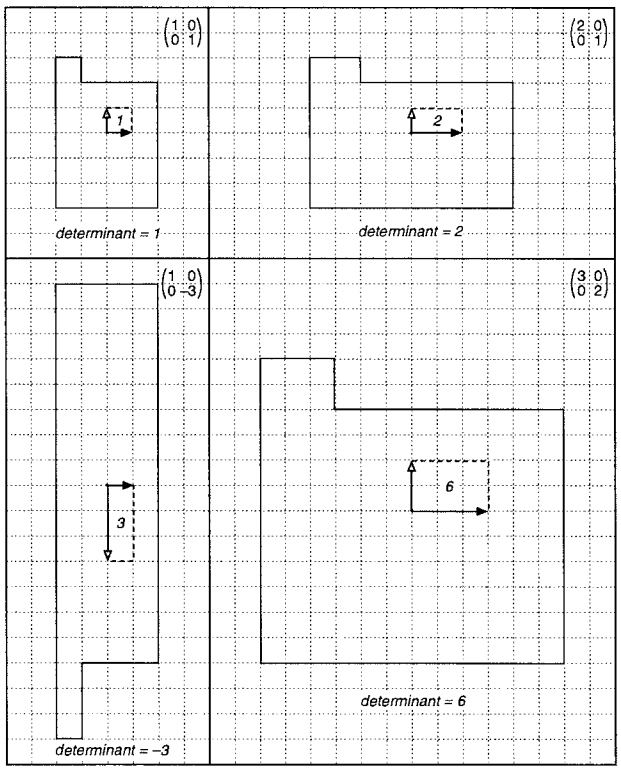
Unit XIV, Activity 3, *Matrices and Determinants*.

Materials

Copies of masters as indicated.

Actions

1. Distribute copies of Master 1 to the students. Ask them to sketch the image of the figure in the upper left-hand corner, including the basis vectors, under each of the given transformations. Have them find the determinant of each of the transformations. Ask them for their observations.



Comments

1. The transformations on Master 1 all have *diagonal* matrices, that is matrices whose only non-zero entries are on the *main* diagonal—the diagonal that extends from the upper left corner to the lower right corner of the matrix.

Master 2 shows the images, including the images of the basis vectors. Each of these transformations maps the basis vectors onto scalar multiples of themselves. Such transformations are called *scalings*; they change the scale of the x and y axes.

The determinant of these transformations can be found by noting the orientation of the basis vectors' images and finding the area of the rectangle they determine.

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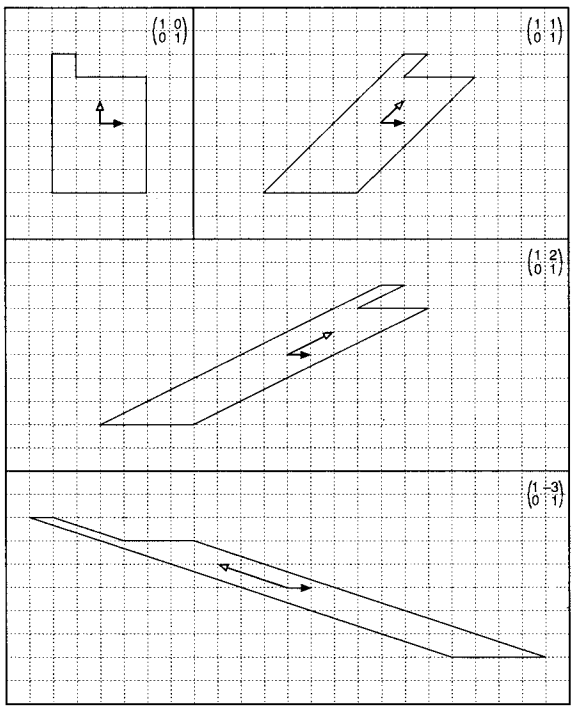
Actions

2. Discuss with the students the effect of the transformation $\begin{pmatrix} k & 0 \\ 0 & m \end{pmatrix}$ where k and m are non-zero real numbers.

3. Repeat Action 1 for Masters 3 and 4.



Shears parallel to x-axis.

Comments

2. The transformation with diagonal matrix $\begin{pmatrix} k & 0 \\ 0 & m \end{pmatrix}$ is a scaling which transforms the unit vectors (1, 0) and (0, 1) to (k, 0) and (m, 0), respectively

If k and m are both positive or both negative, the transformation doesn't change orientation and the area of the rectangle who sides are (k, 0) and (m, 0) is km. Hence, in this case, the determinant is km. On the other hand, if just one of k or m is negative, the orientation is changed and the area of the rectangle, being positive, is $-km$. Hence the determinant is $-(-km)$, or km. Thus, in all cases, the determinant of a diagonal matrix is the product of the entries on the main diagonal.

3. Master 5 and 6 show the images. The given transformations are examples of *shears*. They leave one of the basis vectors fixed and move the tip of the other basis vector along a line parallel to the fixed basis vector. Since the unit square determined by the basis vectors is transformed to a parallelogram whose base and height are 1, areas are unchanged. Also shears do not change the orientation of the basis vectors, so the determinant of a shear is 1.

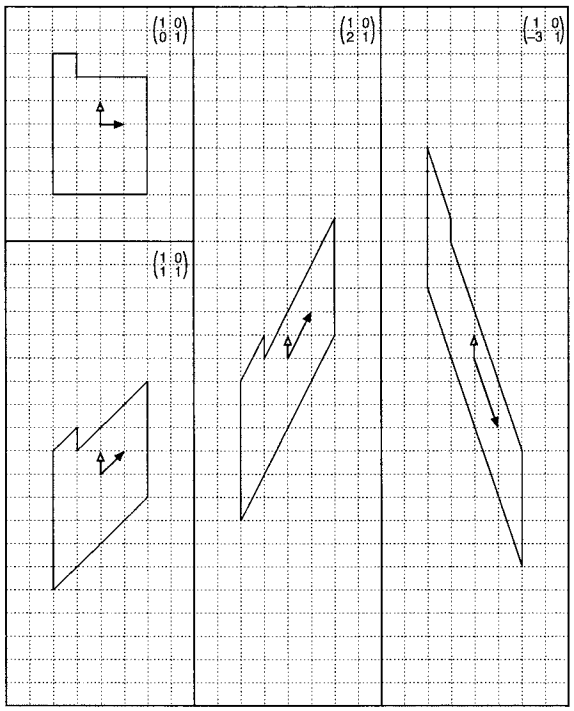
The transformations on Master 3 are shears parallel to the x-axis. They leave the basis vector along the x-axis fixed and move the tip of the other basis vector along the line $y = 1$. Those on Master 4 are shears parallel to the y-axis. They leave the basis vector along the y-axis fixed and move the tip of the other basis vector along the line $x = 1$.

Continued next page.

Actions

Comments

3. *Continued.*



Shears parallel to *y*-axis.

4. Discuss with the students the effect of the transformations $\begin{pmatrix}1 & k\\0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\k & 1\end{pmatrix}$ where k is a non-zero real number.

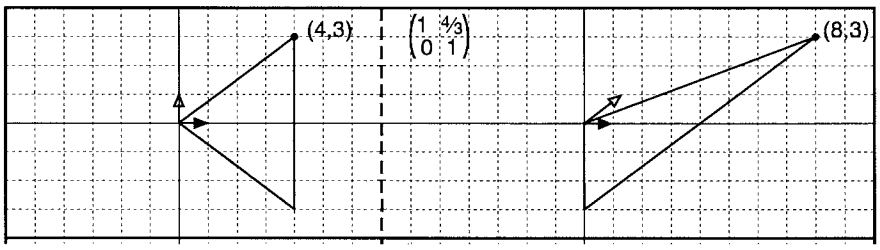
4. The transformation $\begin{pmatrix}1 & k\\0 & 1\end{pmatrix}$ is a shear parallel to the x-axis. It leaves the basis vector along the x-axis fixed. If $k > 0$, it moves the tip of the other basis k units to the right from $(0, 1)$ to $(k, 0)$. If $k < 0$, the movement is to the left. In general, a shear parallel to the x-axis leaves the x-axis fixed and moves other points parallel to the x-axis by an amount proportional to their distance from the axis. Points on opposite sides of the x-axis are moved in opposite directions.

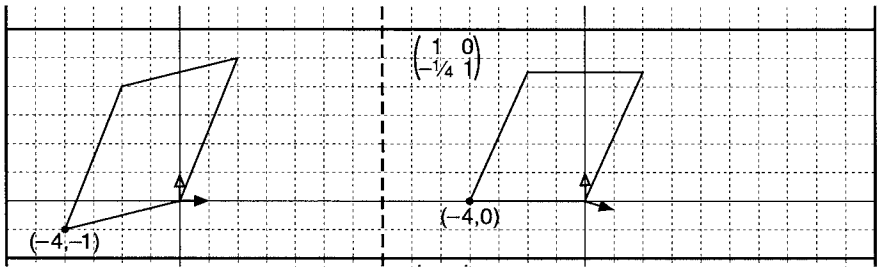
The transformation $\begin{pmatrix}1 & 0\\k & 1\end{pmatrix}$ is a shear parallel to the y-axis. It leaves the basis vector along the y-axis fixed. If $k > 0$, it moves the tip of the other basis k units upward from $(1, 0)$ to $(1, k)$. If $k < 0$, the movement is downward. In general, a shear parallel to the y-axis leaves the y-axis fixed and moves other points parallel to the y-axis by an amount proportional to their distance from the axis. Points on opposite sides of the y-axis are moved in opposite directions.

Since shears do not affect areas or orientation, the determinant of a shear is 1.

Actions

5. Distribute copies of Master 7 to the students. Tell the students that each figure on the left is to be transformed by a shear so that the given point of the figure is transformed to the given point on the right—e.g., for the top figure, the point (4, 3) is to be transformed to (8, 3). For each of the given figures, ask the students to write the matrix of the transformation which does this and sketch the image of the figure under this transformation.





Comments

5. Shears play a major role in subsequent activities—as we shall see, every linear transformation is the product of shears and scalings. This action provides experience in determining shears that transform a given point to a specified location.

Master 8 is a copy of the completed activity. In addition to the asked-for matrices and images, the basis vectors (1, 0) and (0, 1) and their images are also shown.

In the top panel of Master 8, the shear is parallel to the x-axis since the point (4, 3) is moved 4 units to the right to (8, 3). The point (0, 1), being on the same side of the x-axis as (4, 3), is also moved to the right, but since its distance from the x-axis is $1/3$ the distance of (4, 3) from the x-axis, it is only moved $1/3$ the distance that (4, 3) is moved. Thus, the tip of the basis vector (0,1) is moved $4/3$ units to the right to ($4/3$, 1). The basis vector along the x-axis remains fixed at (1, 0). Hence the matrix of the transformation is $\begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{pmatrix}$

In the second to last panel of Master 8, the shear is parallel to the y-axis since the point (−4, −1) is moved upward 1 unit. Hence the point (1, 0), which is on the opposite side of the y-axis from (−4, −1) and $1/4$ the distance from it, is moved downward $1/4$ unit to (1, $-1/4$). The basis vector along the y-axis remains fixed at (0, 1). Hence the matrix is $\begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{pmatrix}$.

Actions

6. Ask the students to make shear transformations to assist them in finding the area of the parallelogram which has successive vertices at:

a. (4, 2), (0, 0), and (–2, 5);
b. (–3, 2), (0, 0), and (–1, –5);
c. (2, 5), (0, 0), and (4, 3).

Comments

6. a. Figure 1 shows the result of shearing the parallelogram parallel to the *y*-axis until the vertex at (4, 2) is moved 2 units downward to the *x*-axis. Since the vertex at (–2, 5) is half as distant from the *y*-axis as (4, 2) and on the opposite side, it is moved 1 unit upward to (–2, 6). The resulting parallelogram has base 4 and height 6. Hence its area is 24. Since shears don't affect area, the area of the original parallelogram is also 24. If one wishes, one can make a second shear parallel to the *x*-axis—thus leaving points on the *x*-axis fixed—to obtain the rectangle shown.

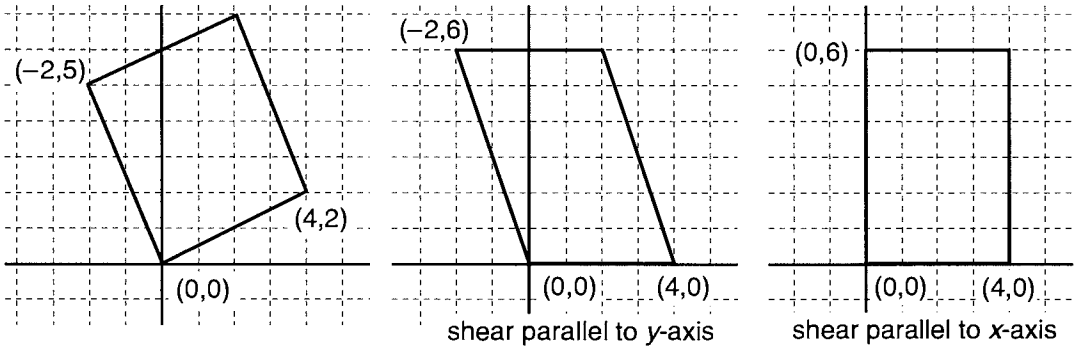


Figure 1

Figure 2 is the result of first shearing the parallelogram parallel to the *x*-axis and then parallel to the *y*-axis. In the first shear, the point (–2, 5) is moved 2 units to the right. Hence the point (4, 2) which is $^2/_5$ of the distance (–2, 5) is from the *x*-axis will be moved $(^2/_5)2$, or $^4/_5$, of a unit to the right. Again, the resulting parallelogram has area 24.

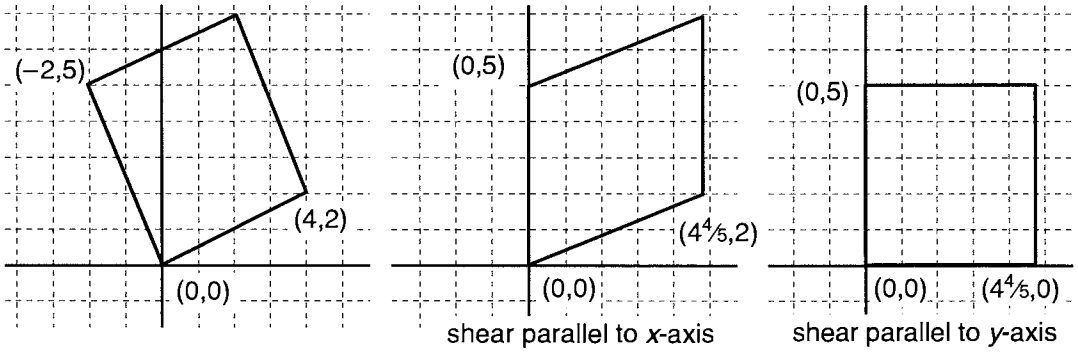


Figure 2

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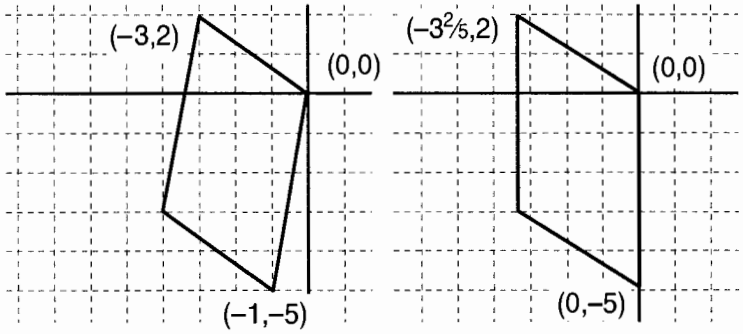


Figure 3

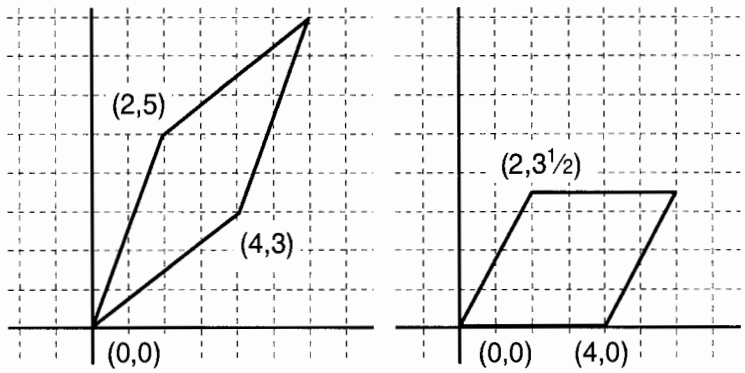
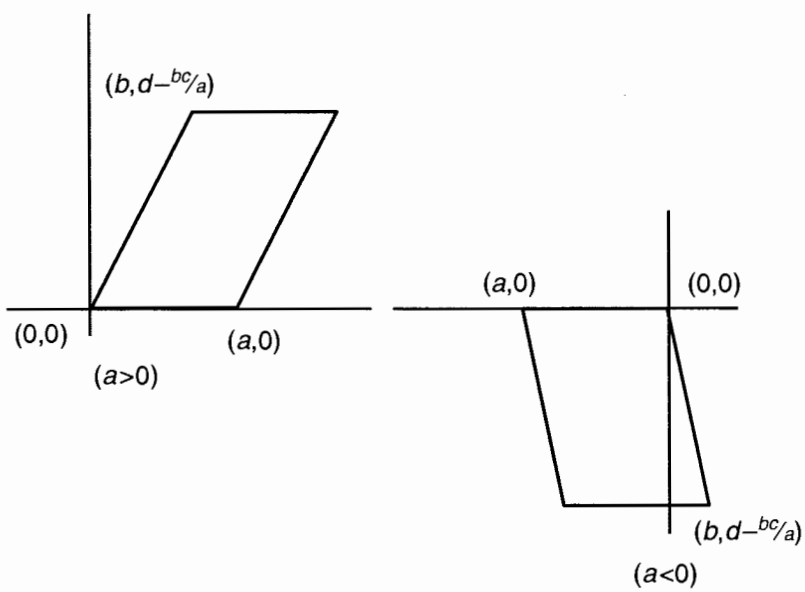


Figure 4

6. *Continued.*
b. As shown in Figure 3, by shearing parallel to the x-axis, the parallelogram is transformed into a parallelogram whose base is 5 and height is $3^2/_5$. Hence, its area is $5(3^2/_5)$, or 17. In this shear, the point $(-1, -5)$ is moved 1 unit to the right. Hence the point $(-3, 2)$ which is $^2/_5$ of the distance $(-1, -5)$ is from the x-axis, and on the opposite side, is moved $^2/_5$ of a unit to the left.

c. In Figure 4, a shear parallel to the y-axis transforms the parallelogram into one with base 4 and height $3^1/_2$. Hence its area is 14. In this shear, the point (4, 3) is moved downward 3 units. Hence the point (2, 5), which is half the distance as (4, 3) from the y-axis and on the same side, is moved $1^1/_2$ units downward.

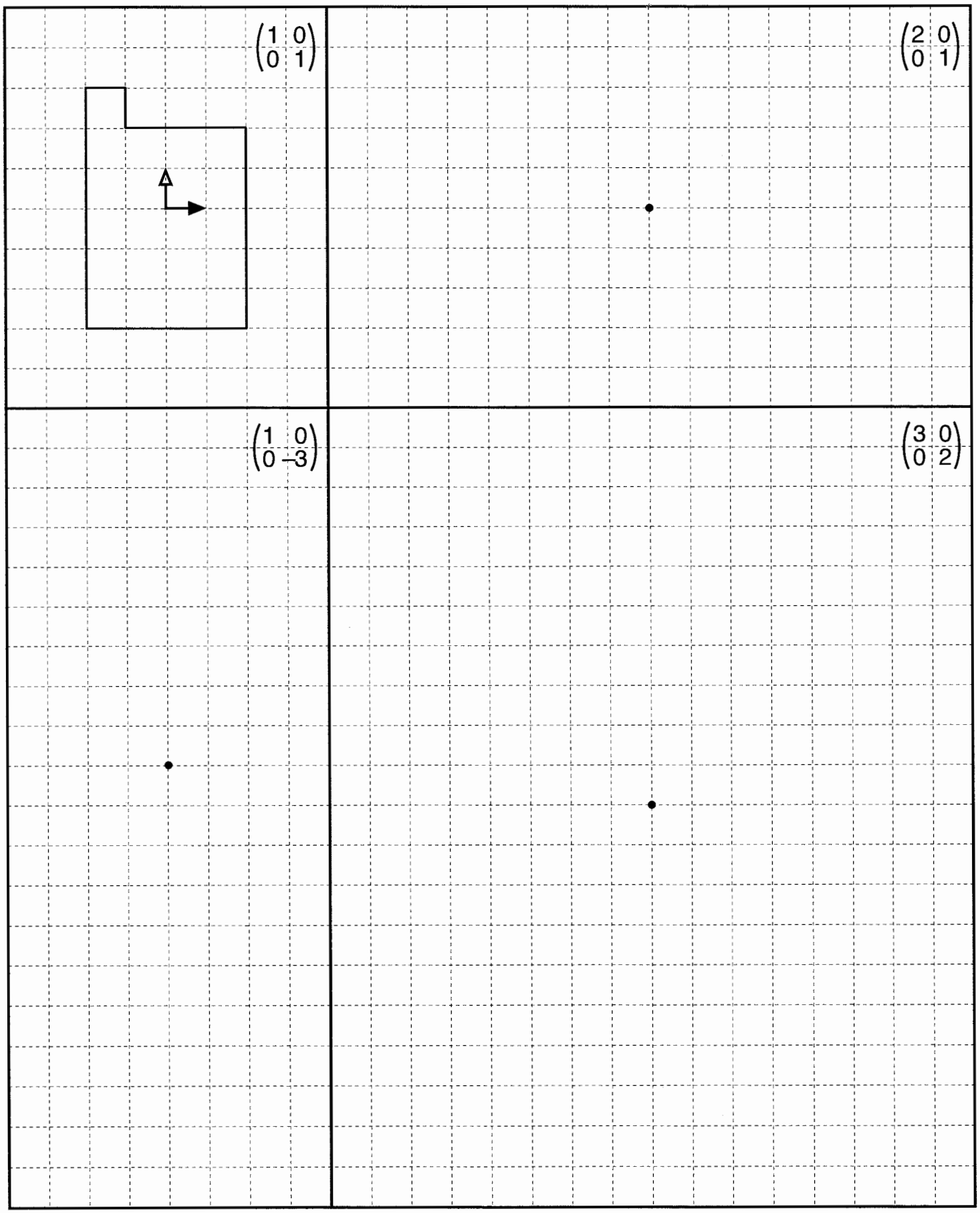
7. (Optional). Ask the students to find the area of the parallelogram with successive vertices at (a, c), $(0, 0)$ and (b, d), assuming that the parallelogram lies on the left as one proceeds from $(0, 0)$ to (a, c). Discuss.



7. One way to proceed is to shear the parallelogram parallel to the y-axis so the vertex at (a, c) is moved to $(a, 0)$. This shear has matrix $\begin{pmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{pmatrix}$. Hence the vertex (b, d) is moved to $\begin{pmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{pmatrix}\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} b \\ d - \frac{bc}{a} \end{pmatrix}$.
Since shears don't change orientation, the resulting parallelogram will lie to the left of the side with vertices $(0, 0)$ and $(a, 0)$. There are two cases, as shown, depending upon whether a is positive or negative. If a is positive, the resulting parallelogram has base a and height $d - {}^{bc}/_a$; if a is negative, the base of the parallelogram is $-a$ and the height is $-(d - {}^{bc}/_a)$. In both cases, the area of the parallelogram is $a(d - {}^{bc}/_a)$, or $ad - bc$.

The given parallelogram is the image of the unit square U under the transformation $\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence, $|\mathbf{T}| = ad - bc$.

If the parallelogram lies on the right as one proceeds from $(0, 0)$ to (a, c), the area of the parallelogram is $-(ad - bc)$. However, in this case $\mathbf{T}$ changes orientation. So $|\mathbf{T}| = -(-(ad - bc)) = ad - bc$.



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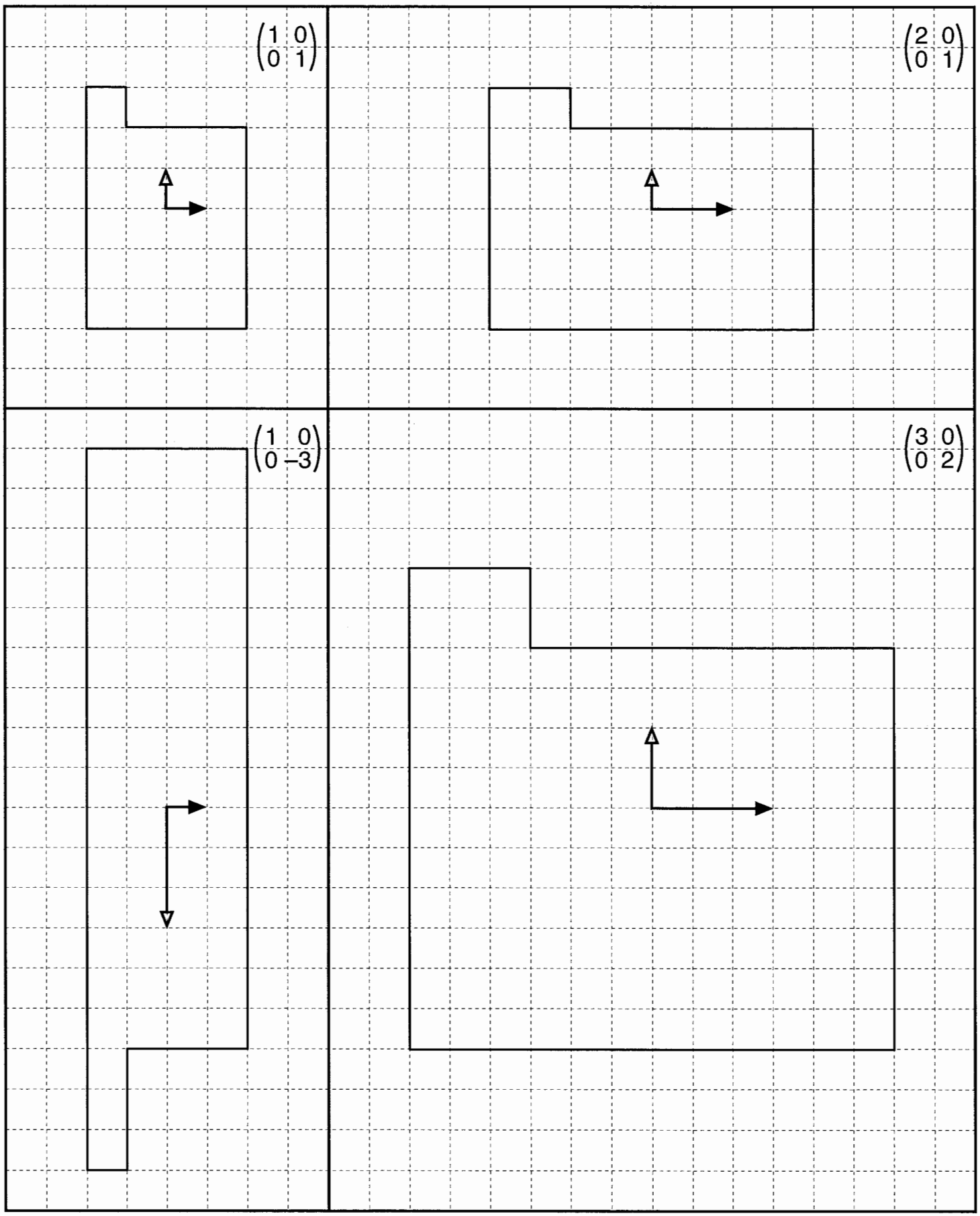

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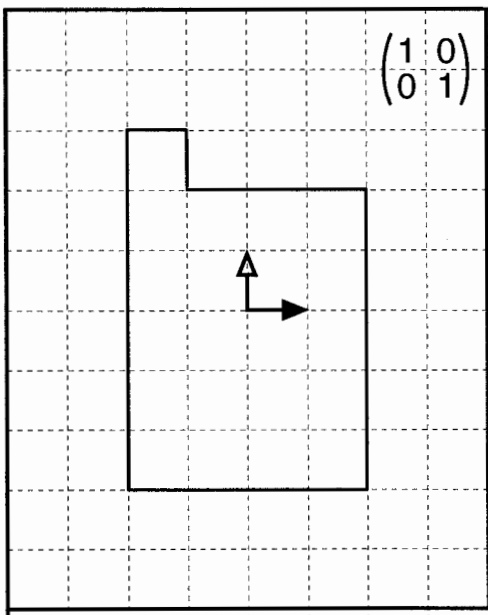

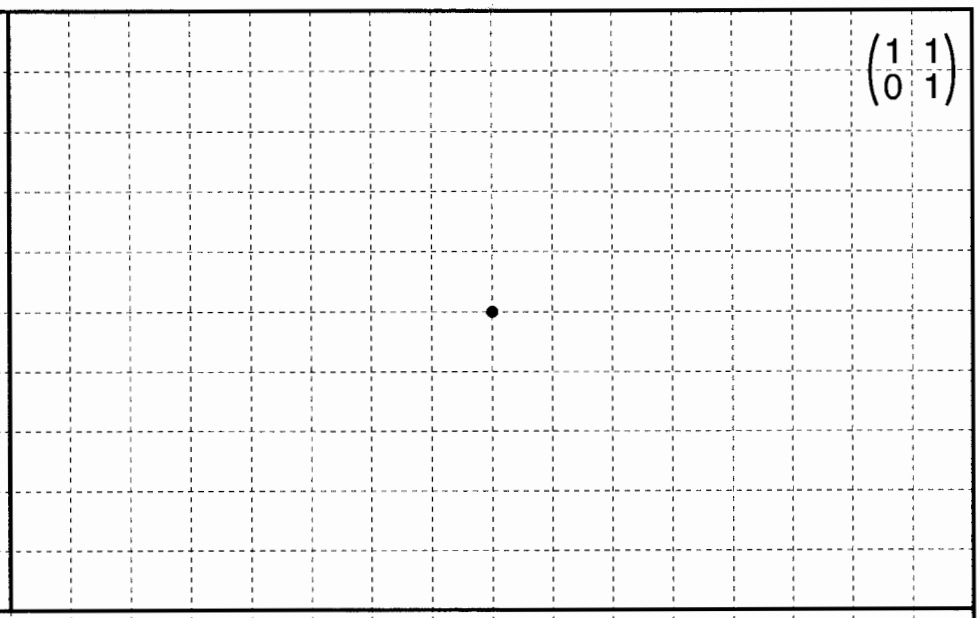

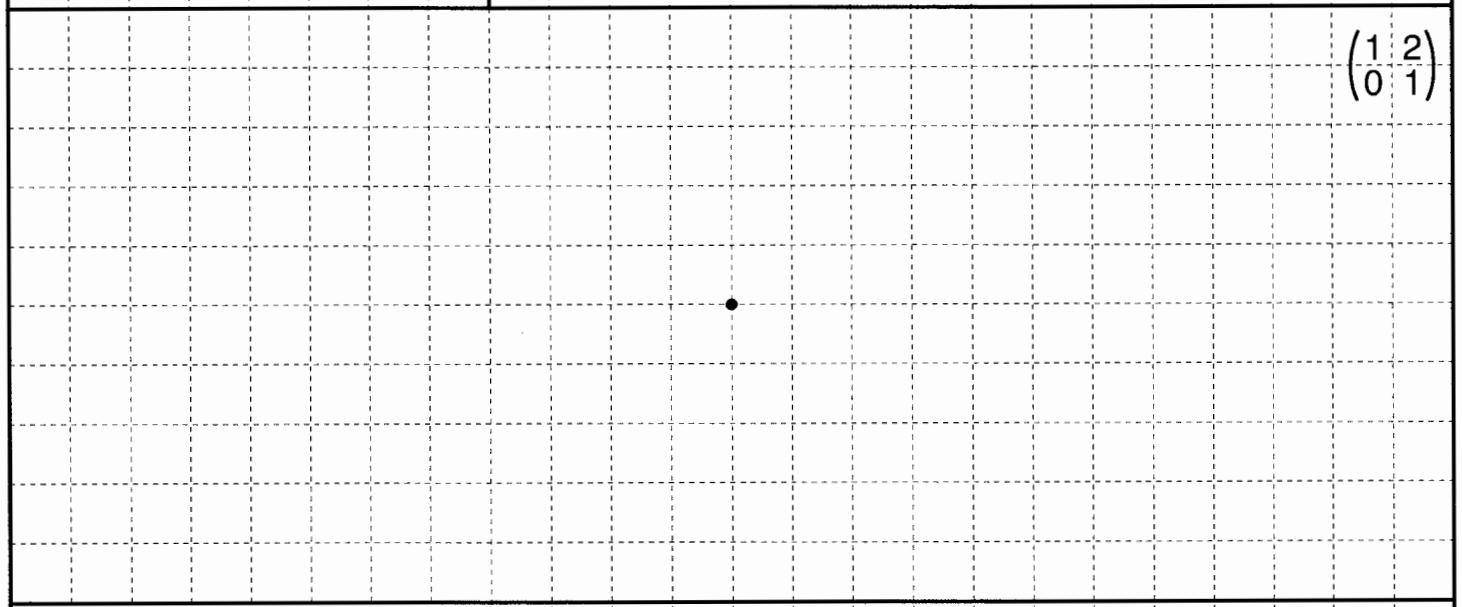

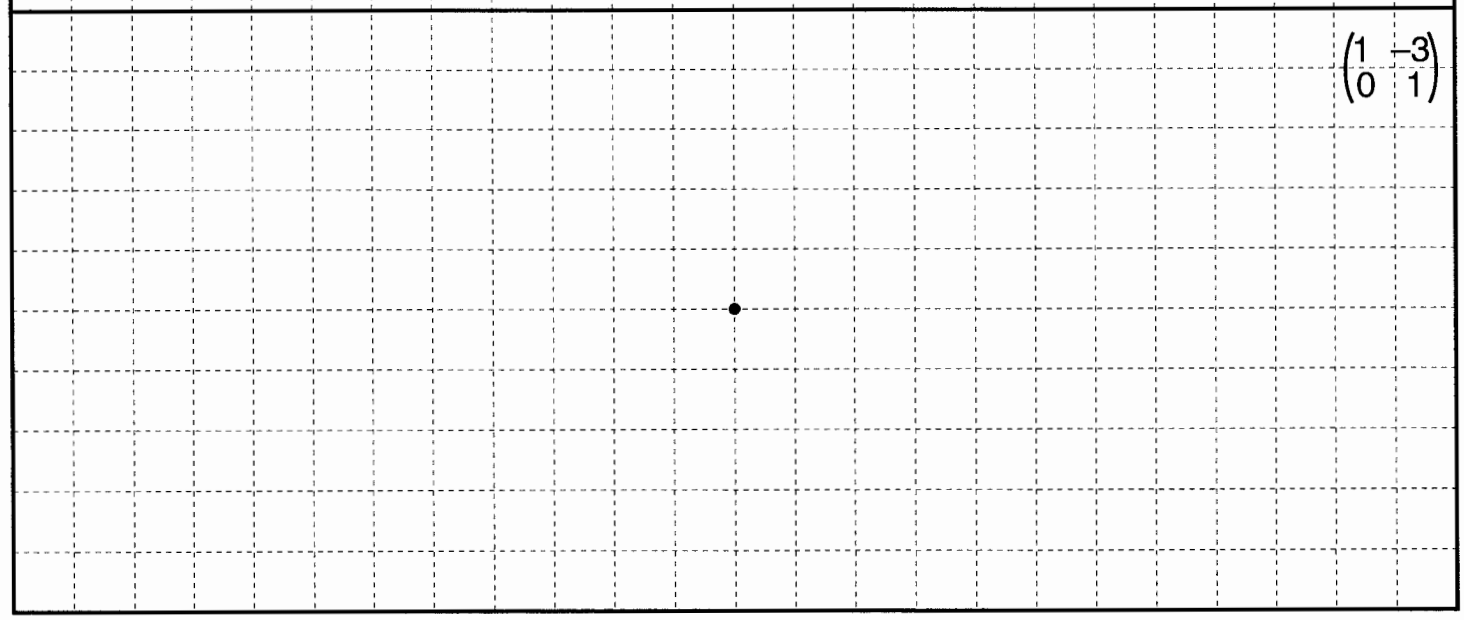

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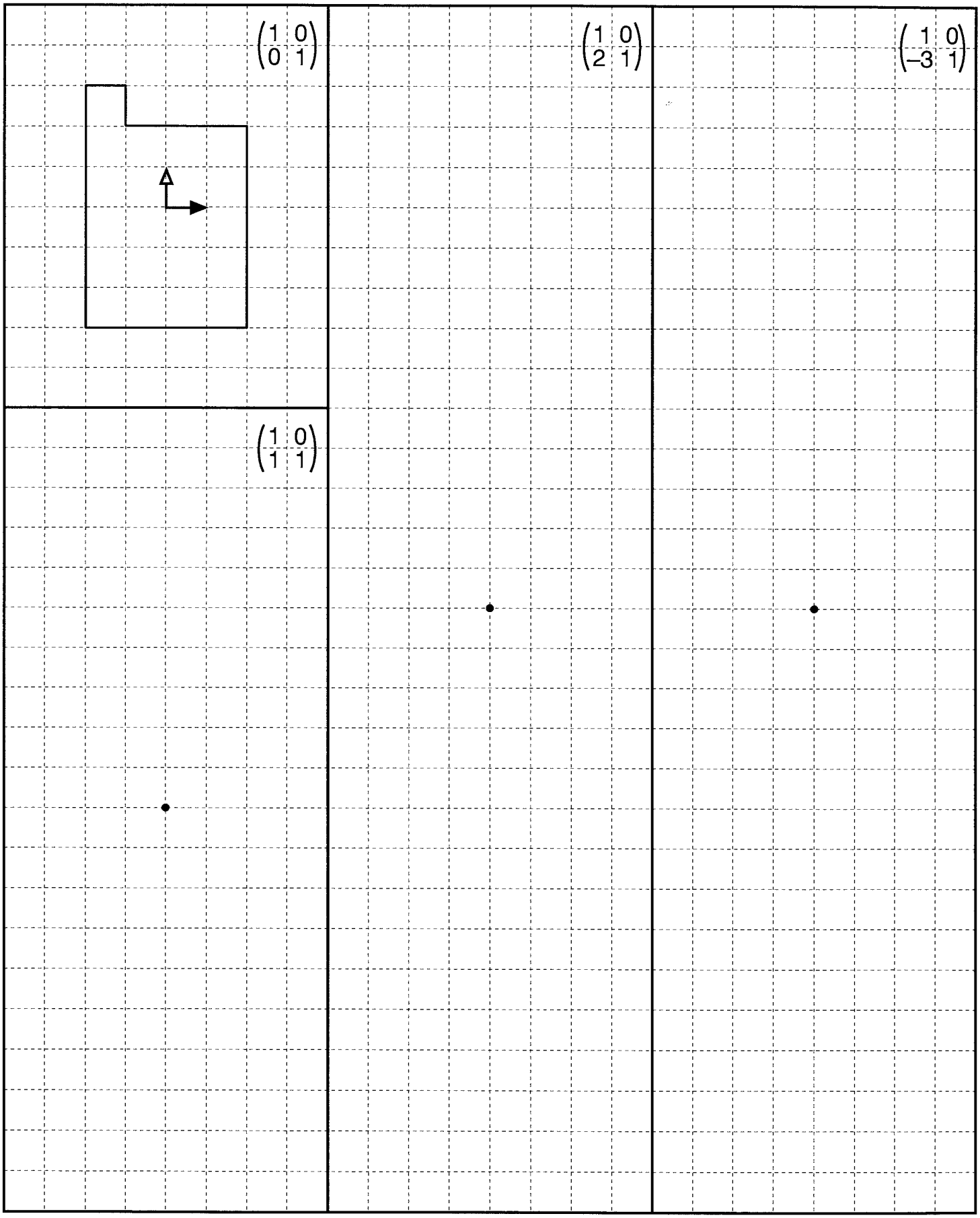

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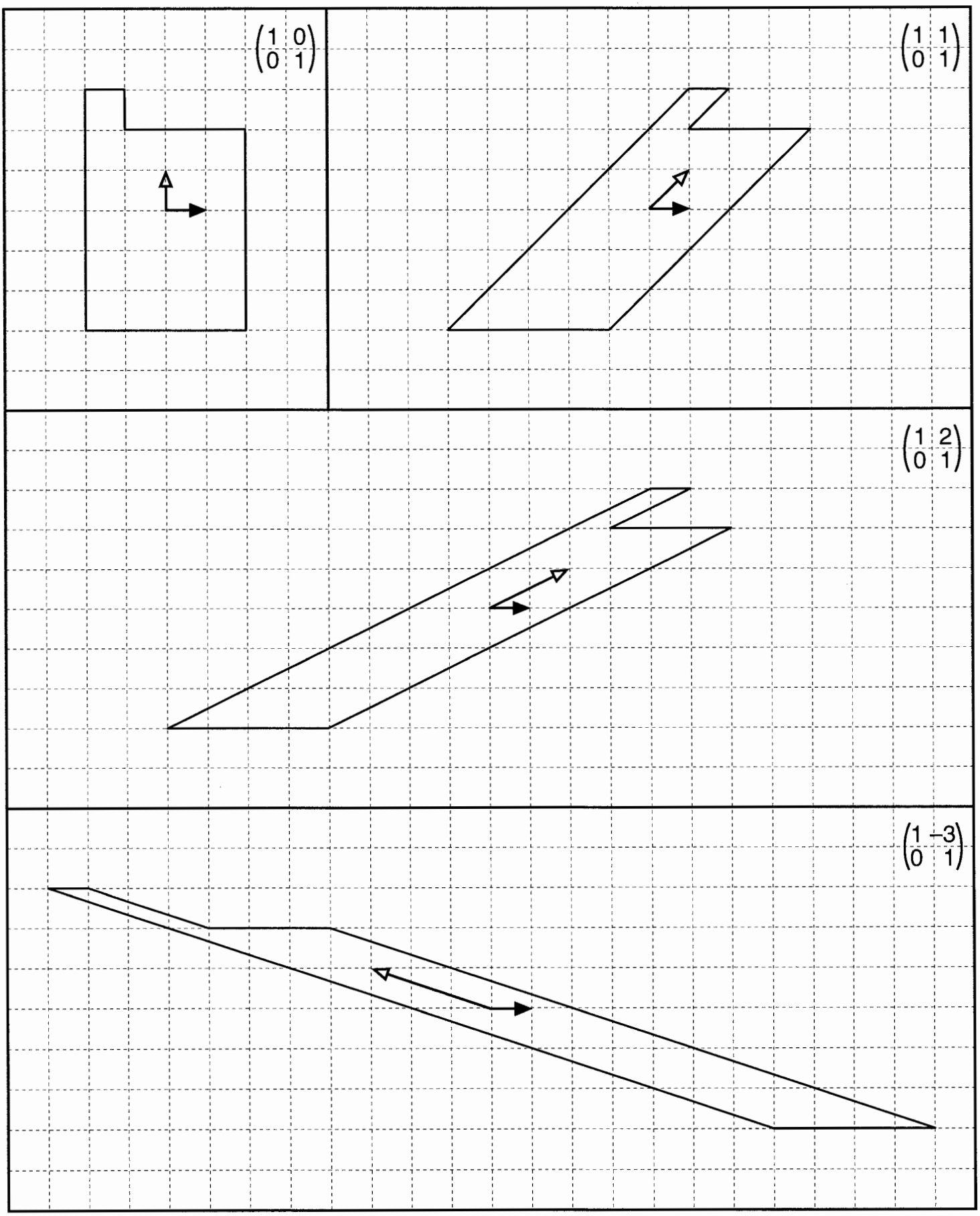

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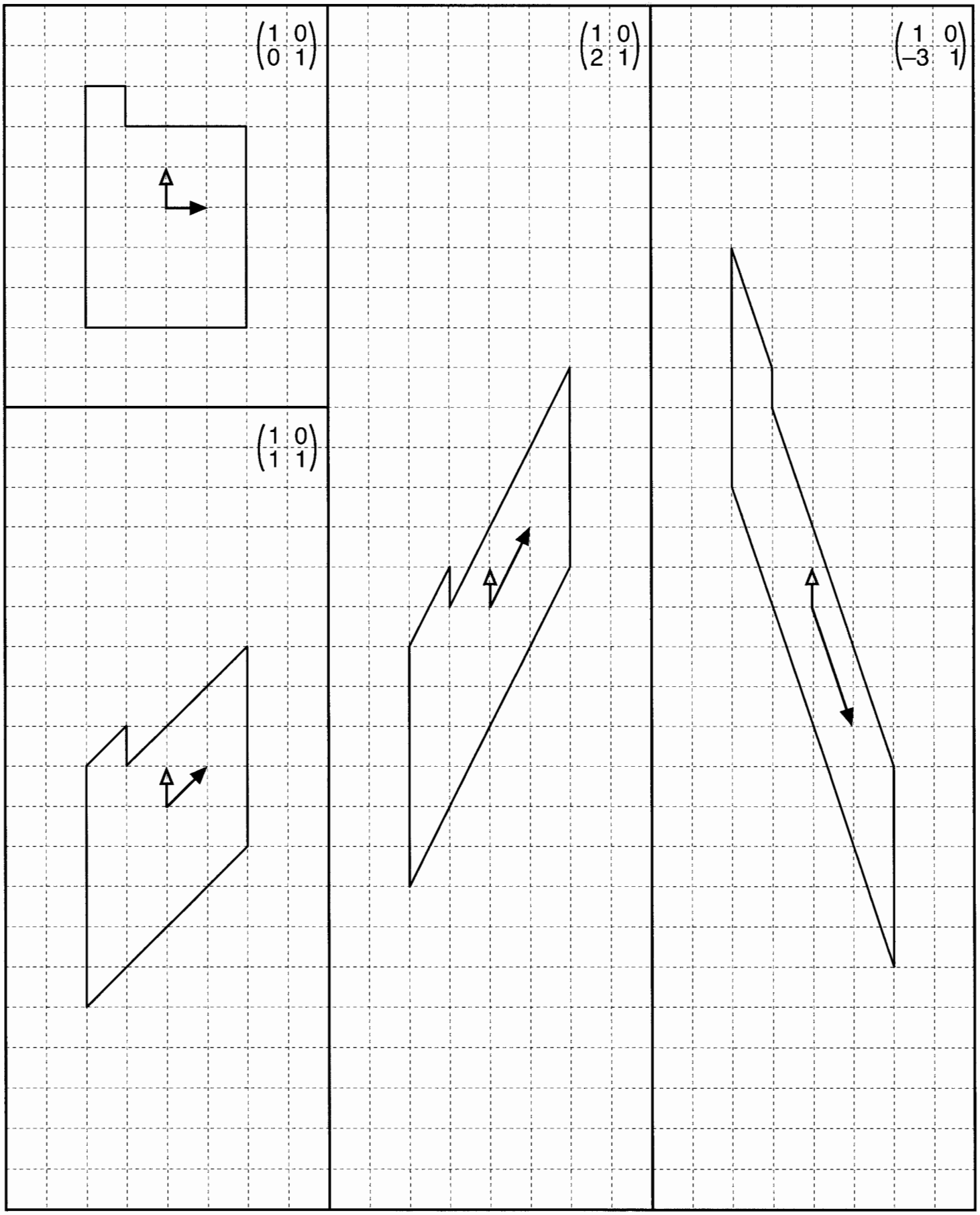

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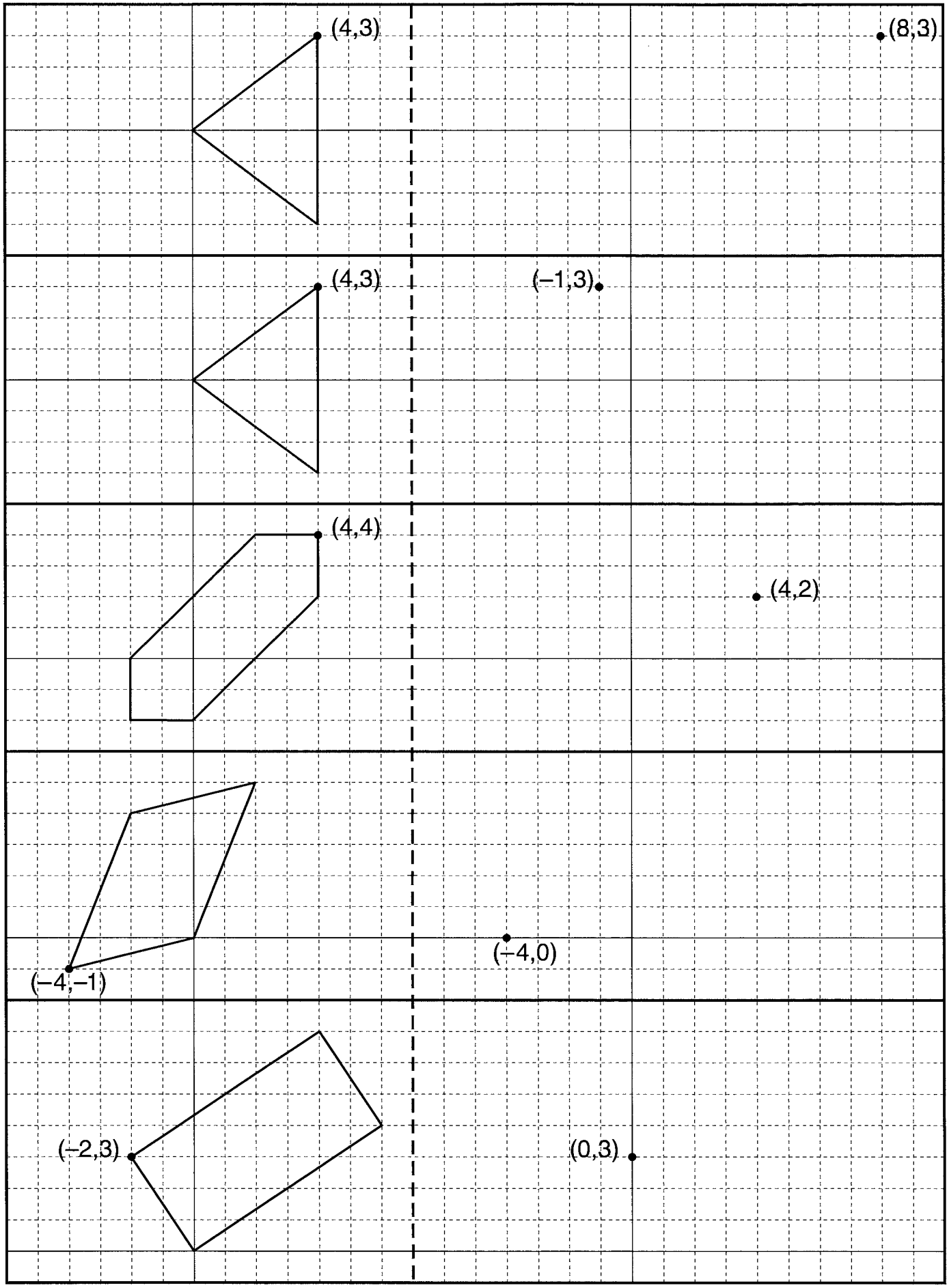

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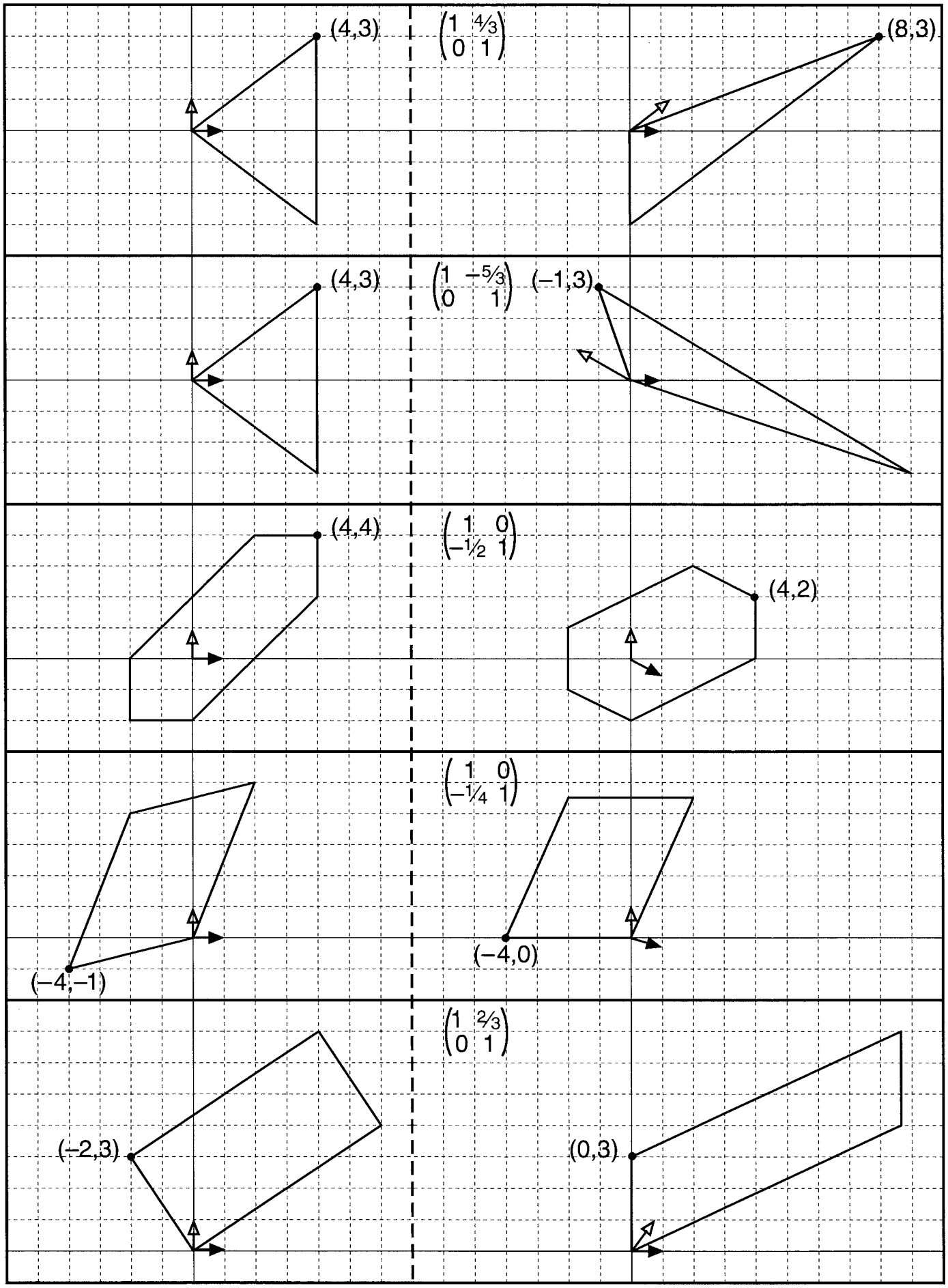

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Inverses

O V E R V I E W

The inverse of a linear transformation is defined and methods of finding inverses are developed.

Prerequisite Activity

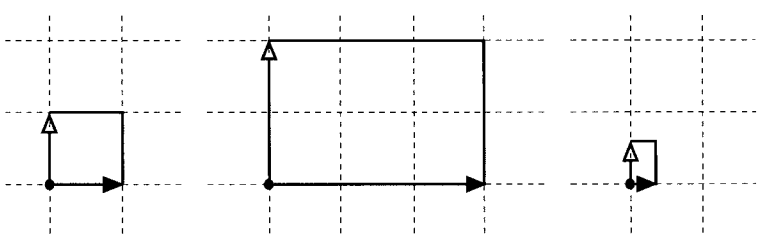
Unit XIV, Activity 4, *Products of Transformations;* Unit XIV, Activity 5, *Scaling and Shearing.*

Materials

Graph paper as needed. Copies of masters as indicated.

Actions

1. Define the *inverse* of a transformation. Then ask the students to find the inverses of the transformations on Masters 1, 3, and 4 from Activity 5, *Scaling and Shearing*. Discuss the inverses of scalings and shears.



u $\qquad$ $\mathbf{S}(u)$ $\qquad$ $\mathbf{S}^{-1}(u)$

Comments

1. Transformations are *inverses* of each other if each undoes what the other does or, more precisely, if their composition is the identity transformation. Thus, transformations **A** and **B** are inverses if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. The inverse of **A** is designated by $\mathbf{A}^{-1}$, read "**A** inverse."

The transformation $\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ leaves the x-axis fixed while reversing the direction of the y-axis and increasing its scale by a factor of 3. Thus, its inverse is $\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$ which leaves the x-axis fixed while reversing the direction of the y-axis and decreasing its scale by a factor of 3.

The transformation $\mathbf{S} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ increases the x- and y-axis by factors of 3 and 2 respectively. Its inverse is $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ which decreases the x-axis and y-axis by factors of 3 and 2 respectively. (See the sketch.)

In general, the inverse of the diagonal matrix $\begin{pmatrix} k & 0 \\ 0 & m \end{pmatrix}$ is the diagonal matrix $\begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$.

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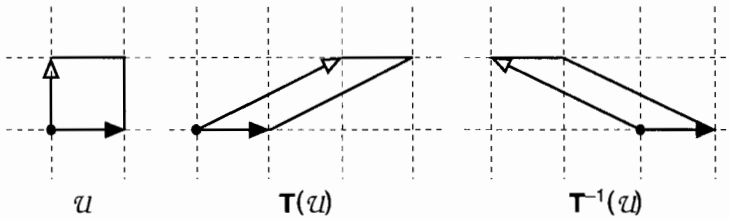
Actions



2. Let $\mathcal{U}$ be the square determined by the basis vectors (1, 0) and (0, 1) and let $\mathbf{T} = \begin{pmatrix} 4 & 2 \\ -2 & 5 \end{pmatrix}$. Ask the students to find two shears, one parallel to each axis, whose product will transform $\mathbf{T}(\mathcal{U})$ into a rectangle which has a vertex at the origin and sides along the x and y axes. Then ask the students to find a scaling which will transform this rectangle to $\mathcal{U}$. Discuss.

Comments

1. *Continued.*

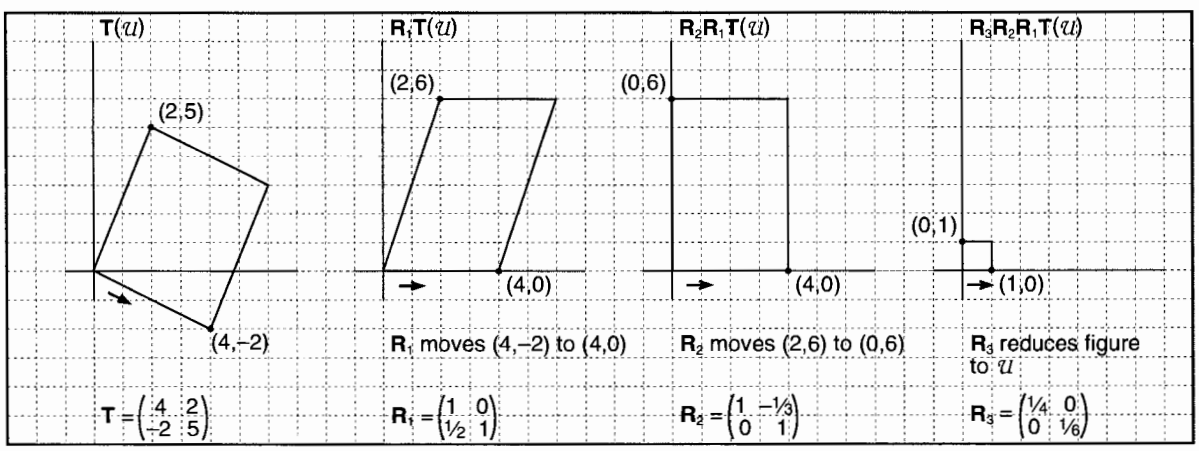
The shear $\mathbf{T} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ leaves the x-axis fixed and moves the tip of the basis vector (0, 1) two units to the right. Its inverse is $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, which leaves the x-axis fixed and moves the tip of (0, 1) two units to the left. (See the sketch.)

In general, the inverse of $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$ and the inverse of $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$.

2. Actions 2, 3, and 4 illustrate a method for finding the inverse of a transformation by expressing it as a product of shears and a scaling.

$\mathbf{T}(\mathcal{U})$ is a parallelogram which has vectors (2, 5) and (4, –2) as sides.

There are two possibilities, depending on whether one first shears parallel to the y-axis or the x-axis. Master 1, attached, shows both possibilities. The arrow by each figure indicates orientation.

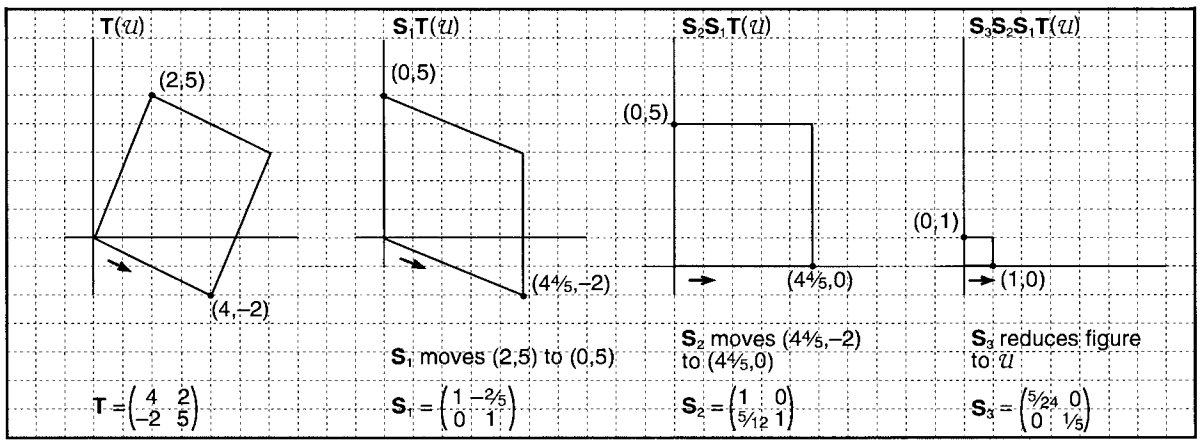


In the top half, $\mathbf{R}_1$ is a shear parallel to the y-axis which takes the point (4, –2) to (4, 0). Since $\mathbf{R}_1$ moves a point 4 units to the right of the y-axis up 2 units, it will move a point 2 units to the right of the y-axis up 1 unit. Thus $\mathbf{R}_1$ moves (2, 5) to (2, 6). Also, $\mathbf{R}_1$ will move (1, 0) up $^1/_2$ unit to (1, $^1/_2$). Since it leaves (0, 1) fixed, its matrix is $\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$.

Continued next page.

2. *Continued.* Thus $\mathbf{R}_1\mathbf{T}(\mathcal{U})$ is a parallelogram with vectors (2, 6) and (4, 0) as sides. $\mathbf{R}_2$ is a shear parallel to the x-axis which moves (2, 6) to (0, 6). Since it moves a point 6 units above the x-axis 2 units to the left, it will move (0, 1) $^1/_3$ unit to the left to ($-^1/_3$, 1). It leaves (1, 0) fixed. Hence its matrix is $\begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}$. Thus $\mathbf{R}_2\mathbf{R}_1\mathbf{T}(\mathcal{U})$ is a rectangle with sides along the axes. The transformation $\mathbf{R}_3 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$, which reduces the x and y dimensions by factors of 4 and 6, respectively, then transforms this rectangle to $\mathcal{U}$. Thus $\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1\mathbf{T}(\mathcal{U}) = \mathcal{U}$.

On the bottom half of Master 1, the transformation of $\mathbf{T}(\mathcal{U})$ to $\mathcal{U}$ is accomplished by



first shearing parallel to the x-axis, then shearing parallel to the y-axis, and then scaling. If $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$ are the respective matrices of these transformations, then

$\mathbf{S}_3\mathbf{S}_2\mathbf{S}_1\mathbf{T}(\mathcal{U}) = \mathcal{U}$ where $\mathbf{S}_1 = \begin{pmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{pmatrix}$,

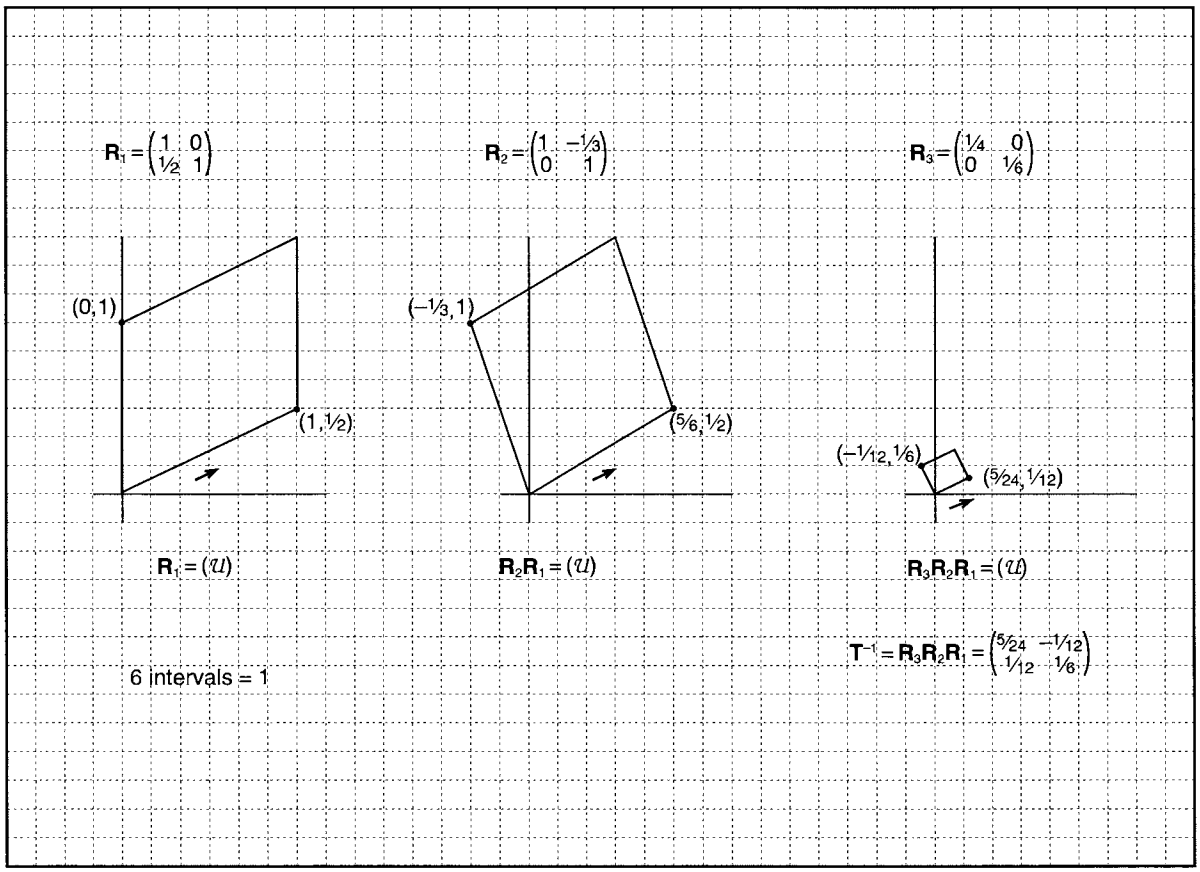
$\mathbf{S}_2 = \begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 1 \end{pmatrix}$, and $\mathbf{S}_3 = \begin{pmatrix} \frac{5}{24} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$.

Actions

3. Ask the students to use the previous results to find the matrix and determinant of $\mathbf{T}^{-1}$. Discuss.

Comments

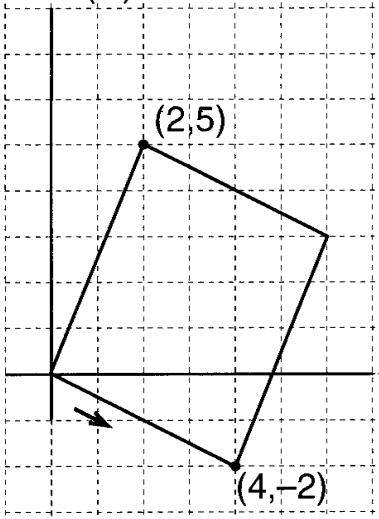
3. Since $\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1\mathbf{T}$ maps $\mathcal{U}$ onto itself without changing its orientation, it maps (1, 0) and (0, 1) onto themselves and hence $\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1\mathbf{T} = \mathbf{I}$. Thus, $\mathbf{T}^{-1} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1$. The matrix of $\mathbf{T}^{-1}$ can be found by finding the product $\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1$ using matrix multiplication (cf. Action 8, Activity 4) or by determining the effect of this product on the unit square $\mathcal{U}$, as shown in Master 2. (Note that the scales on Masters 1 and 2 are different.) In either case, one finds $\mathbf{T}^{-1} = \begin{pmatrix} \frac{5}{24} & -\frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{pmatrix}$.



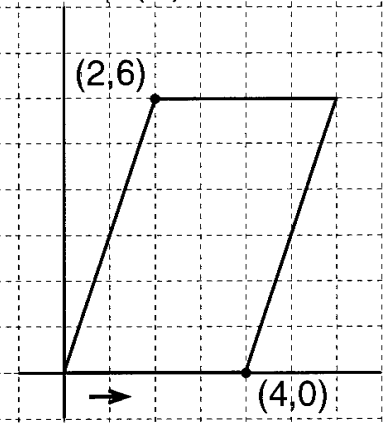
Since shears have determinate 1, $|\mathbf{T}^{-1}| = |\mathbf{R}_3| = \frac{1}{24}$. Also, $|\mathbf{T}| = |\mathbf{R}_2\mathbf{R}_1\mathbf{T}| = 24$ (the area of $\mathbf{R}_2\mathbf{R}_1\mathbf{T}(\mathcal{U})$ on Master 1). Thus $|\mathbf{T}^{-1}| = |\mathbf{T}|^{-1}$, that is, the determinant of the inverse of $\mathbf{T}$ is the inverse of the determinant of $\mathbf{T}$. This is true in general since $\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$ and the determinant of a product is the product of determinants (see Comment 7, Activity 4).

4. Have the students find the matrix and determinant of $\mathbf{T}^{-1}$ if $\mathbf{T} = \begin{pmatrix} 8 & -3 \\ -2 & -6 \end{pmatrix}$.

4. In this case, as illustrated in Master 3, $\mathbf{T}^{-1} = \begin{pmatrix} \frac{1}{9} & -\frac{1}{18} \\ -\frac{1}{27} & -\frac{4}{27} \end{pmatrix}$ and $|\mathbf{T}|^{-1} = -\frac{1}{54}$.

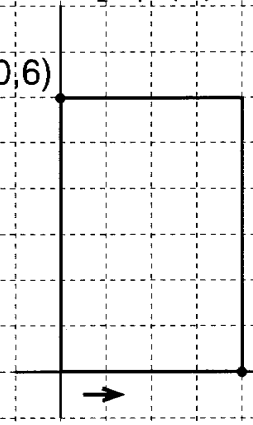


$\mathbf{T} = \begin{pmatrix} 4 & 2 \\ -2 & 5 \end{pmatrix}$



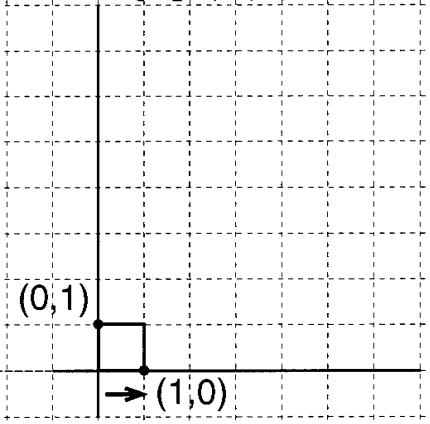
$\mathbf{R}_1$ moves (4,–2) to (4,0)

$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$



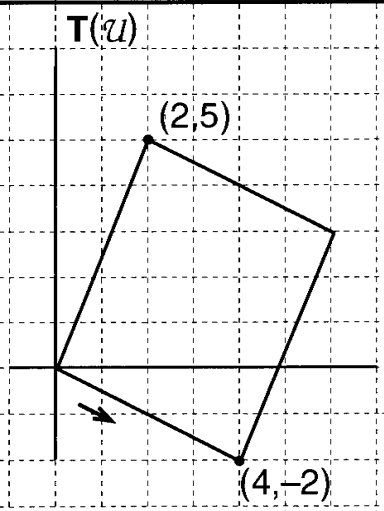
$\mathbf{R}_2$ moves (2,6) to (0,6)

$\mathbf{R}_2 = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}$

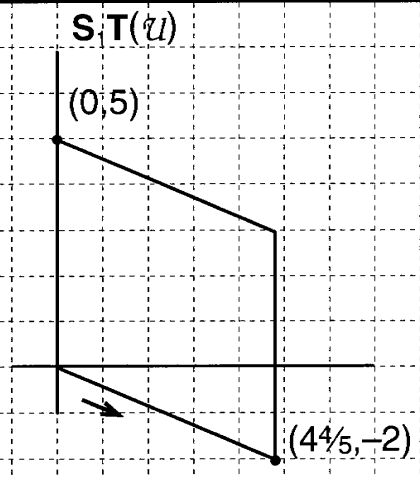


$\mathbf{R}_3$ reduces figure to $\mathcal{U}$

$\mathbf{R}_3 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$

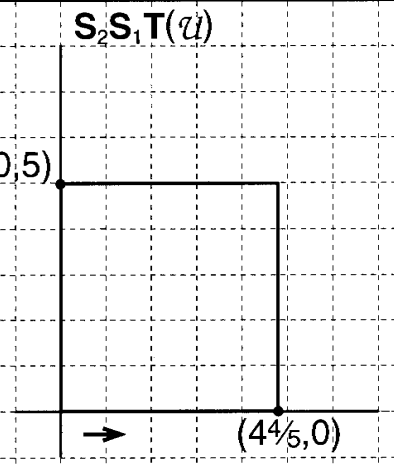


$\mathbf{T} = \begin{pmatrix} 4 & 2 \\ -2 & 5 \end{pmatrix}$



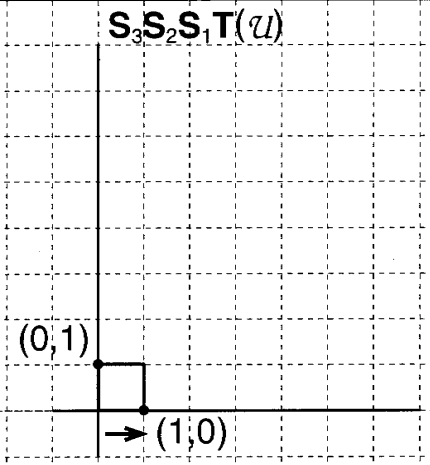
$\mathbf{S}_1$ moves (2,5) to (0,5)

$\mathbf{S}_1 = \begin{pmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{pmatrix}$



$\mathbf{S}_2$ moves (4⅘,–2) to (4⅘,0)

$\mathbf{S}_2 = \begin{pmatrix} 1 & 0 \\ \frac{5}{12} & 1 \end{pmatrix}$



$\mathbf{S}_3$ reduces figure to $\mathcal{U}$

$\mathbf{S}_3 = \begin{pmatrix} \frac{5}{24} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$

$\mathbf{T} = \begin{pmatrix} 4 & 2 \\ -2 & 5 \end{pmatrix}$

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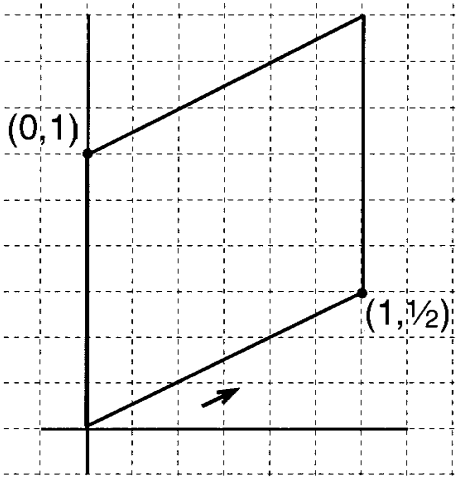
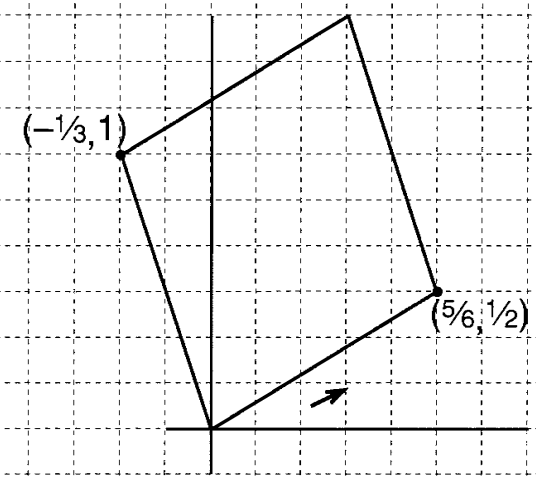
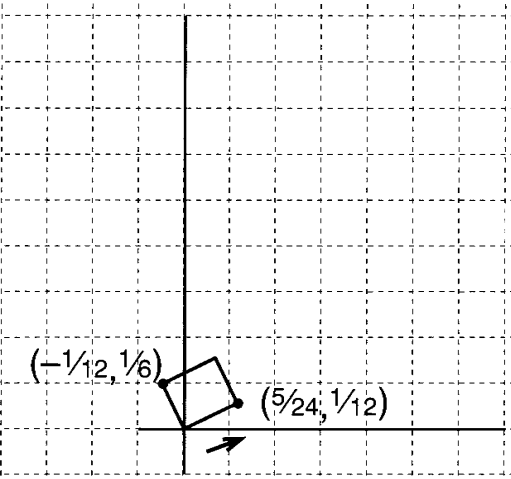
$$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$$

$$\mathbf{R}_2 = \begin{pmatrix} 1 & -1/3 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_3 = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/6 \end{pmatrix}$$







$\mathbf{R}_1 = (\mathcal{U})$

$\mathbf{R}_2\mathbf{R}_1 = (\mathcal{U})$

$\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1 = (\mathcal{U})$

$$\mathbf{T}^{-1} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1 = \begin{pmatrix} 5/24 & -1/12 \\ 1/12 & 1/6 \end{pmatrix}$$

6 intervals = 1

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Actions

5. (Optional) Repeat Action 4 if $\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, and d are positive real numbers.

6. (Optional) Point out to the students that expressing the inverse of a transformation **T** as a product of a two shears and a scaling implies that the transformation itself can be expressed as a product of two shears and a scaling. Ask the students to do this for the transformations in Actions 2 and 4.

Comments

5. Master 4 shows a product of transformations which gives $\mathbf{T}^{-1}$. Note that $ad - bc$ must be unequal to 0 for the inverse to exist. As pointed out in Action 7 of Activity 5, $ad - bc = |\mathbf{T}|$. Thus, if one follows the convention that, for a matrix **A**, $k\mathbf{A}$ is the matrix in which every element of **A** is multiplied by k, from Master 4, one has $\mathbf{T}^{-1} = \frac{1}{|\mathbf{T}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

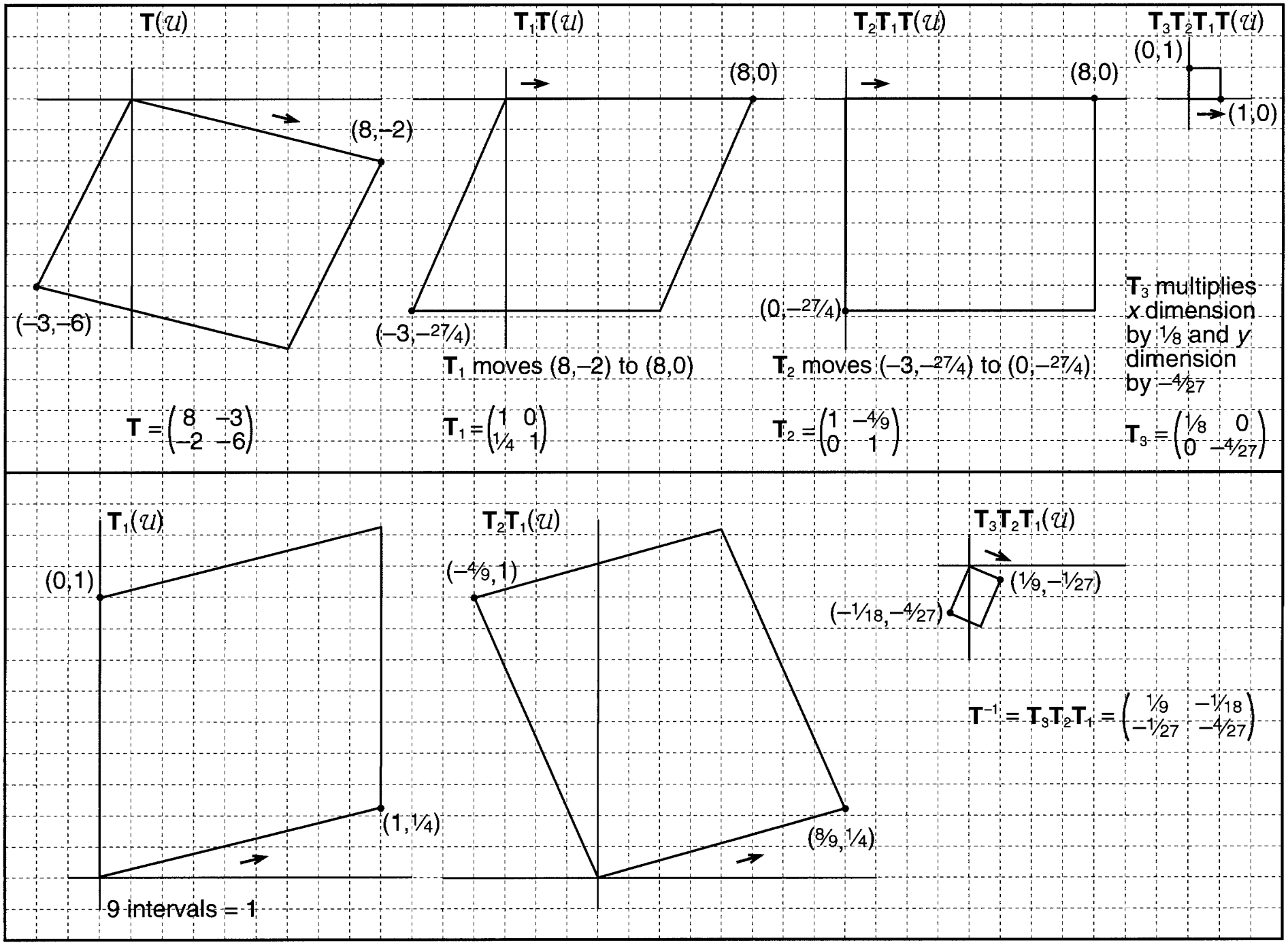
The sketches in Master 4 suppose a, b, c, and d are all positive. However, one can verify that the result holds for any values of a, b, c, and d, provided $ad - bc \neq 0$, by forming the product $\mathbf{TT}^{-1}$ and noting it equals **I**.

6. For the matrix **T** of Action 2, one has, from comment 3, $\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1\mathbf{T} = \mathbf{I}$. Thus **T** is the inverse of the product $\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1$, that is **T** undoes the effect of carrying out these three transformations, which is accomplished by carrying out their inverses in the reverse order, that is $\mathbf{T} = \mathbf{R}_1^{-1}\mathbf{R}_2^{-1}\mathbf{R}_3^{-1}$. Since the inverse of a shear is a shear and the inverse of a scaling is a scaling, **T** is the product of two shears and a scaling. Using the results of Actions 1 and 2, one has

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}.$$

For the matrix **T** of Action 4,

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{4}{9} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & -\frac{27}{4} \end{pmatrix}.$$



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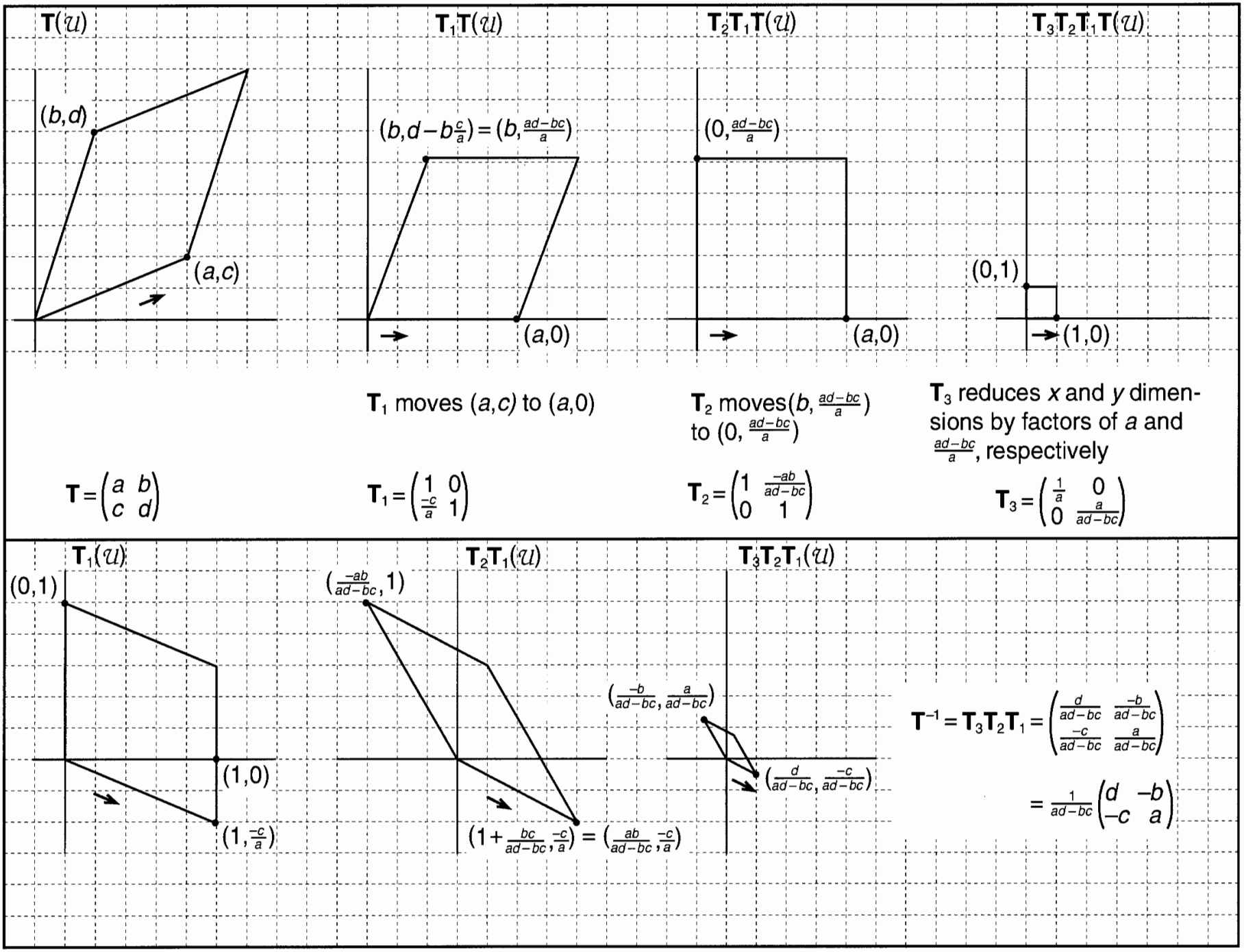

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Systems of Equations

O V E R V I E W

Methods of solving systems of two linear equations in two unknowns are developed.

Prerequisite Activity

Unit XIV, Activity 4, *Products of Transformations.*

Materials

Activity sheet as noted in Action 3. Graph paper as needed. Copies of Masters as needed.

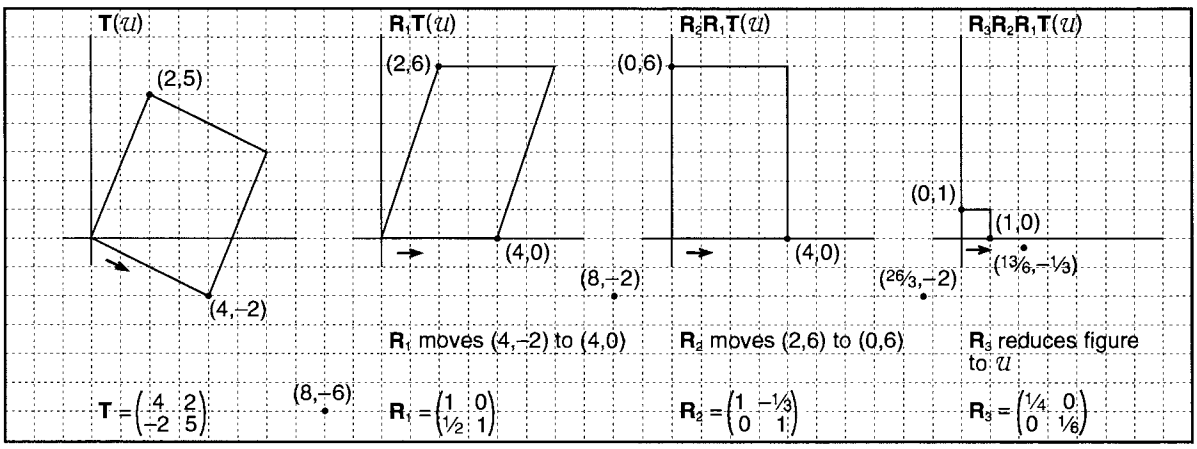
Actions

1. Ask the students to find the point (x, y) such that $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$ where $\mathbf{T}$ is the transformation of Action 2 of the previous activity, that is $\mathbf{T} = \begin{pmatrix} 4 & 2 \\ -2 & 5 \end{pmatrix}$. Discuss.

Comments

1. Since $\mathbf{T}$ maps (x, y) to $(8, -6)$, $\mathbf{T}^{-1}$ maps $(8, -6)$ to (x, y), that is, $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{T}^{-1}\begin{pmatrix} 8 \\ -6 \end{pmatrix}$.

$\mathbf{T}^{-1}$ is the product of the sequence of transformations that transforms $\mathbf{T}(\mathcal{U})$ to $\mathcal{U}$, described in Comment 2 of Activity 6. Hence, the effect of $\mathbf{T}^{-1}$ on $(8, -6)$ can be determined by transforming $(8, -6)$ by this sequence of transformations. This is illustrated below using Master 1 from Activity 6. The shear $\mathbf{R}_1$ moves $(8, -6)$ up 4 units to $(8, -2)$, $\mathbf{R}_2$ moves this point $\frac{2}{3}$ unit to the right to $(\frac{26}{3}, -2)$ which is then transformed by the scaling $\mathbf{R}_3$ to $(\frac{13}{6}, -\frac{1}{3})$.



Alternatively, (x, y) can be found by directly applying $\mathbf{T}^{-1}$, as determined in Action 3 of Activity 6, to $(8, -6)$:

$$\begin{pmatrix} \frac{5}{24} & -\frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 8 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{5}{24}(8) - \frac{1}{12}(-6) \\ \frac{1}{12}(8) + \frac{1}{6}(-6) \end{pmatrix} = \begin{pmatrix} \frac{13}{6} \\ -\frac{1}{3} \end{pmatrix}$$

The students may use other methods to determine (x, y).

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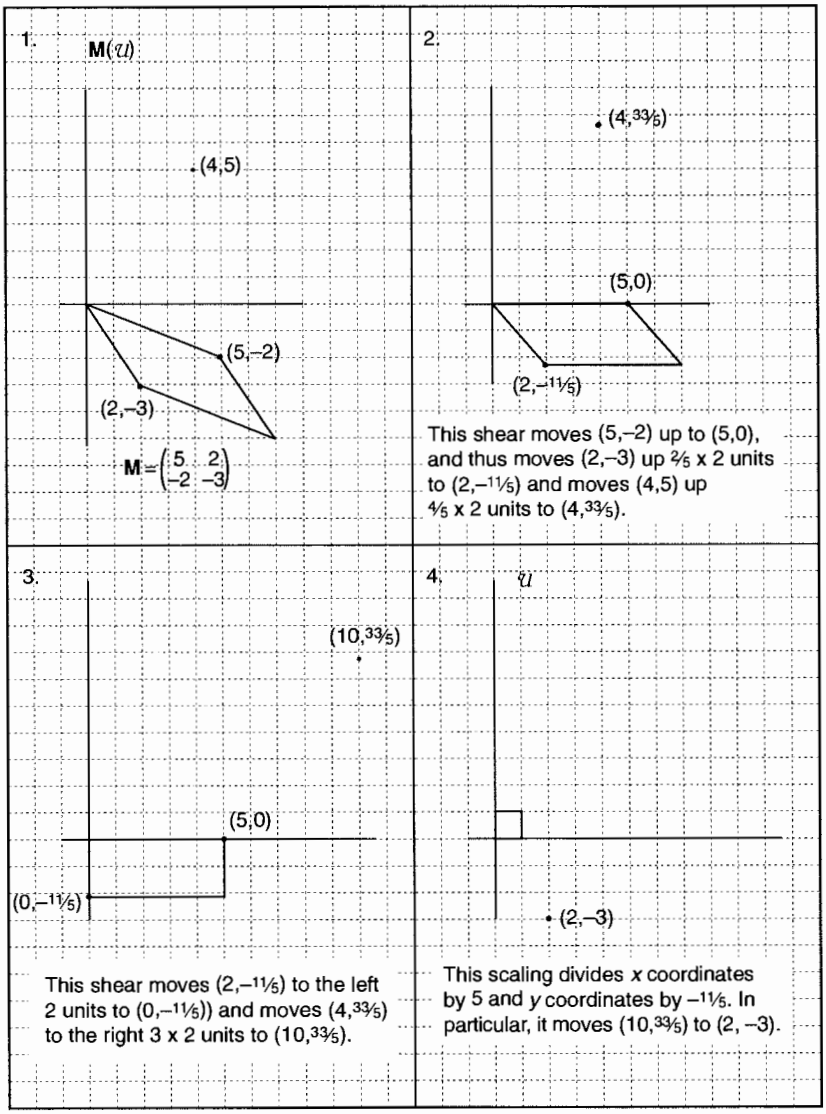
Actions

2. Ask the students to find x and y such that

$$5x + 2y = 4,$$
$$-2x - 3y = 5.$$

Discuss.



Comments

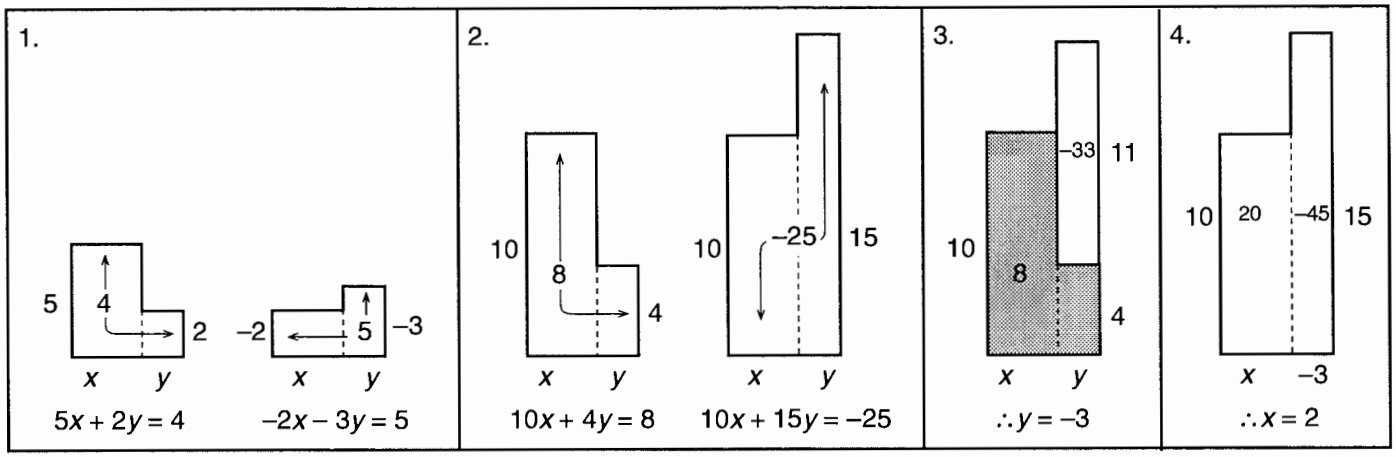
2. The equations may be written in matrix form as $\begin{pmatrix} 5 & 2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. Thus, if we let $\mathbf{M} = \begin{pmatrix} 5 & 2 \\ -2 & -3 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

The effect of $\mathbf{M}^{-1}$ on (4, 5) can be determined by transforming (4, 5) by the shears and scaling required to transform $\mathbf{M}(\mathcal{U})$ to $\mathcal{U}$ as shown in the sketch (a transparency of the sketch may be made from the attached Master 1).

Alternatively, $\mathbf{M}^{-1}$ can be computed directly using the formula developed in Action 5 of Activity 6 and then applied to (4,5) to obtain

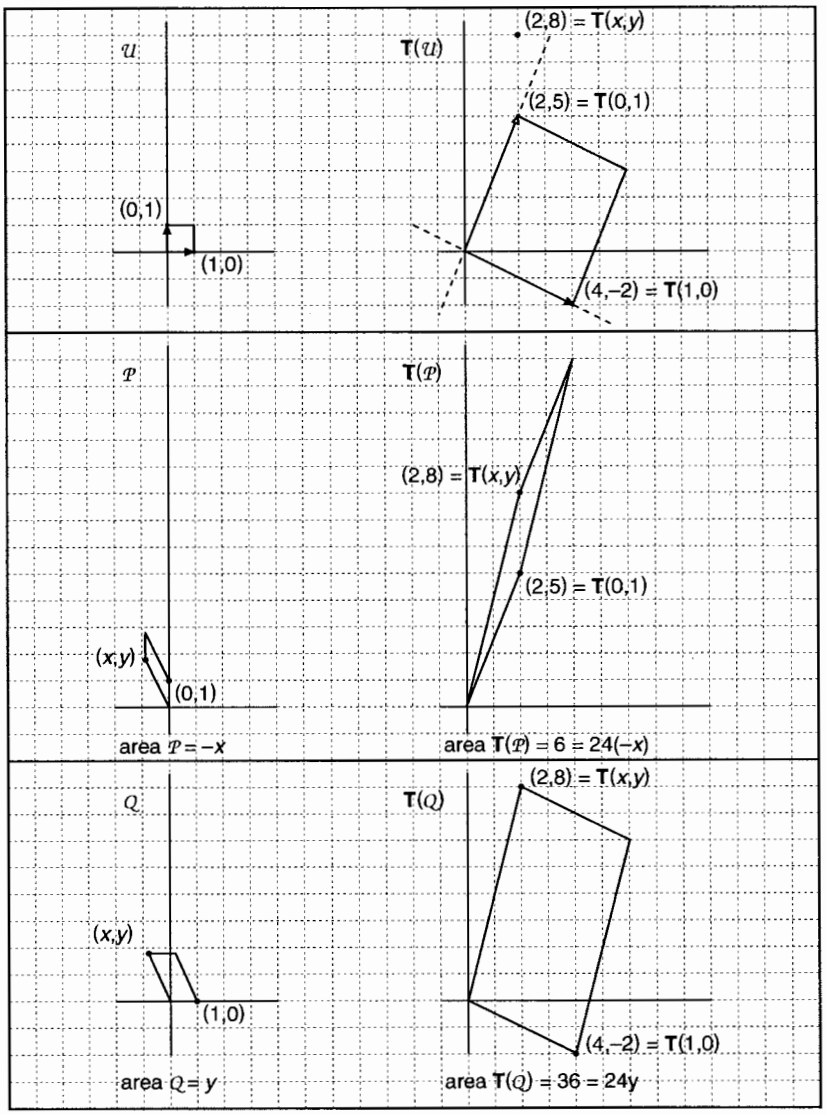
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{2}{11} & -\frac{5}{11} \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Students who have had previous experience with *Math and the Mind's Eye* materials may use sketches to arrive at a solution. One possibility is shown below, where the initial sketches of the two equations are modified so the "x regions" are comparable.



Actions

3. Distribute Activity Sheet XIV-7 to the students. Ask them to work in groups to carry out the described actions.



4. Ask the students to use to the process of Action 3 to find x and y such that

$$8x - 3y = 2,$$
$$-2x - 6y = 8.$$

Comments

3. It will be helpful if the students work on graph paper.

Note that the transformation **T** in the activity sheet is that of Action 1. As pointed out in Comment 3 of Activity 6, $|\mathbf{T}| = 24$.

Master 2, attached, shows the significant sketches. The top panel of the master shows the location of $\mathbf{T}(x, y)$. Its location relative to the coordinate system determined by $\mathbf{T}(1, 0)$ and $\mathbf{T}(0, 1)$ indicates that (x, y) is in the quadrant of the **I**-grid in which x-coordinates are negative and y-coordinates are positive.

The middle panel shows parallelograms $\mathcal{P}$ and $\mathbf{T}(\mathcal{P})$. Parallelogram $\mathcal{P}$ has a base of 1 and altitude $-x$ (since x is negative). Hence its area is $-x$. As determined from its vertices, the area of $\mathbf{T}(\mathcal{P})$ is 6. But, since $|\mathbf{T}|$ is 24, the area of $\mathbf{T}(\mathcal{P})$ is also 24 times the area of $\mathcal{P}$, or $-24x$. Hence, $-24x = 6$ and $x = -1/4$.

In the bottom panel, one vertex of parallelogram Q is $(1, 0)$ rather than $(0, 1)$. Its area is y, so the area of $\mathbf{T}(Q)$ is $24y$. On, the other hand, as determined by its vertices, the area of $\mathbf{T}(Q)$ is 36. Hence $24y = 36$ and $y = 3/2$.

4. Written in matrix form, the set of equations is $\begin{pmatrix} 8 & -3 \\ -2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

As shown on Master 3, attached, $-54x = 12$ and $-54y = 68$. Hence $x = -2/9$ and $y = -34/27$.

Note that the scales of the figures on the left and right sides of Master 3 are different.

Actions

5.(Optional.) Ask the students to use the procedures used in Actions 3 and 4 to obtain formulas for x and y if

$$ax + by = h,$$
$$cx + dy = k.$$

Discuss.

Comments

5. Written in matrix form, the set of equations is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix}$.

On Master 4, attached, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is designated as **T**. The master shows the derivation of a formula for x for all possible signs, positive or negative, of $|\mathbf{T}|$ and x. In every case, $x = \dfrac{hd - bk}{|\mathbf{T}|}$.

Changing the parallelogram $\mathcal{P}$ by replacing the vertex at (0, 1) with a vertex at (1, 0), one obtains $y = \dfrac{ak - ch}{|\mathbf{T}|}$.

Note that $hd - bk$ is the determinant of the matrix $\begin{pmatrix} h & b \\ k & d \end{pmatrix}$ and $ak - ch$ is the determinant of the matrix $\begin{pmatrix} a & h \\ c & k \end{pmatrix}$. This gives rise to *Cramer's Rule*, named after Gabriel Cramer (1704–1752): The value of each variable is the determinant of the matrix obtained by replacing the coefficients of the variable being evaluated by the constant terms, divided by the determinant of the matrix of coefficients.

Note that the given solution assumes $|\mathbf{T}| \neq 0$, that is $ad - bc \neq 0$. If $ad - bc = 0$, the system either has no solution, e.g.,

$$3x + 2y = 4,$$
$$9x + 6y = 10,$$

or the system has infinitely many solutions, e.g.,

$$3x + 2y = 4,$$
$$9x + 6y = 12.$$

Name _____

Suppose $\mathbf{T} = \begin{pmatrix} 4 & 2 \\ -2 & 5 \end{pmatrix}$ and let (x, y) be the point whose image is (2, 8), that is, $\mathbf{T}(x, y) = (2, 8)$. Let $\mathcal{P}$ be the parallelogram with successive vertices at (x, y), (0, 0), and (0, 1).

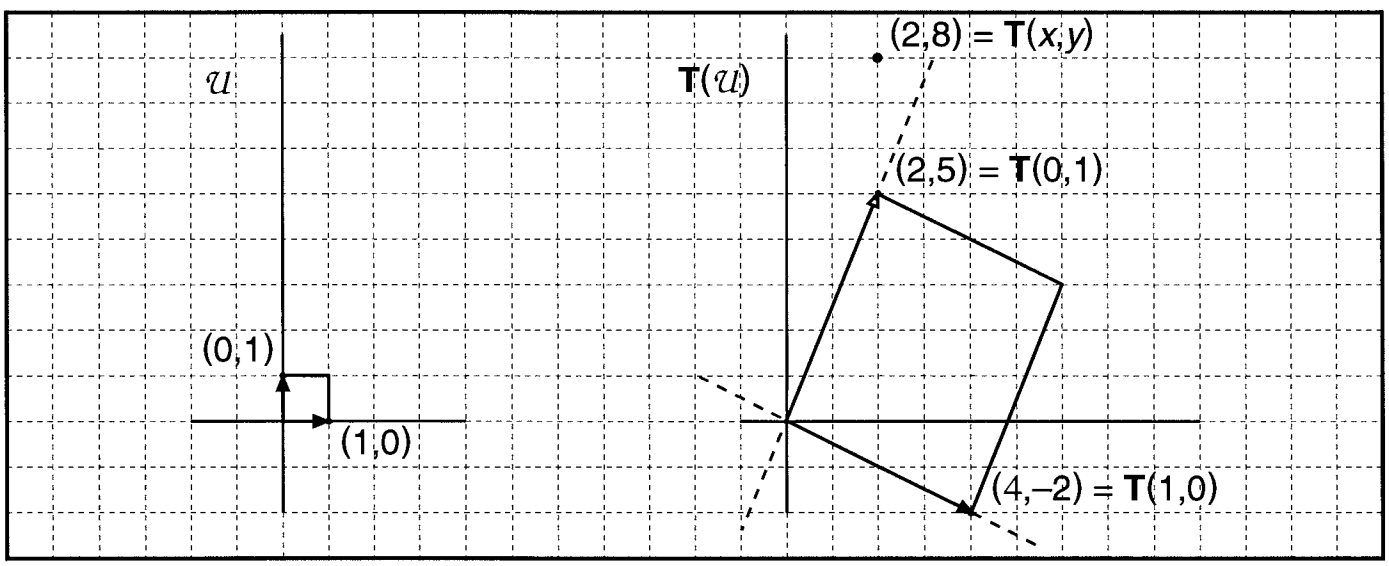
1. Determine the quadrant of the x,y plane in which (x, y) lies.

2. Express the area of $\mathcal{P}$ in terms of x.

3. Sketch $\mathbf{T}(\mathcal{P})$ and find its area.

4. Since $|\mathbf{T}| = 24$, the area of $\mathbf{T}(\mathcal{P})$ is 24 times the area of $\mathcal{P}$. Use this relationship and the results of 2 and 3 to find the value of x.

5. Adapt the above procedure to find the value of y.



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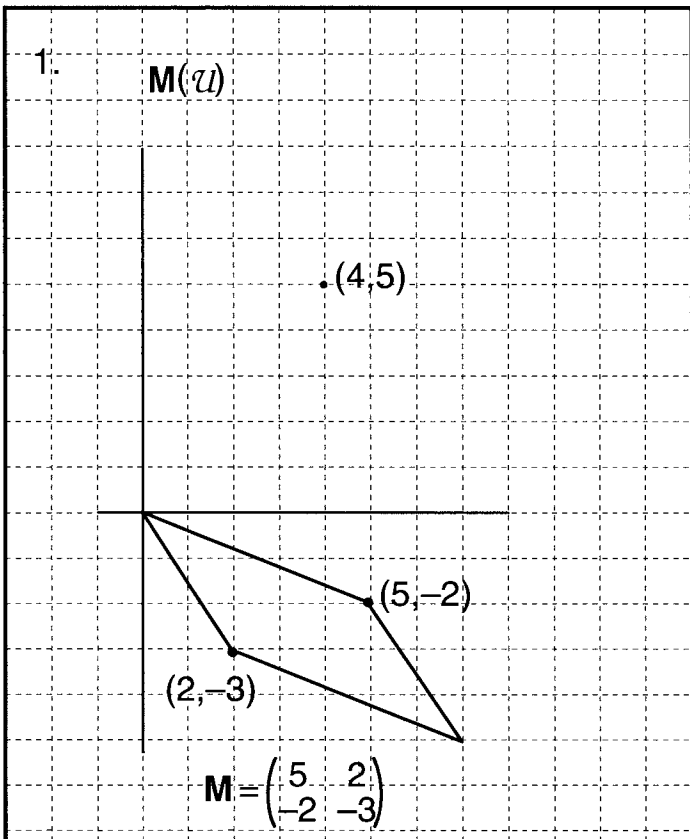
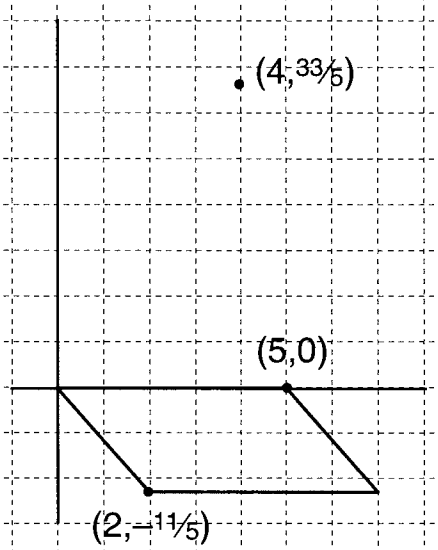
1.



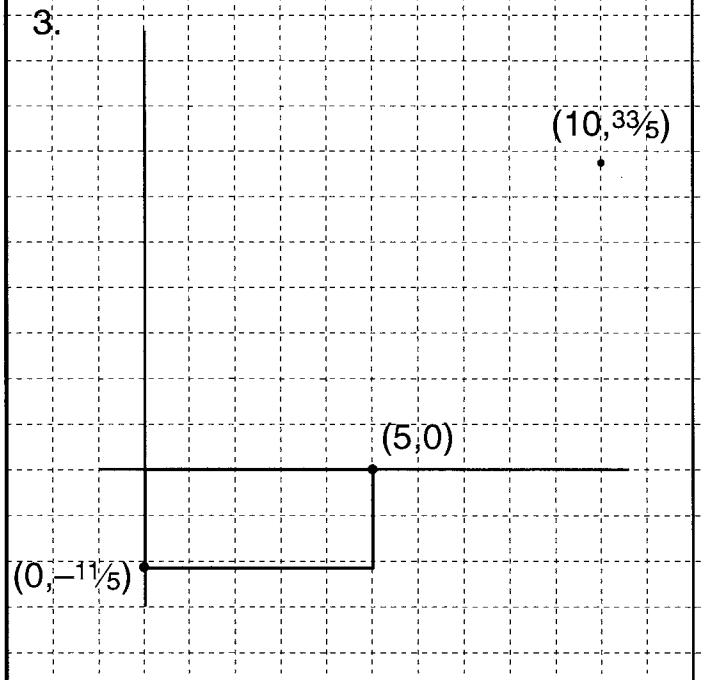
$$\mathbf{M} = \begin{pmatrix} 5 & 2 \\ -2 & -3 \end{pmatrix}$$

2.



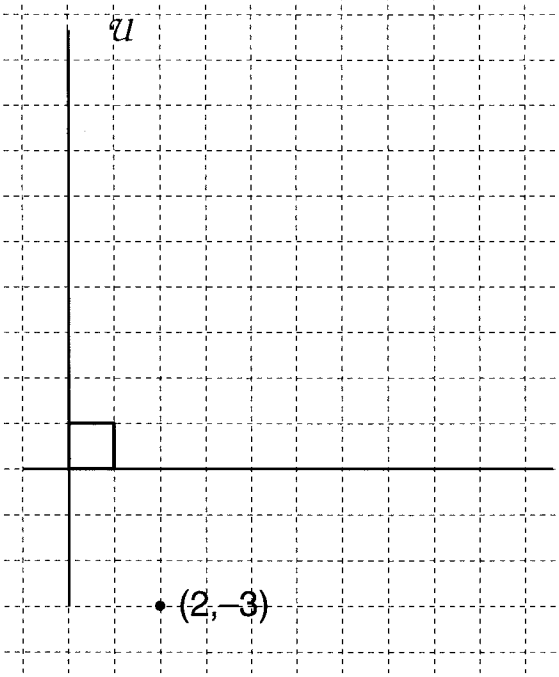
This shear moves (5,–2) up to (5,0), and thus moves (2,–3) up $\frac{2}{5}$ x 2 units to (2,$-\frac{11}{5}$) and moves (4,5) up $\frac{4}{5}$ x 2 units to (4,$\frac{33}{5}$).

3.



This shear moves (2,$-\frac{11}{5}$) to the left 2 units to (0,$-\frac{11}{5}$)) and moves (4,$\frac{33}{5}$) to the right 3 x 2 units to (10,$\frac{33}{5}$).

4.



This scaling divides *x* coordinates by 5 and *y* coordinates by $-\frac{11}{5}$. In particular, it moves (10,$\frac{33}{5}$) to (2, –3).

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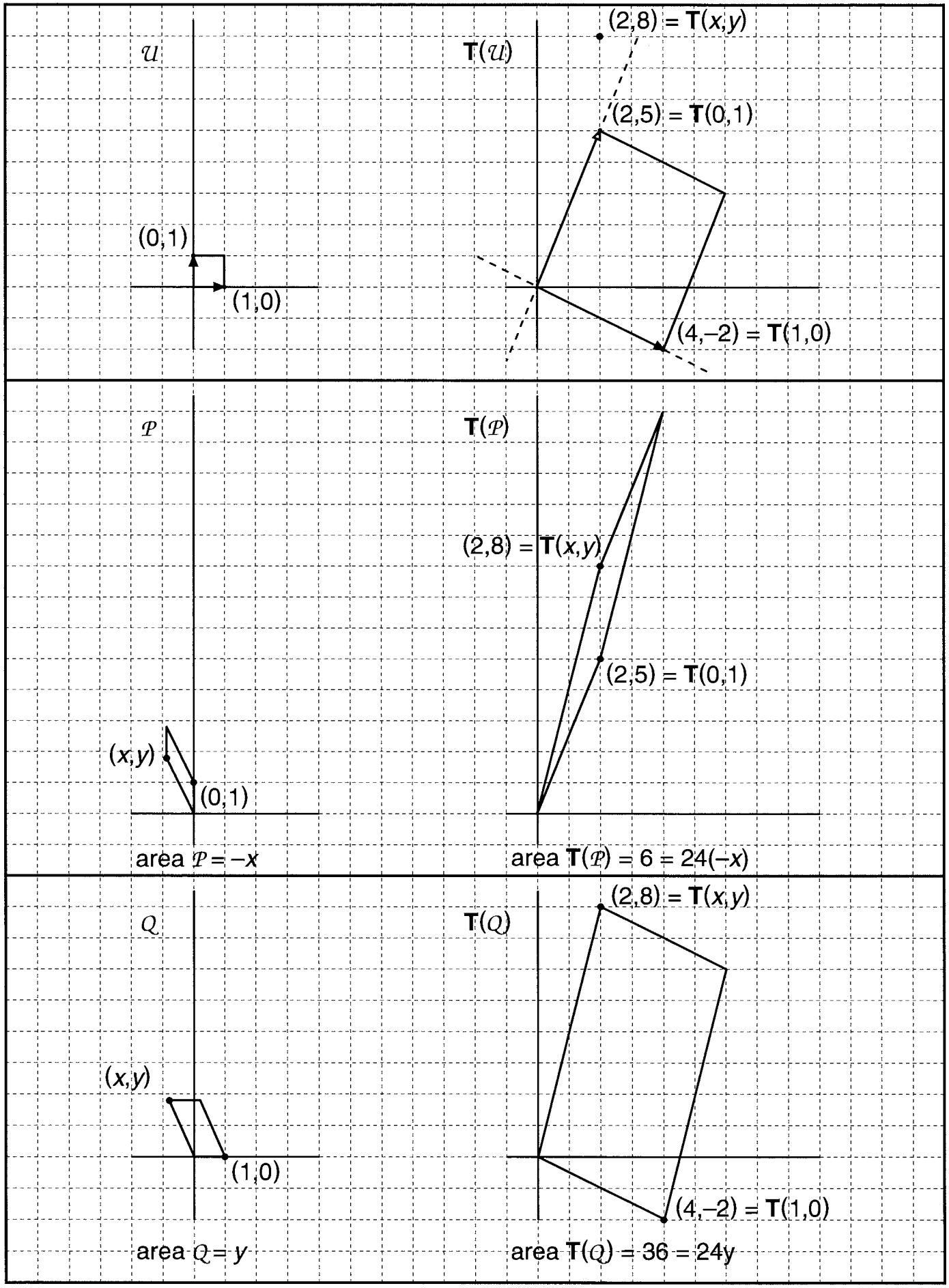

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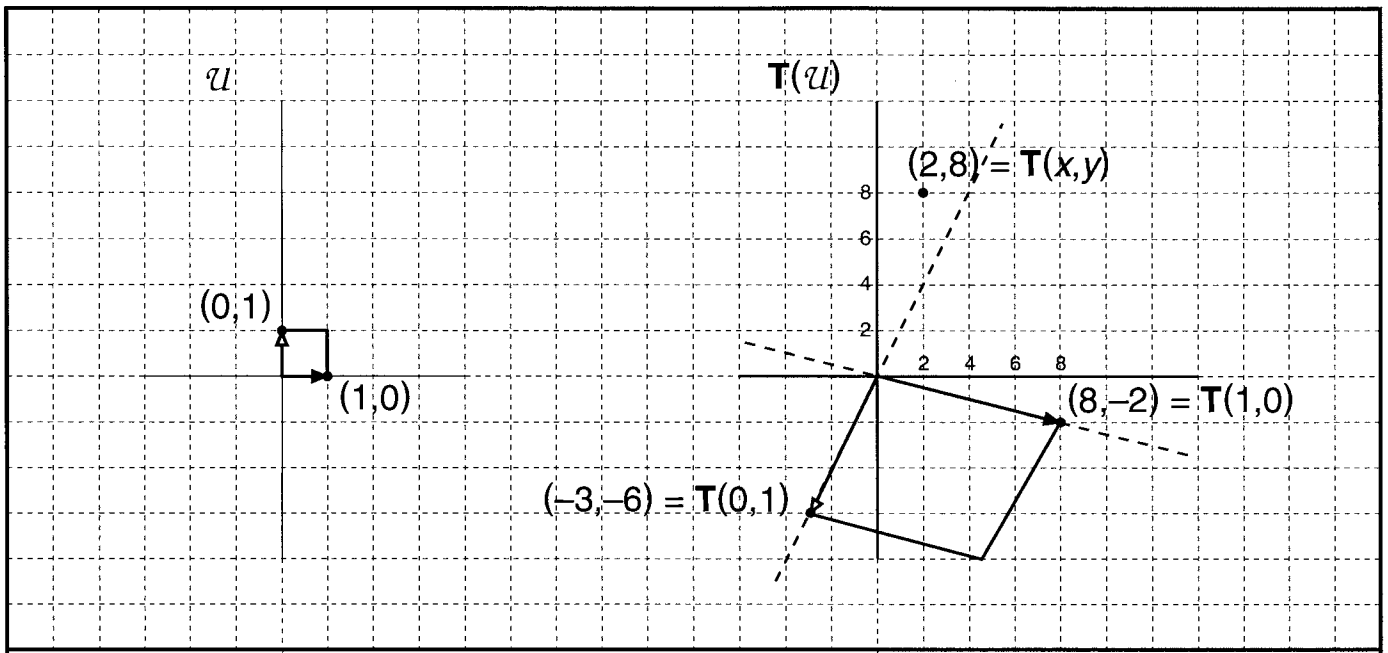

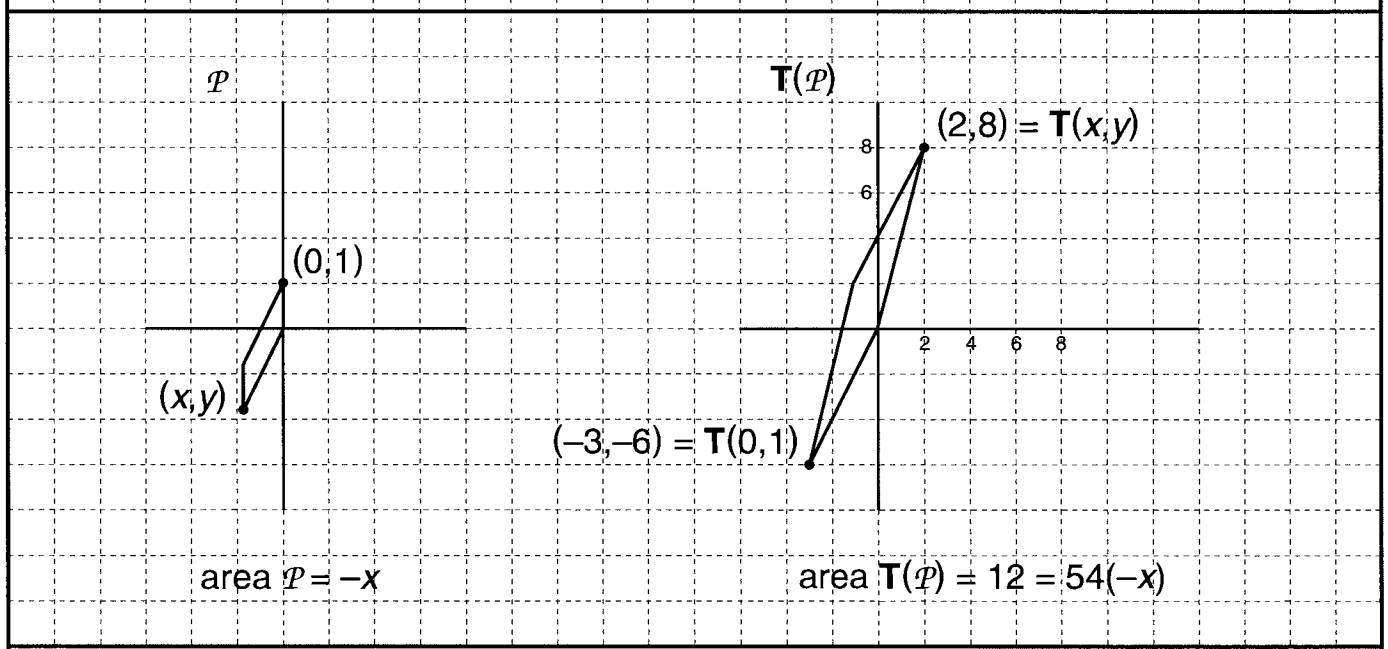

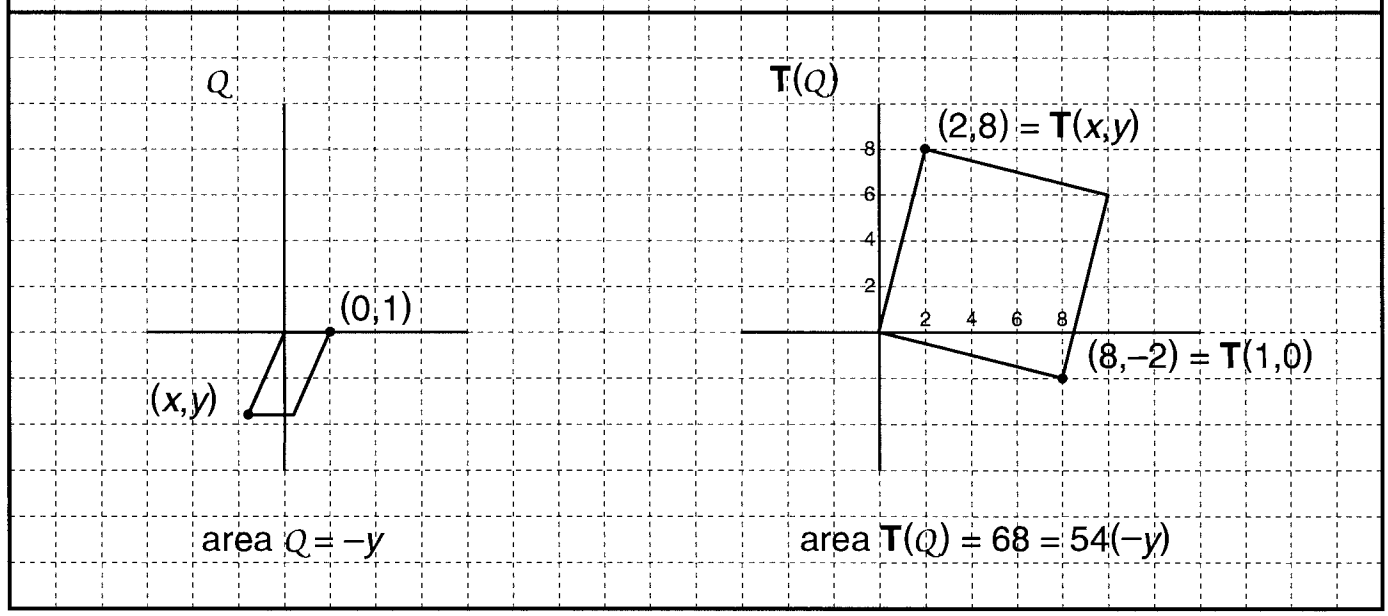

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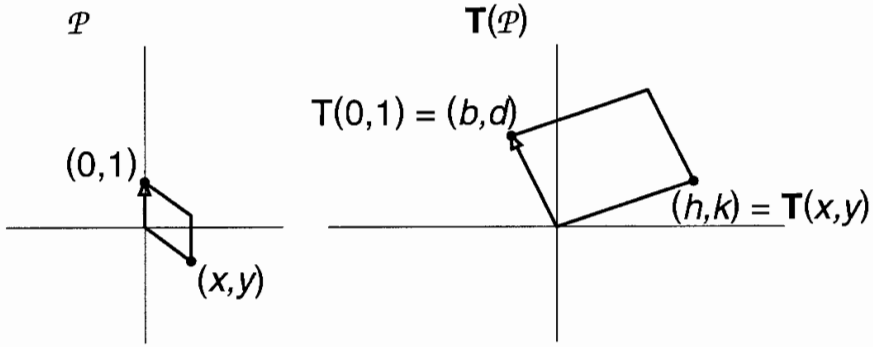
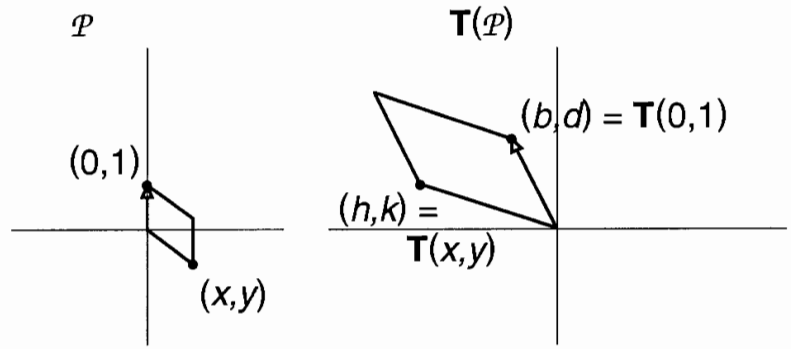
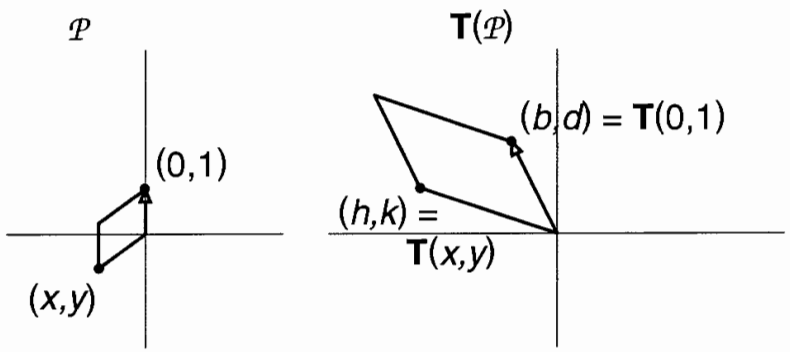
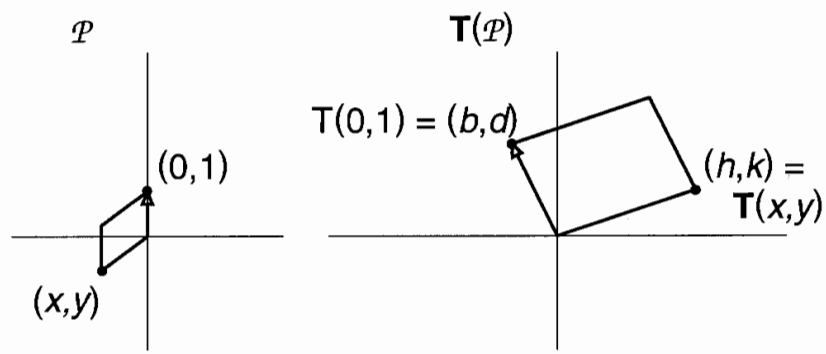
$\lvert\mathbf{T}\rvert > 0$; i.e., $\mathbf{T}$ does not change orientation	$\lvert\mathbf{T}\rvert < 0$; i.e., $\mathbf{T}$ changes orientation
Case 1. $x > 0$  area $\mathcal{P} = x$ area $\mathbf{T}(\mathcal{P}) = hd - kb = x\lvert\mathbf{T}\rvert$ $x = \frac{hd-kb}{\lvert\mathbf{T}\rvert}$	Case 1. $x > 0$  area $\mathcal{P} = x$ area $\mathbf{T}(\mathcal{P}) = bk - dh = x(-\lvert\mathbf{T}\rvert)$ $x = \frac{hd-bk}{\lvert\mathbf{T}\rvert}$
Case 2. $x < 0$  area $\mathcal{P} = -x$ area $\mathbf{T}(\mathcal{P}) = bk - dh = -x\lvert\mathbf{T}\rvert$ $x = \frac{hd-bk}{\lvert\mathbf{T}\rvert}$	Case 2. $x < 0$  area $\mathcal{P} = -x$ area $\mathbf{T}(\mathcal{P}) = hd - kb = (-x)(-\lvert\mathbf{T}\rvert)$ $x = \frac{hd-bk}{\lvert\mathbf{T}\rvert}$

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Invariance

O V E R V I E W

An *invariant* set of points is defined and examples given. *Eigenvectors* and *eigenvalues* are introduced.

Prerequisite Activity

Unit XIV, Activity 5, *Scaling and Shearing*.

Materials

Copies of Masters and grid paper as indicated.

Actions

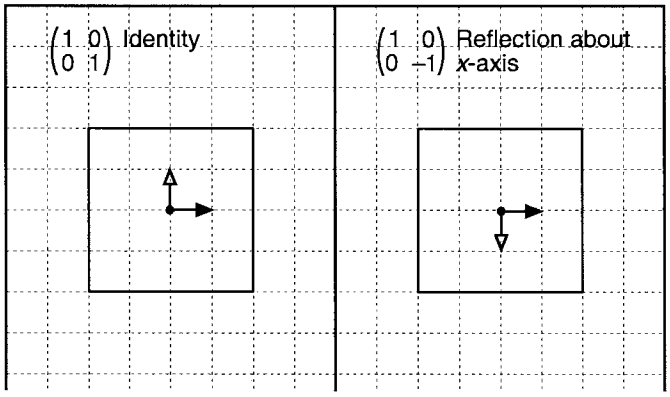
1. Distribute copies of Masters 1 and 2 to the students. On an overhead transparency of Master 1, in one of the spaces provided, write the matrix of a transformation which maps the given square onto itself. Show the transformed basis vectors and give a description, in words, of the effect of the transformation. On a transparency of Master 2, show the image of the given figure under this transformation.

Comments

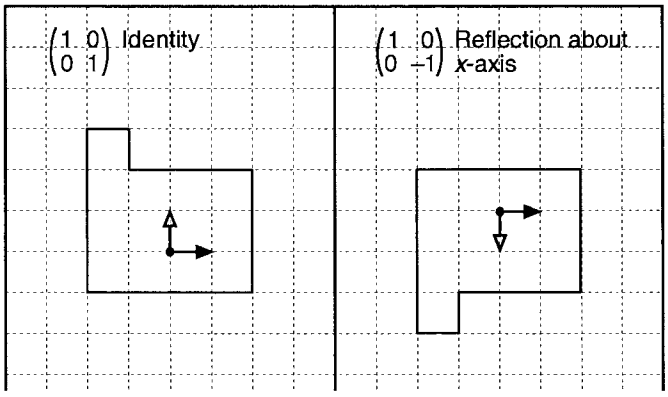
1. The transformation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a reflection about the x-axis. Its effect on the basis vectors and the figures on Masters 1 and 2 are shown below.

This transformation moves all points except those on the x-axis. However, the set of images of points on the given square on Master 1 form the same square. That is, the transformation maps the square onto itself.

The figure on Master 2 and its image under the above transformation are not the same figure. The effect of the transformation on a set of points is better portrayed by its action on this figure than its action on the square of Master 1.



From Master 1



From Master 2

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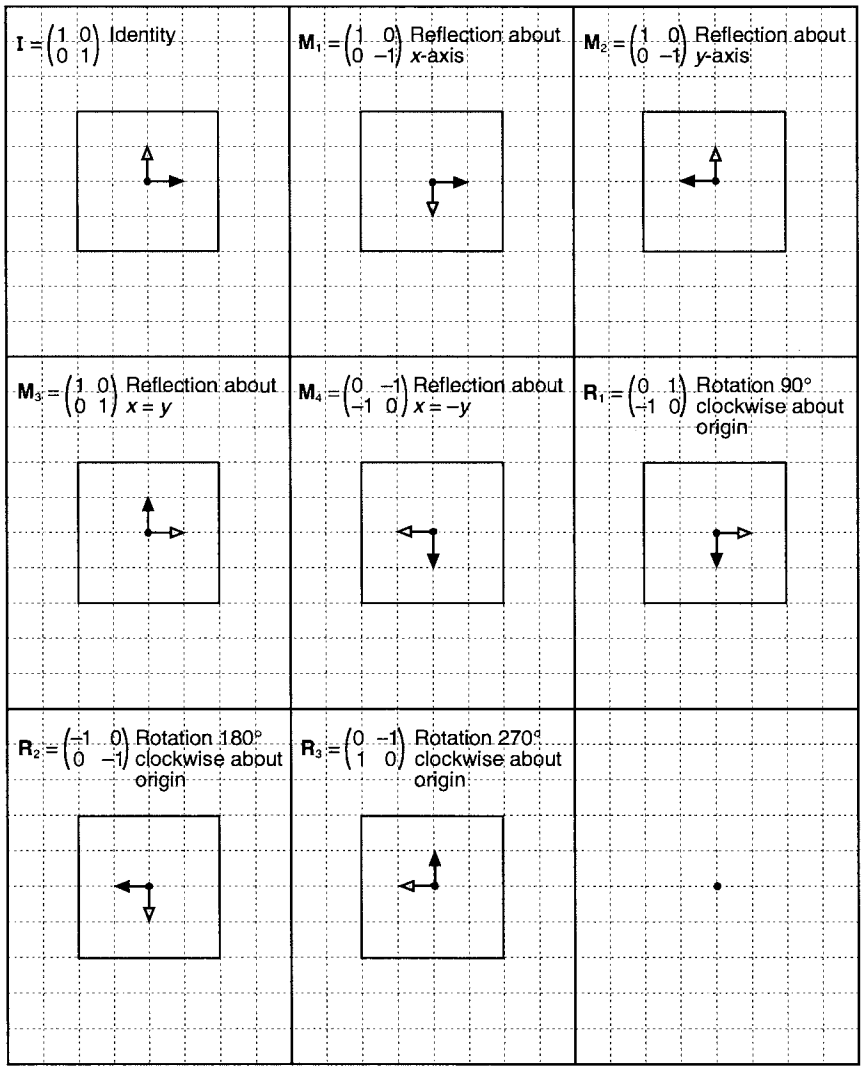
Actions

2. Ask the students to record in like fashion on Master 1 other transformations which map the given square onto itself. Have them record on Master 2 the effects of each of these transformations on the given figure.

3. (Optional.) Ask the students to form a "multiplication table" for the transformations found in Action 3.



Comments

2. There are 8 transformations which map the square of Master 1 onto itself. These are shown on Master 3, attached at the end of this unit. Master 4 shows the effect of these transformations on the figure of Master 2.

These 8 transformations are known as "symmetries of the square." They form a so-called "group" where the "group operation" is the composition of transformations. A "group table" is formed in the next action.

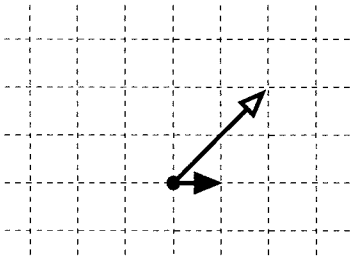
The students may be interested in finding the groups of symmetries of other figures, such as a non-square rectangle, a swastika, and various regular polygons. The transformations can be described in words. Writing the matrices of some of the rotations that occur may require a knowledge of trigonometry.

3. It is helpful in forming a table to assign letter names to the transformations. Shown on the left is one way of assigning letters. The table is constructed so the entry in the table in the row headed by a matrix, say **A**, and the column headed by a matrix, say **B**, is the matrix **AB**.

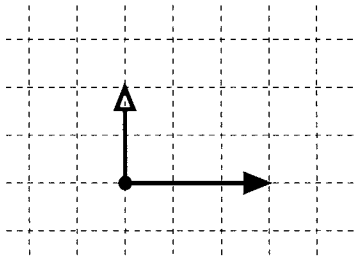
	I	R_1	R_2	R_3	M_1	M_2	M_3	M_4
I	I	R_1	R_2	R_3	M_1	M_2	M_3	M_4
R_1	R_1	R_2	R_3	I	M_4	M_3	M_1	M_2
R_2	R_2	R_3	I	R_1	M_2	M_1	M_4	M_3
R_3	R_3	I	R_1	R_2	M_3	M_4	M_2	M_1
M_1	M_1	M_3	M_2	M_4	I	R_2	R_1	R_3
M_2	M_2	M_4	M_1	M_3	R_2	I	R_3	R_1
M_3	M_3	M_2	M_4	M_1	R_3	R_1	I	R_2
M_4	M_4	M_1	M_3	M_2	R_1	R_3	R_2	I

Actions

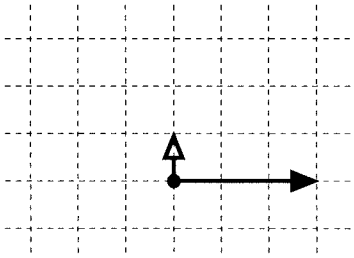
4. Define when a set of points $\mathcal{S}$ is said to be *invariant* under a transformation **T** and when it is said to be *fixed*. Ask the students to describe sets of points which are invariant under shears and scalings. Discuss.



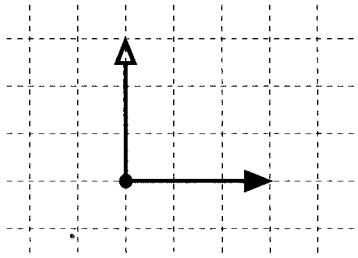
x-axis is fixed; lines parallel to *x*-axis are invariant



x- and *y*-axis are invariant



y-axis is fixed; lines perpendicular to *y*-axis are invariant



all lines through origin are invariant

5. Distribute copies of an **S**-grid (Master 5) to the students where $\mathbf{S} = \begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix}$. Tell them that **S** leaves two lines through the origin invariant. Ask the students to look for these lines. Discuss.

Comments

4. A set of points $\mathcal{S}$ is invariant under a transformation **T** if $\mathbf{T}(\mathcal{S}) = \mathcal{S}$, that is, the set $\mathcal{S}$ is transformed onto itself by the transformation **T**. The set $\mathcal{S}$ is said to be fixed if for every point $\mathcal{P}$ in $\mathcal{S}$, $\mathbf{T}(\mathcal{P}) = \mathcal{P}$, that is, **T** acts like the identity transformation on points of $\mathcal{S}$. Note that if the set $\mathcal{S}$ is invariant under **T**, the points of $\mathcal{S}$ may be shuffled about by **T**, but if $\mathcal{S}$ is fixed under **T**, all points of $\mathcal{S}$ are unchanged by **T**.

The square on Master 3 is invariant under all of the transformations shown there, however only the identity transformation leaves it fixed.

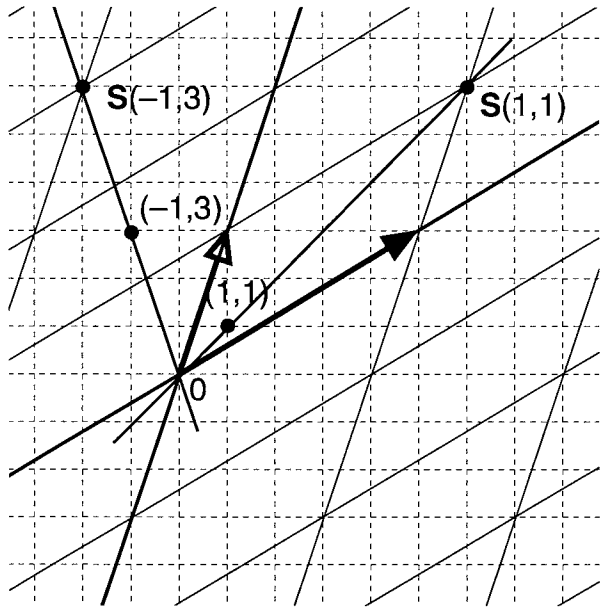
For a shear parallel to the *x*-axis, all lines parallel to the *x*-axis are invariant, but only the *x*-axis is fixed.

The *x* and *y* axes are invariant under a scaling. A scaling may only effect one axis (one of its non-zero entries may be 1); in which case the other axis is fixed and all lines perpendicular to it are invariant. For some scalings, e.g., $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, all lines through the origin are invariant.

5. Master 5, attached, shows an **S**-grid, with basis vectors $\mathbf{S}(1, 0) = (5, 3)$ and $\mathbf{S}(0, 1) = (1, 3)$, superimposed on an **I**-grid of dotted lines.

Continued next page.

Actions



6. (Optional) Define *eigenvector* and *eigenvalue*. Distribute copies of Master 6 to the students. Describe the grid shown on the master and then ask the students to locate on the grid the following points and the images of these points under **S**:

$$\mathcal{A} = (-2, 6),\ \mathcal{B} = (2, 2),\ \mathcal{C} = (5, -11),\ \mathcal{D} = (-2, 4).$$

Comments

5. *Continued.*
A line through the origin will be invariant under **S** if, for some point (x, y) on the line, $\mathbf{S}(x, y)$ is also on the line, that is (x, y), $\mathbf{S}(x, y)$ and the origin are collinear. By examining the various vertices of the **S**-grid, the students may discover that $\mathbf{S}(1, 1)$ and $(1, 1)$ lie on the same line through the origin, as do $\mathbf{S}(-1, 3)$ and $(-1, 3)$. (See the figure.) The equations of these lines are $y = x$ and $y = -3x$, respectively.

Note that $\mathbf{S}(1, 1)$ is six times the distance from the origin that $(1, 1)$ is. Hence, **S** maps $y = x$ onto itself but increases distances along the line by a factor of 6. **S** increases distances along $y = -3x$ by a factor of 2.

The invariant lines may be found algebraically as shown in the appendix at the end of this unit.

6. An *eigenvector*, (also called a *characteristic vector)* of a transformation **T** is a vector which lies on a line that is invariant under **T**, thus an eigenvector is a vector such that it and all of its scalar multiples constitute a line invariant under **T**. The factor by which distances on this line are changed by **T** is called an *eigenvalue* (also called a *characteristic value.*) From Action 5, we see that $(1, 1)$ and $(-1, 3)$ are eigenvectors of **S** with eigenvalues 6 and 2, respectively.

Master 6 is a grid determined by the eigenvectors $(1, 1)$ and $(-1, 3)$ of **S**. We will refer to this grid as the **E**-grid. In specifying **E**-grid coordinates, the coordinate associated with the closed-arrow vector will be given first.

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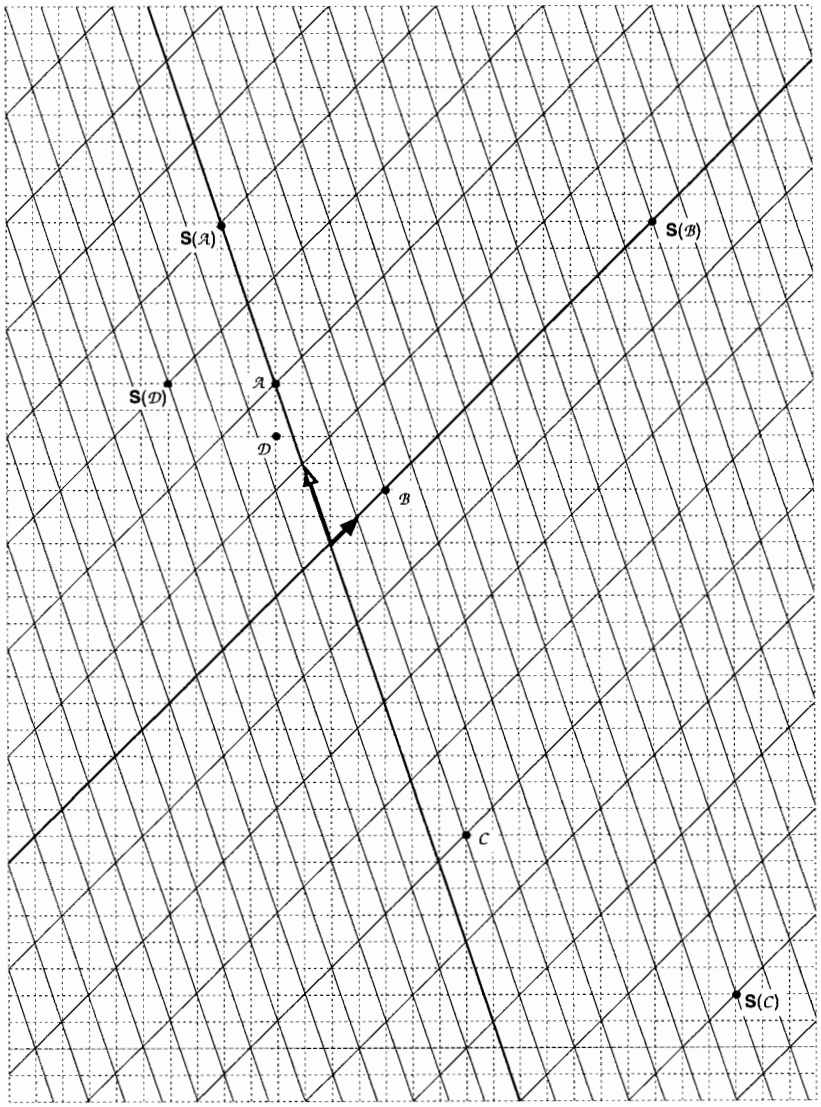
Actions



Comments

6. *Continued.*
Point $\mathcal{A}$ lies on the line containing eigenvector $(-1, 3)$. Hence, **S** maps $\mathcal{A}$ onto a point on this line twice as far from the origin. Thus, $\mathbf{S}(\mathcal{A}) = (-4, 6)$. Point $\mathcal{B}$ lies on the line containing eigenvector $(1, 1)$. Hence, **S** maps $\mathcal{B}$ onto a point on this line six times as far from the origin. Thus, $\mathbf{S}(\mathcal{B}) = (12, 12)$.

The **E**-grid coordinates of C are $\mathbf{E}(1, -4)$. Since **S** multiplies the first **E**-grid coordinate by 6 and the second by 2, $\mathbf{S}(C) = \mathbf{E}(6, -8) = (14, -18)$.

Since $\mathcal{D} = (-2, 4) = \mathbf{E}(-\frac{1}{2}, \frac{3}{2})$, $\mathbf{S}(\mathcal{D}) = \mathbf{E}(-6(\frac{1}{2}), 2(\frac{3}{2})) = \mathbf{E}(-3, 3) = (-6, 6)$.

Students may verify the location of the images by locating them on an **S**-grid (Master 5).

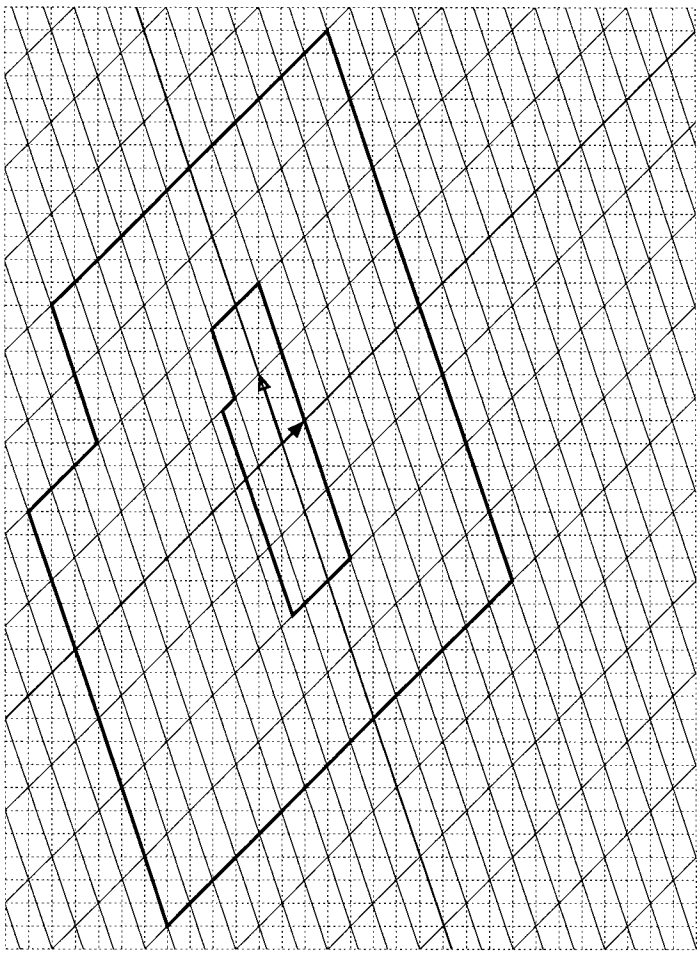
Actions

7. (Optional). Point out to the students that the linear transformation **S** can be thought of as a "scaling" in the direction of its eigenvectors, increasing distances in the direction of (1, 1) by a factor of 6 and distances in the direction of (–1, 3) by a factor of 2. Distribute a copy of Master 7 to the students and ask them to enlarge the given figure by the aforementioned scaling factors. Then ask them to draw the **S**-image of this figure on a sheet of **S**-grid paper.

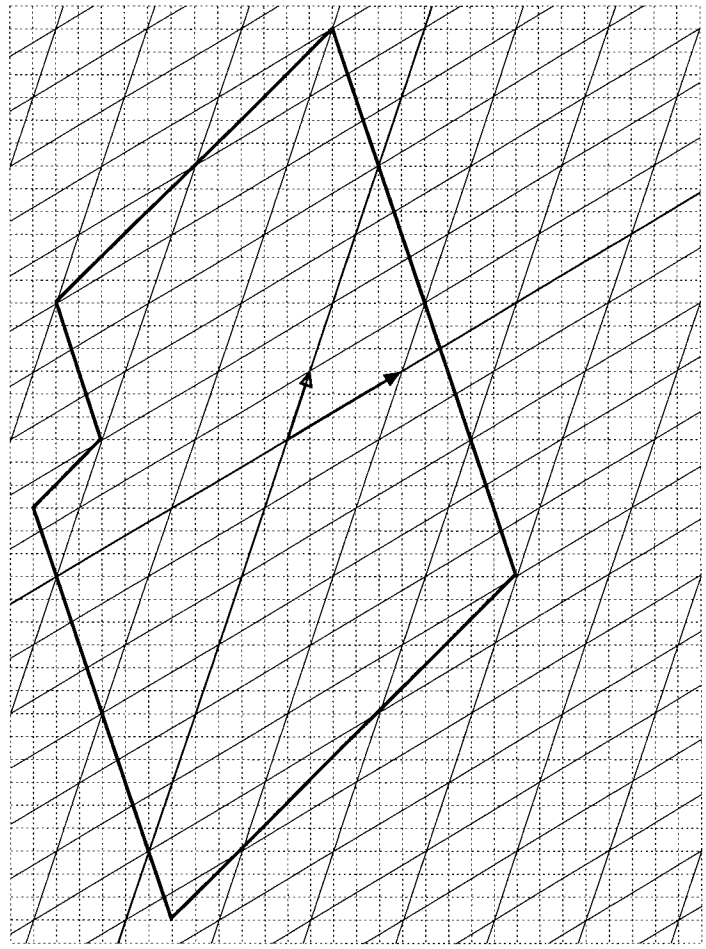
Comments

7. Master 8 shows the original figure and the result of enlarging it six times in the direction of the closed-arrow eigenvector and two times in the direction of the open-ended eigenvector.

Master 9 shows the image of the figure drawn on an **S**-grid. The **S** image was determined by finding the **I**-grid coordinates of the vertices of the figure and transforming them to **S**-grid coordinates. For example, the uppermost vertex of the figure is at (–1, 7) on an **I**-grid. The **S** image of this vertex is at **S**(–1, 7).



Master 8



Master 9

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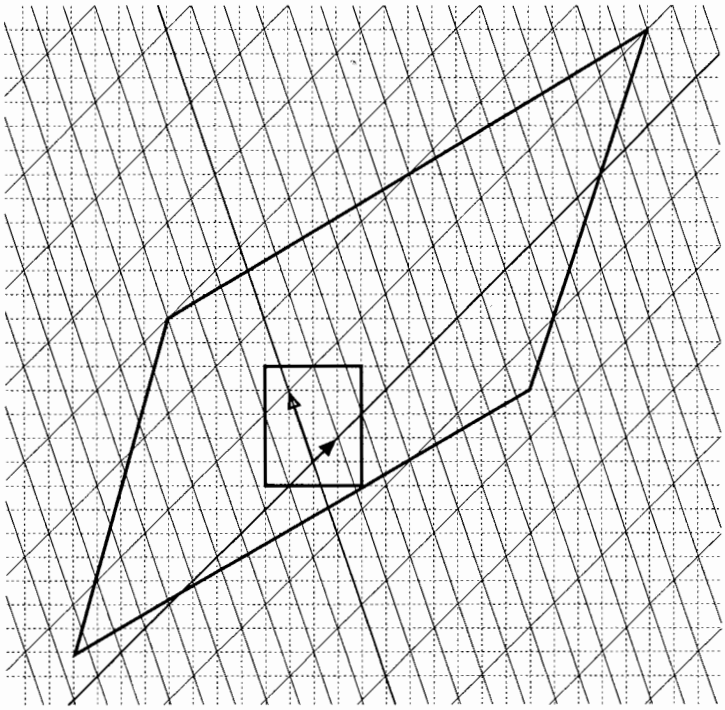
Actions



Comments

7. *Continued.*
Drawing the **E**-grid scaling of the figure is simplified by the figure having sides which are parallel to the directions of the eigenvectors. The students may find it interesting to draw the scaling of a figure whose sides are not parallel to the eigenvectors. Shown here is a square which has been "magnified" by a factor of 6 in the direction of the first eigenvector and by a factor of 2 in the direction of the second.

APPENDIX

The vectors $\mathbf{S}(x, y)$ and (x, y) are collinear if they are scalar multiples of each other. This will be the case if there is a number k such that $\mathbf{S}(x, y) = k(x, y)$. Since $\mathbf{S}(x, y) = (5x + y, 3x + 3y)$ and $k(x, y) = (kx, ky)$, the vectors $\mathbf{S}(x, y)$ and $k(x, y)$ will be equal provided $5x + y = kx$ and $3x + 3y = ky$, or

(1) $y = x(k - 5)$ and (2) $3x = y(k - 3)$.

Since $(x, y) \neq (0, 0)$, not both x and y are zero. Suppose $x \neq 0$. Replacing y in (2) by its value from (1) one has $3x = x((k - 3)(k - 5)$, or since $x \neq 0$,

(3) $3 = (k - 3)(k - 5)$.

(If $y \neq 0$, replacing x in (2) by its value from (2) and then dividing by y will lead to the same result.)

Solving quadratic equation (3) for k, one finds $k = 6$ or $k = 2$. From (1) it follows that $y = x$ or $y = -3x$. Hence, if (x, y) is on the line $y = x$ or on the line $y = -3x$, **S** maps it onto a scalar multiple of itself. Thus these two lines are invariant under **S**.

Notice the values of k are the eigenvalues of **S**. Since the quadratic equation in k has two roots, **S** has two eigenvalues and there are exactly two lines through the origin which are invariant under **S**.

One may use this procedure to find the eigenvalues for any non-singular linear transformation **T**. The resulting quadratic in k will have at most two solutions and may only have one. The solutions may be complex numbers in which case **T** has no invariant lines through the origin. However, in this case, the transformation obtained by applying a reflection across an axis to **T** will have invariant lines.

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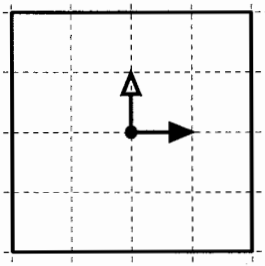
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity



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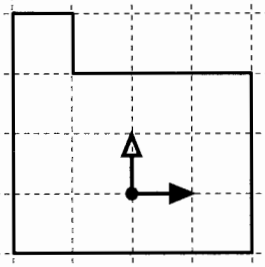
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity



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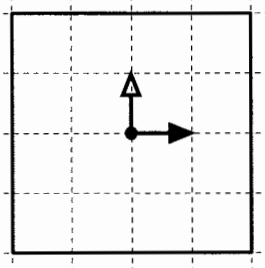
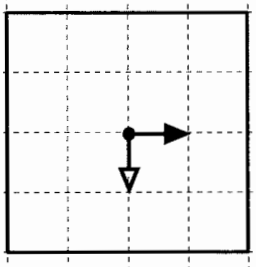
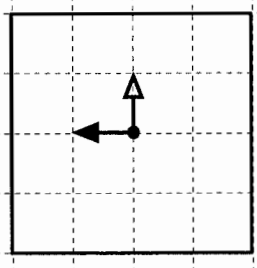
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity



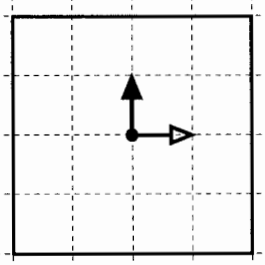
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection about x-axis



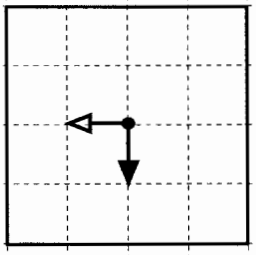
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection about y-axis



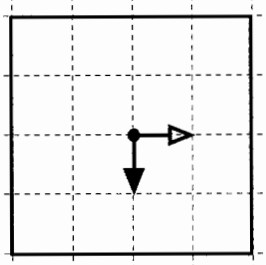
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Reflection about $x = y$



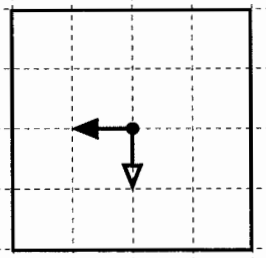
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ Reflection about $x = -y$



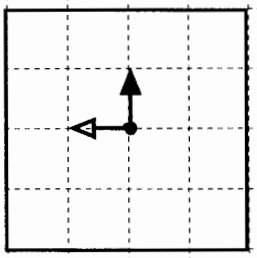
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Rotation 90° clockwise about origin



$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Rotation 180° clockwise about origin



$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Rotation 270° clockwise about origin



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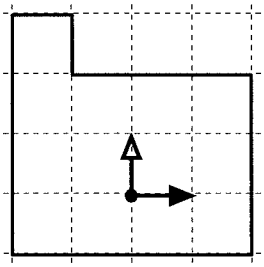
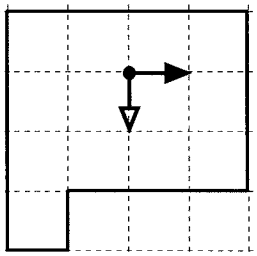
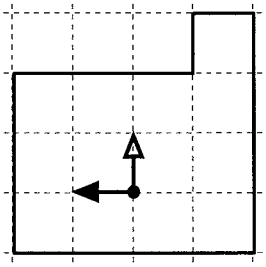
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity



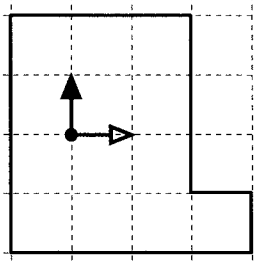
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection about x-axis



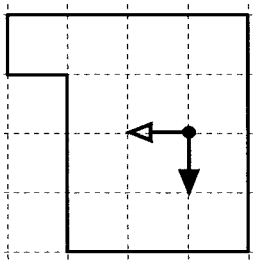
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection about y-axis



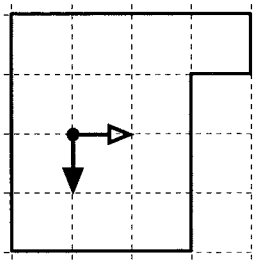
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Reflection about $x = y$



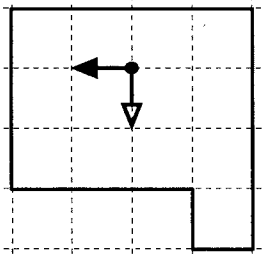
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ Reflection about $x = -y$



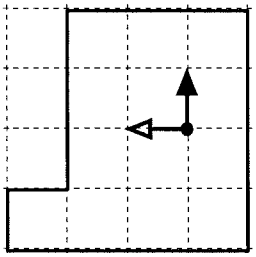
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Rotation 90° clockwise about origin



$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Rotation 180° clockwise about origin



$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Rotation 270° clockwise about origin



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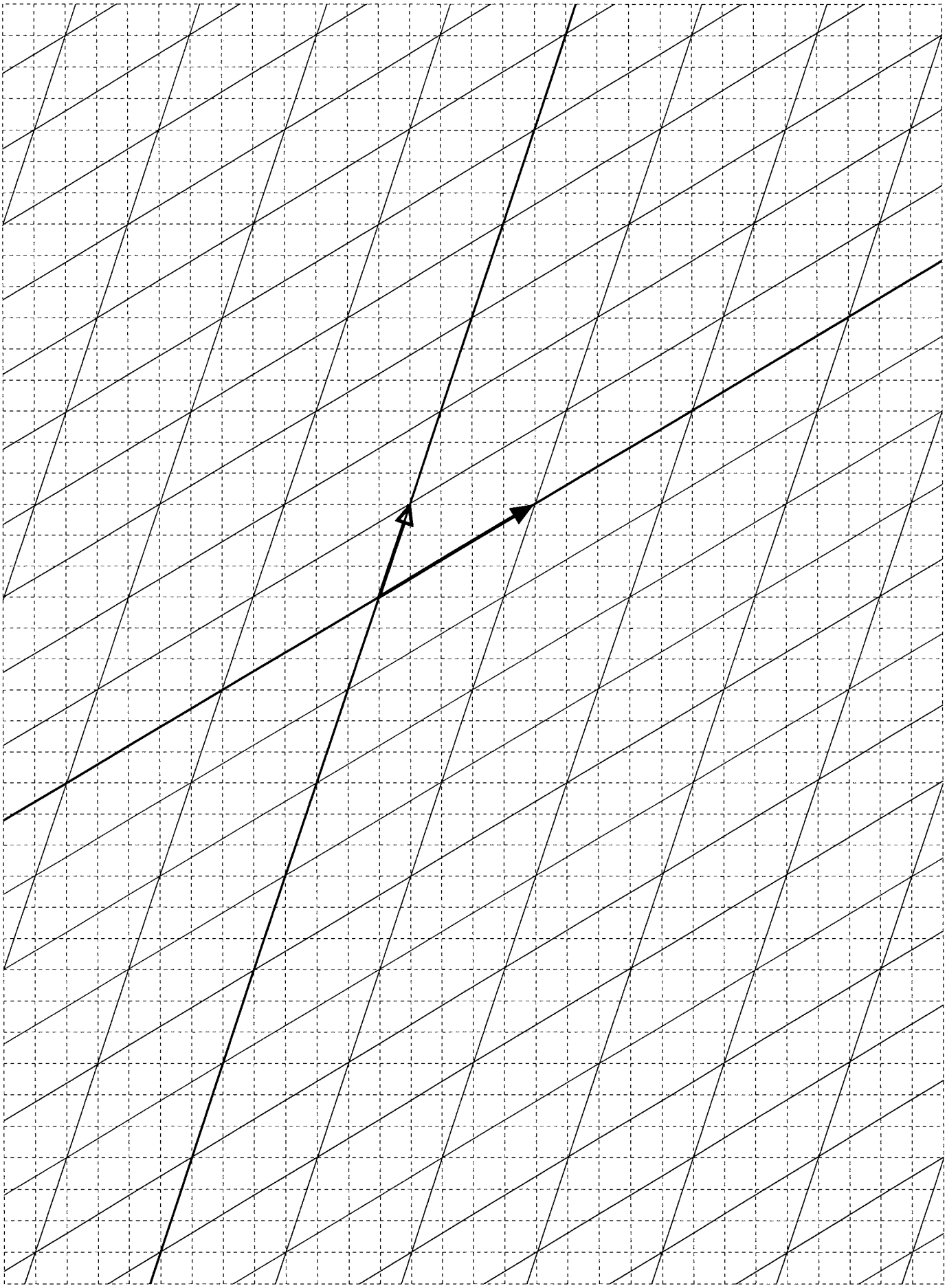
S-Grid



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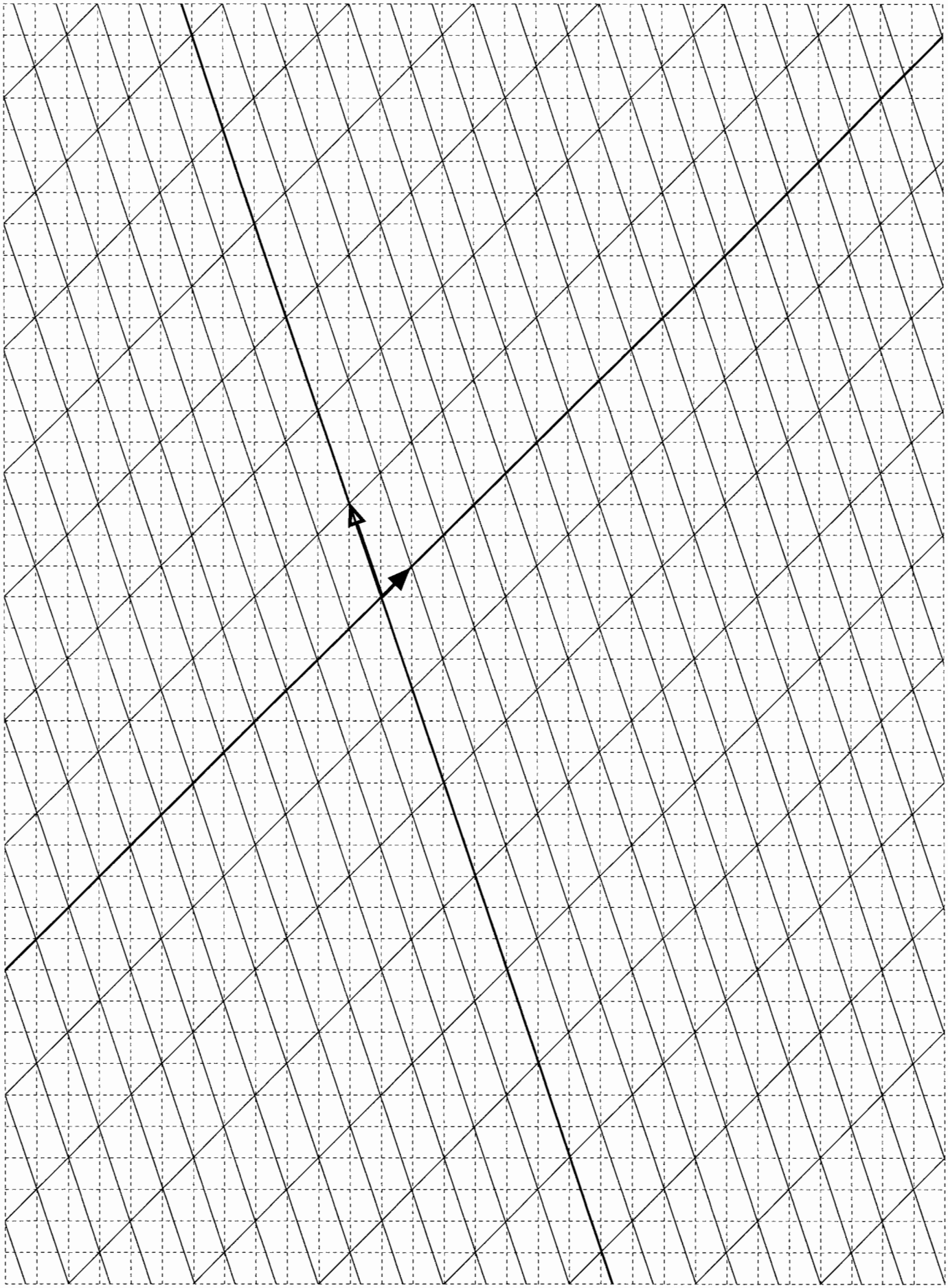
E-Grid



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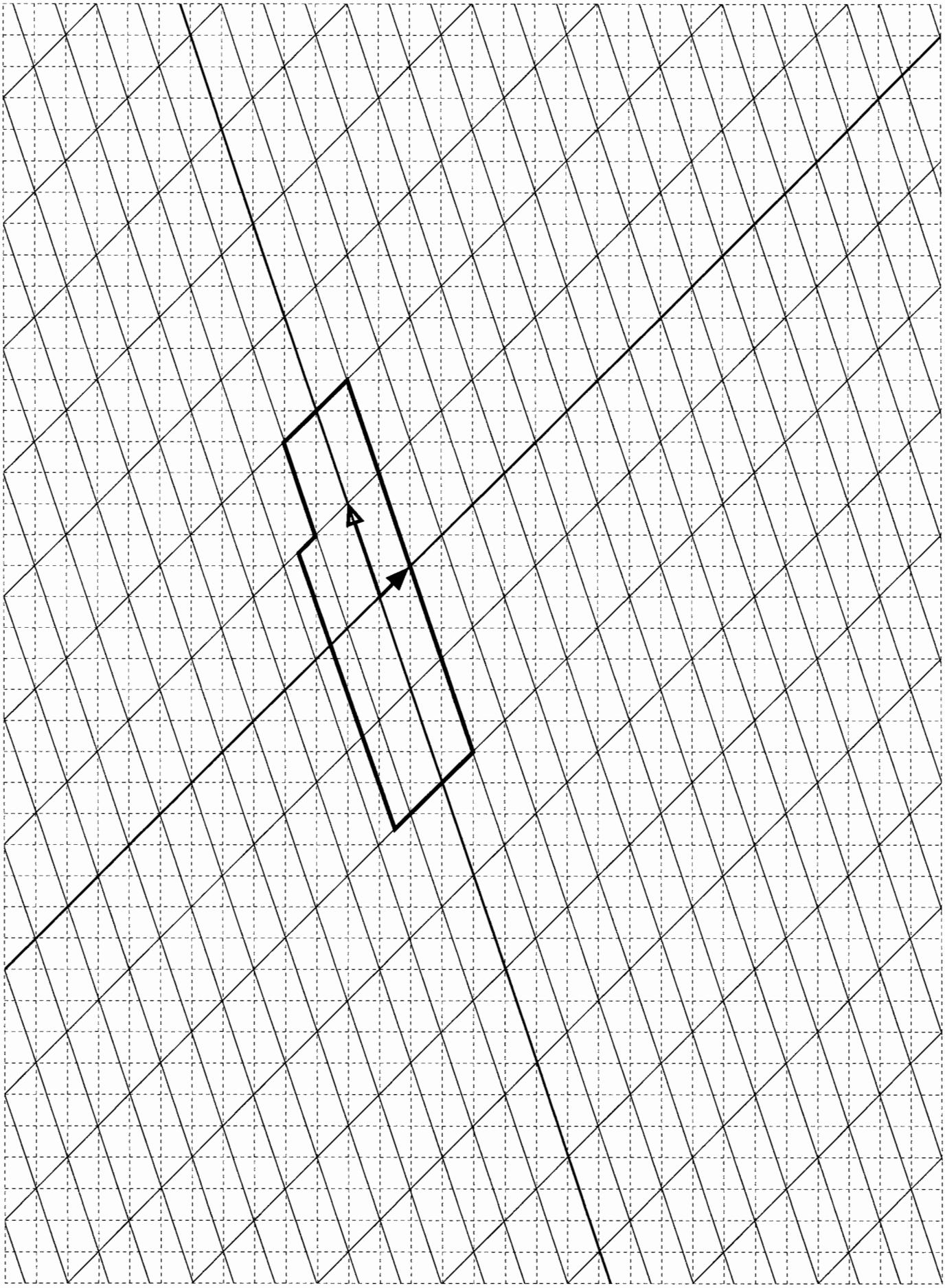
E-Grid



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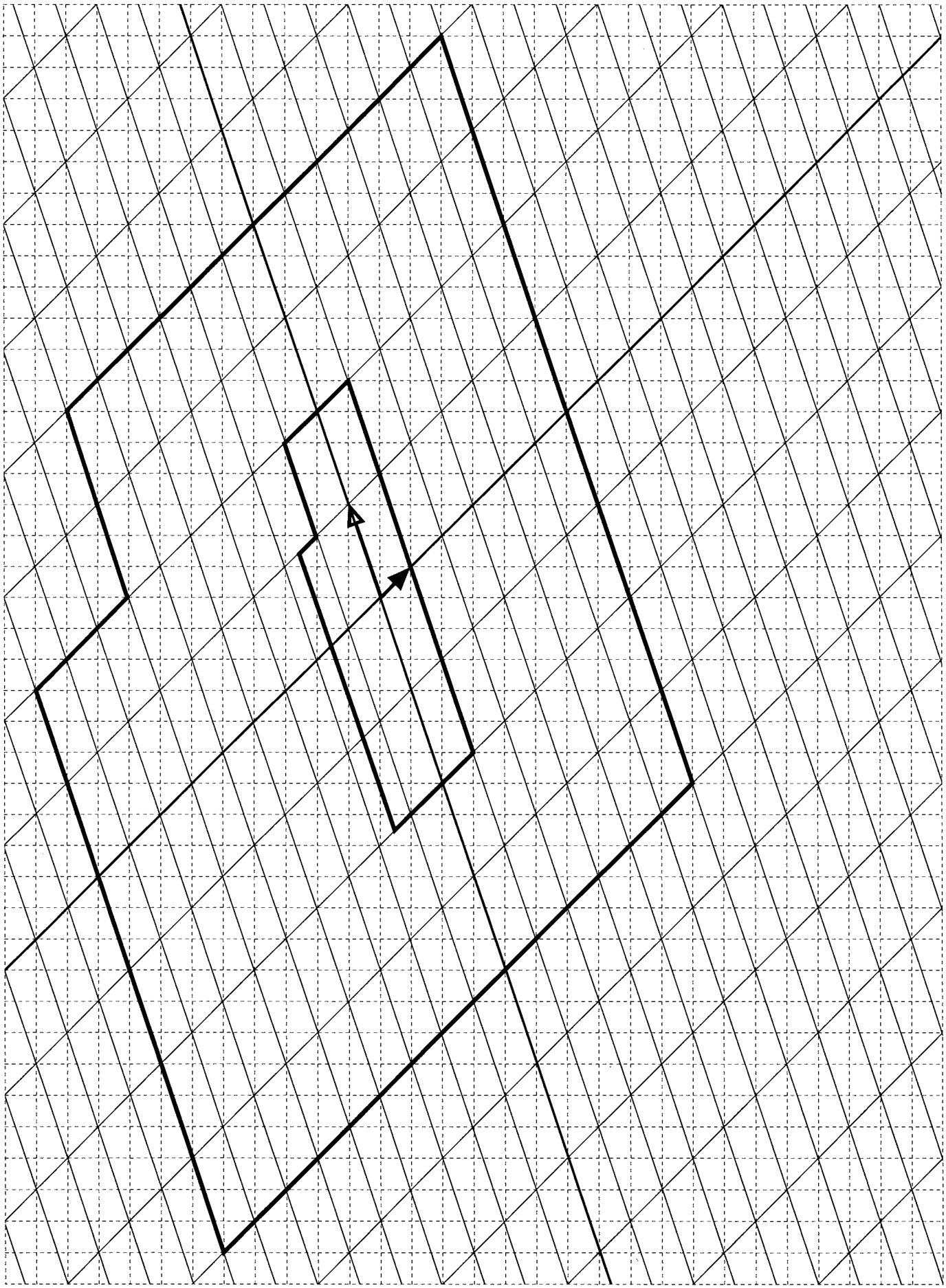
E-Grid



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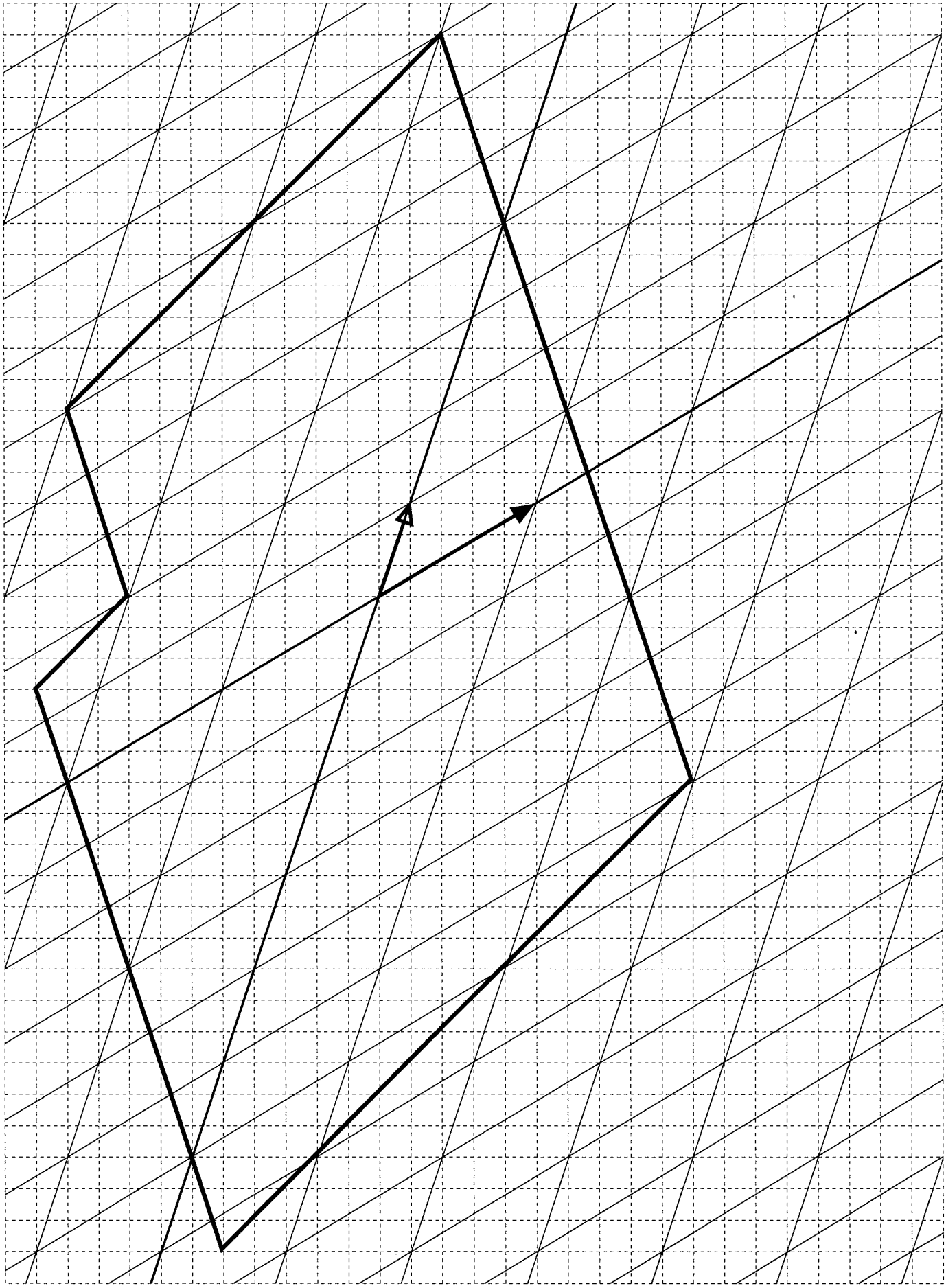
S-Grid



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Space Transformations

O V E R V I E W

Concepts introduced in previous activities are extended to three-dimensional settings.

Prerequisite Activity

Prerequisite: Unit XIV, Activity 6, *Inverses*.

Materials

Model of a parallelepiped (see Comment 1), grid paper, copies of Masters.

Actions

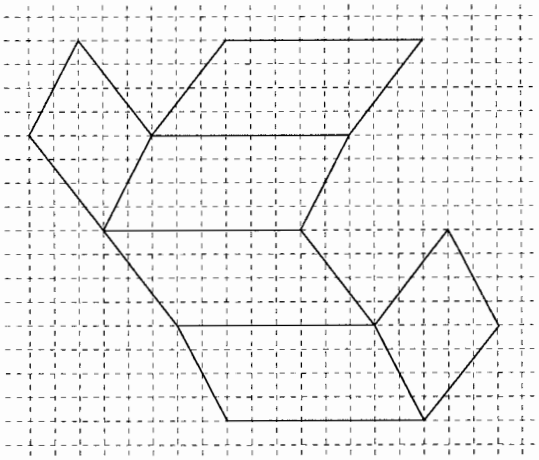
1. Distribute grid paper to the students. Show them a model of a parallelepiped that has no rectangular faces. Ask them to draw a net that will fold up to form a parallelepiped that has no rectangular faces.

2. Describe a three-dimensional "identity" grid.

Comments

1. Master 1, attached, shows a net that folds up to form a parallelepiped. A fairly sturdy model can be made by making a copy of the master on cardstock (a half of a file folder will do). After the net is cut out, straight folds can be made by first scoring the fold lines using a ruler and the point of a compass.

Shown below is an example of a net drawn on a grid.



2. Master 2, attached, is a perspective drawing of a three-dimensional "identity" grid. A transparency of the master can be shown to facilitate the discussion.

Continued next page.

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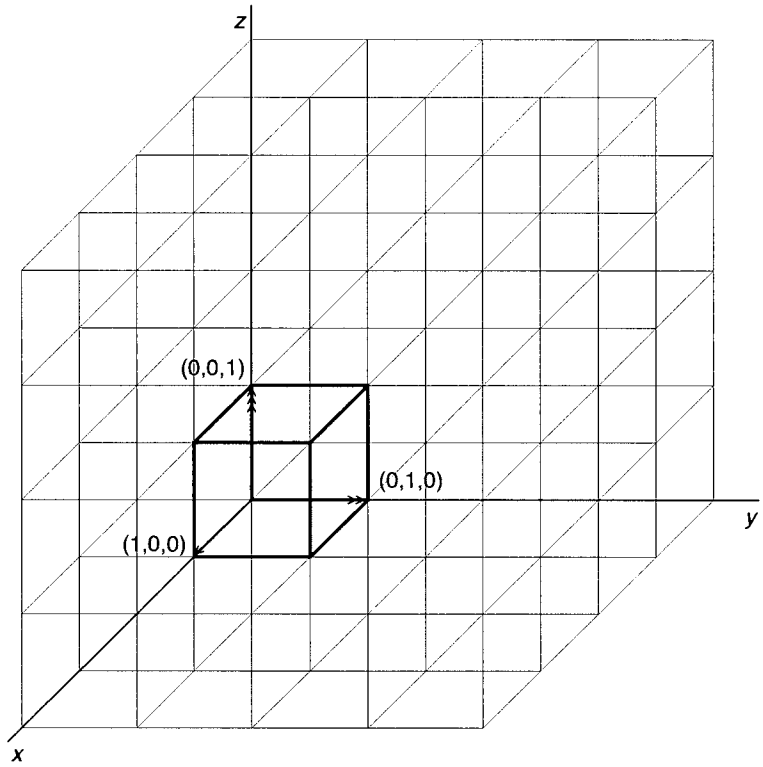
Actions

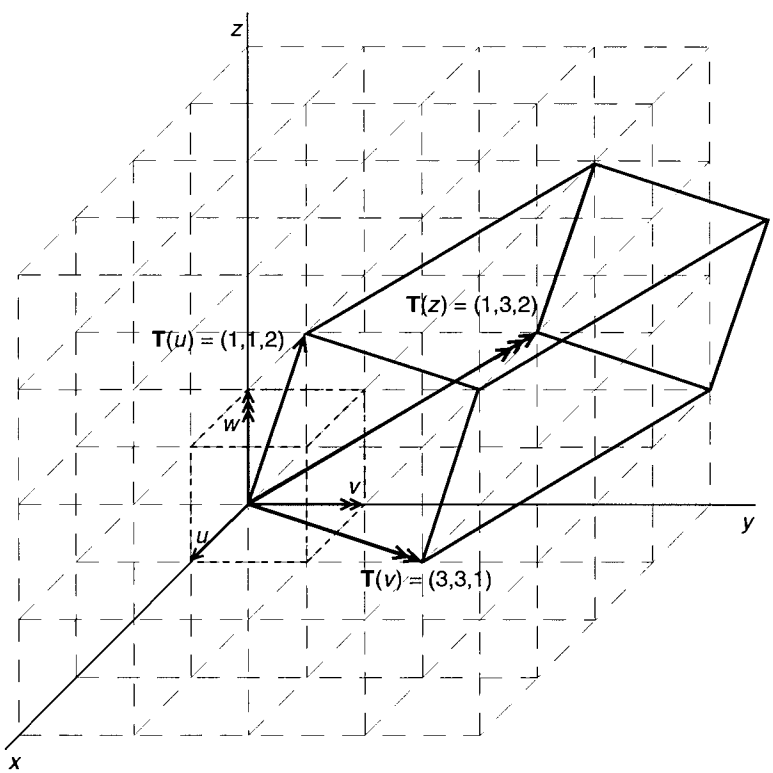


Comments

2. *Continued.*
The grid is generated by three mutually independent vectors of unit length emanating from a common point known as the *origin*. These vectors are called *unit* vectors; they are the *basis* vectors of the identity grid. The lines containing these vectors are referred to as axes and will be designated x, y, and z. The unit vectors along these axes will be designated u, v, and w, respectively, and their endpoints are given the coordinates (1, 0, 0), (0, 1, 0), and (0, 0, 1). As before, vectors emanating from the origin will be identified with the coordinates of their endpoints. Thus, $u = (1, 0, 0)$, $v = (0, 1, 0)$, and $w = (0, 0, 1)$.

The *unit* cube is the cube which has the unit vectors as edges. Its volume is 1. The "identity" grid may be thought of as a network of unit cubes.

3. Discuss with the students three-dimensional linear transformations and the effects they produce. Define the matrix of a transformation.



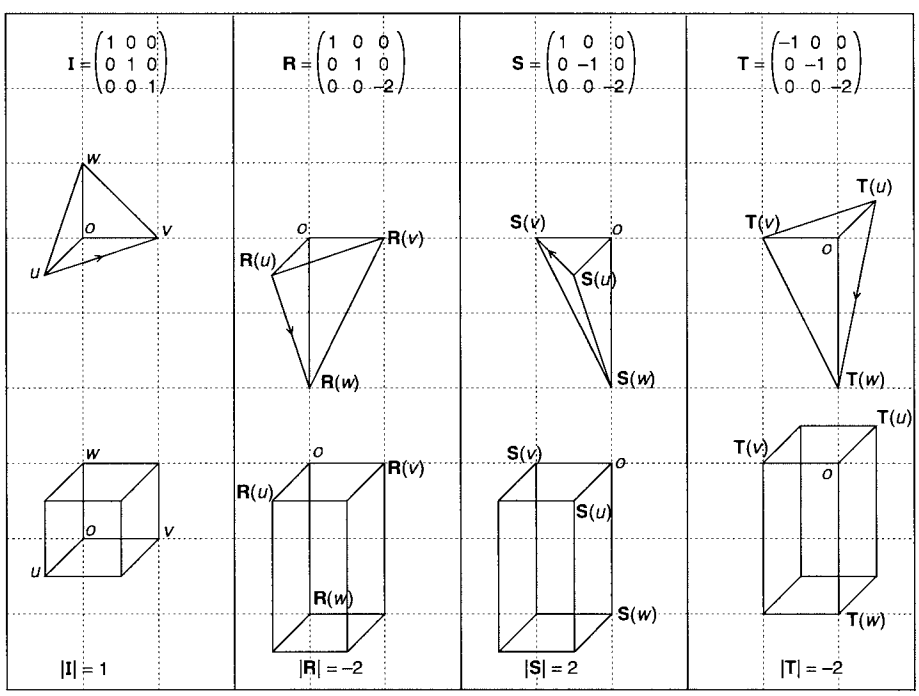
3. A three-dimensional linear transformation transforms the unit basis vectors into a set of vectors that become the basis for the transformed space. If none of the vectors in this new set of basis vectors are collinear, they form the edges of a parallelepiped which is the image of the unit cube. (See the sketch.) A coordinate grid of the transformed space may be thought of as a network of these parallelepipeds.

If all of the transformed basis vectors are collinear, the transformed space is a line; if two of the transformed basis vectors are collinear, the transformed space is a plane. In these two cases, the transformations is said to be *singular*.

Continued next page.

Actions

4. Define the determinant of a three-dimensional linear transformation. Ask the students to investigate the determinant of diagonal matrices.



Comments

3. *Continued.*
The matrix of a linear transformation of three-dimensional space is a three column matrix in which the columns are successively the images of u, v, and w. As in the two-dimensional case, we will identify a transformation with its matrix representation.

For example, the matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 3 & 3 \\ 2 & 1 & 2 \end{pmatrix}$$

is the transformation for which $\mathbf{T}(1, 0, 0) = (1, 1, 2)$, $\mathbf{T}(0, 1, 0, 0) = (3, 3, 1)$ and $\mathbf{T}(0, 0, 1) = (1, 3, 2)$.

Master 3 shows the coordinate grid determined by **T** in solid lines and the original grid in dotted lines.

4. The numerical value of the determinant of a linear transformation of space is the factor by which the transformation changes volumes; the determinant is positive or negative depending on whether or not the transformation changes the orientation of the coordinate system.

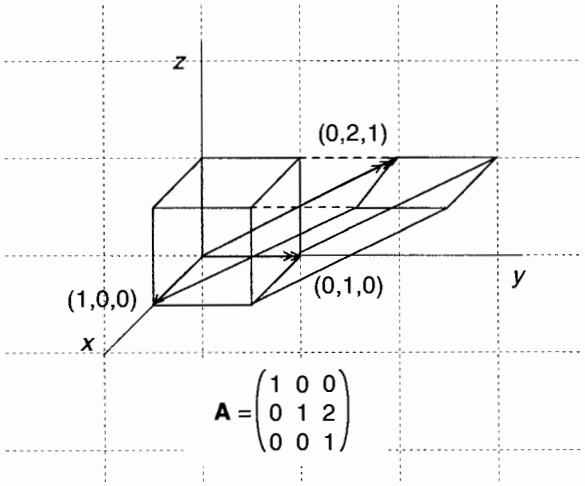
Assuming the volume of the unit cube is 1, the factor by which a linear transformation changes volume will be the volume of the parallelepiped into which the unit cube is transformed. A transformation **T** leaves orientation unchanged if the triangle $\mathcal{T}$ with vertices at the ends of u, v, and w has the same orientation as the triangle $\mathbf{T}(\mathcal{T})$, that is, when viewed from the origin and proceeding in a clockwise direction, the vertices of **T** and the corresponding images of these vertices on $\mathbf{T}(\mathcal{T})$ occur in the same order.

As shown, in the sketch, the transformations **R** and **T** change orientations while **S** does not. (In the sketches, the arrows indicate a clockwise direction when the triangles are viewed from the origin.) Since all three of the transformations double the volume of the unit cube, the numerical value of their determinants is 2.

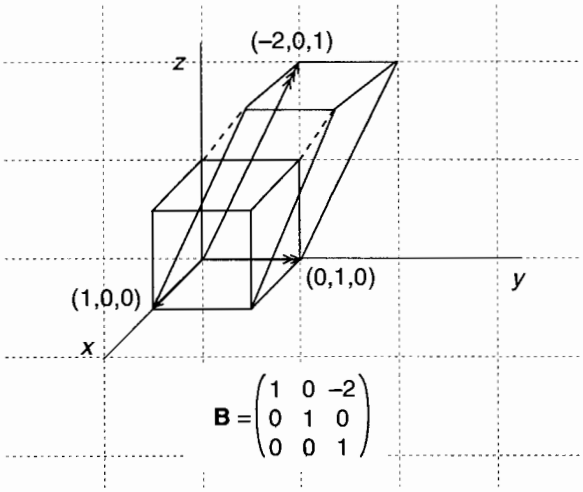
Continued next page.

Actions

5. Tell the students that the transformation **A** leaves the *xy*-plane fixed and shears the face of the unit cube parallel to the *xy*-plane 2 units in the positive *y* direction. Ask them to write the matrix of **A** and to find its determinant. Ask them to give examples of the matrices of other shears parallel to a coordinate plane.



$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\mathbf{B} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Comments

4. *Continued*
A transparency of this sketch can be made from master 4, attached.

A transformation whose matrix has non-zero elements on the main diagonal and zeroes elsewhere is a *scaling* transformation. Its determinant is the product of the elements on its main diagonal. Notice that if 1 or 3 elements of the diagonal are negative, the transformation changes orientation (cf. **R** and **T** above). Otherwise, the transformation does not change orientation (cf. **S** above).

5. As shown in the sketch, **A** leaves the unit vectors in the *x* and *y* directions unchanged, and moves the unit vector in the *z* direction to (0, 2, 1).

Thus, $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

Since **A** leaves the base and altitude of the unit cube unchanged, **A** doesn't change its volume. Also **A** does not change orientations. Hence, $|\mathbf{A}| = 1$.

The transformation $\mathbf{B} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ leaves the *xy*-plane fixed and shears the face of the unit cube parallel to the *xy*-plane 2 units in the negative *x* direction, as shown in the sketch.

The transformation $\mathbf{C} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ leaves the *xz*-plane fixed and shears the face of the cube parallel to the *xz*-plane 2 units in the negative *x* direction.

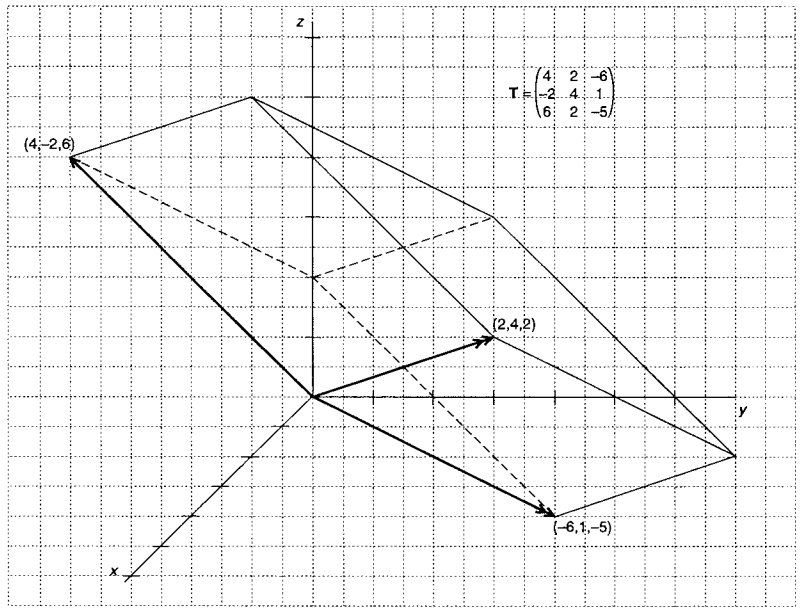
A shear parallel to one of the coordinate planes leaves volume and orientation unchanged and, hence, has determinant 1.

Actions

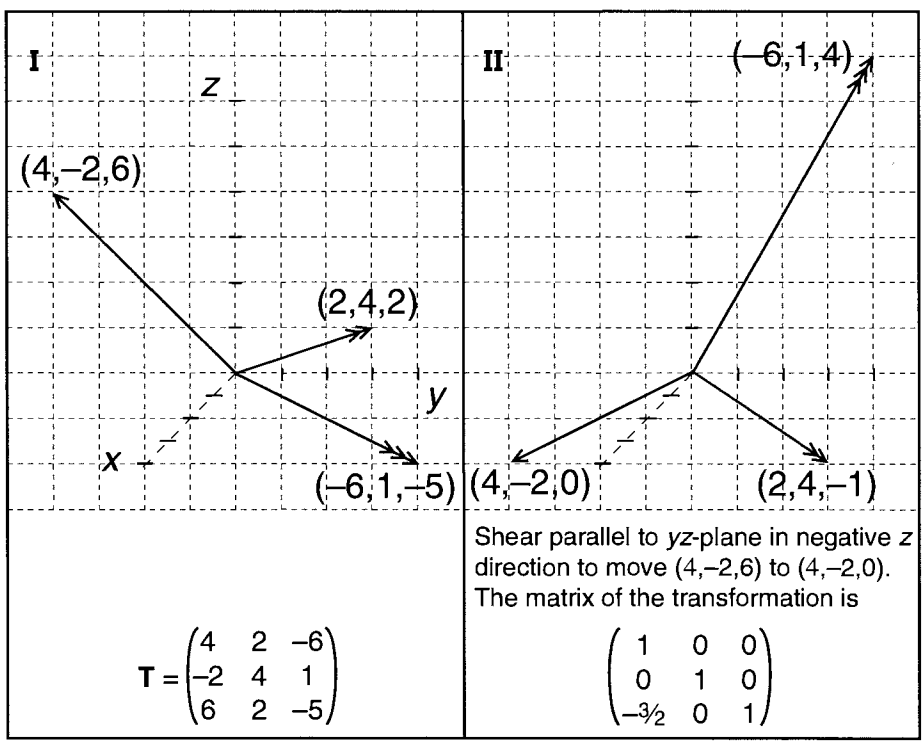
6. Discuss with the students how the matrix $\mathbf{T} = \begin{pmatrix} 4 & 2 & -6 \\ -2 & 4 & 1 \\ 6 & 2 & -5 \end{pmatrix}$ can be expressed as the product of shears and a scaling.

Comments

6. Master 5, attached, shows $\mathbf{T}(\mathcal{U})$ where $\mathcal{U}$ is the unit cube. In the sketch, the single-arrowed, double-arrowed, and triple-arrowed vectors are the images of the unit vectors (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively.



Master 6 shows how the images of the unit vectors can be transformed back to the unit vectors by a series of shears followed by a magnification.



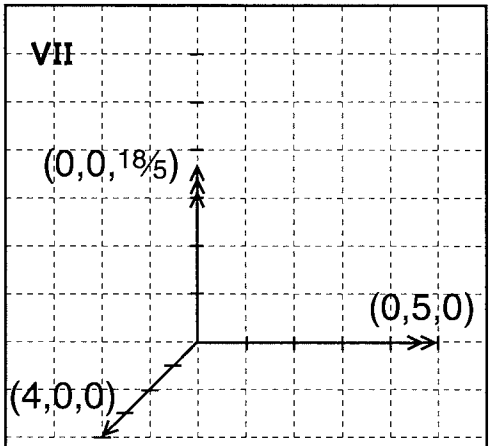
Panel I of Master 6 shows the image vectors. Panel II shows the position of the image vectors after (4, 2, –6) has been moved to (4, –2, 0) by a shear parallel to the yz-plane. Since this shear moves a point 4 x-units from the yz-plane 6 units in the negative z direction, for every x-unit a point is from the yz-plane it will move $-3/2$ units in the z direction. Since (2, 4, 2) is 2 x-units from the yz plane, its z coordinate changes from 2 to $2 + 2(-3/2)$, or –1. Thus the shear moves (2, 4, 2) to (2, 4, –1). Since (–6, 1, –5) is –6 x-units from the yz plane, its z coordinate changes from –5 to $-5 + (-6)(-3/2)$, or 4.

The shear in Panel II moves (1, 0, 0) to $(1, 0, -3/2)$ and leaves (0, 1, 0) and (0, 0, 1) unchanged. Hence its matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix}.$$

Continued next page.

Actions



Shear parallel to *xy*-plane in positive *y* direction to move $(0,-2,{}^{18}/_{5})$ to $(0,0,{}^{18}/_{5})$. The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5/9 \\ 0 & 0 & 1 \end{pmatrix}$$

Comments

6. *Continued*

The effects of the shears in panels III-VII are determined similarly. In panel VII, shown here, the image vectors have been aligned along the coordinate axes. A final scaling transformation in panel VIII reduces them to unit length.

If one begins with the unit vectors of panel VIII of Master 6 and transforms them successively to the vectors in the previous panel, one will arrive, after a scaling and series of shears, at the **T** images of the unit vectors in panel I.

To begin with, the scaling that transforms the unit vectors in Panel VIII to the vectors in panel VII multiples x, y, and z values, respectively by 4, 5, and ${}^{18}/_{5}$. Hence its matrix is $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{18}{5} \end{pmatrix}$.

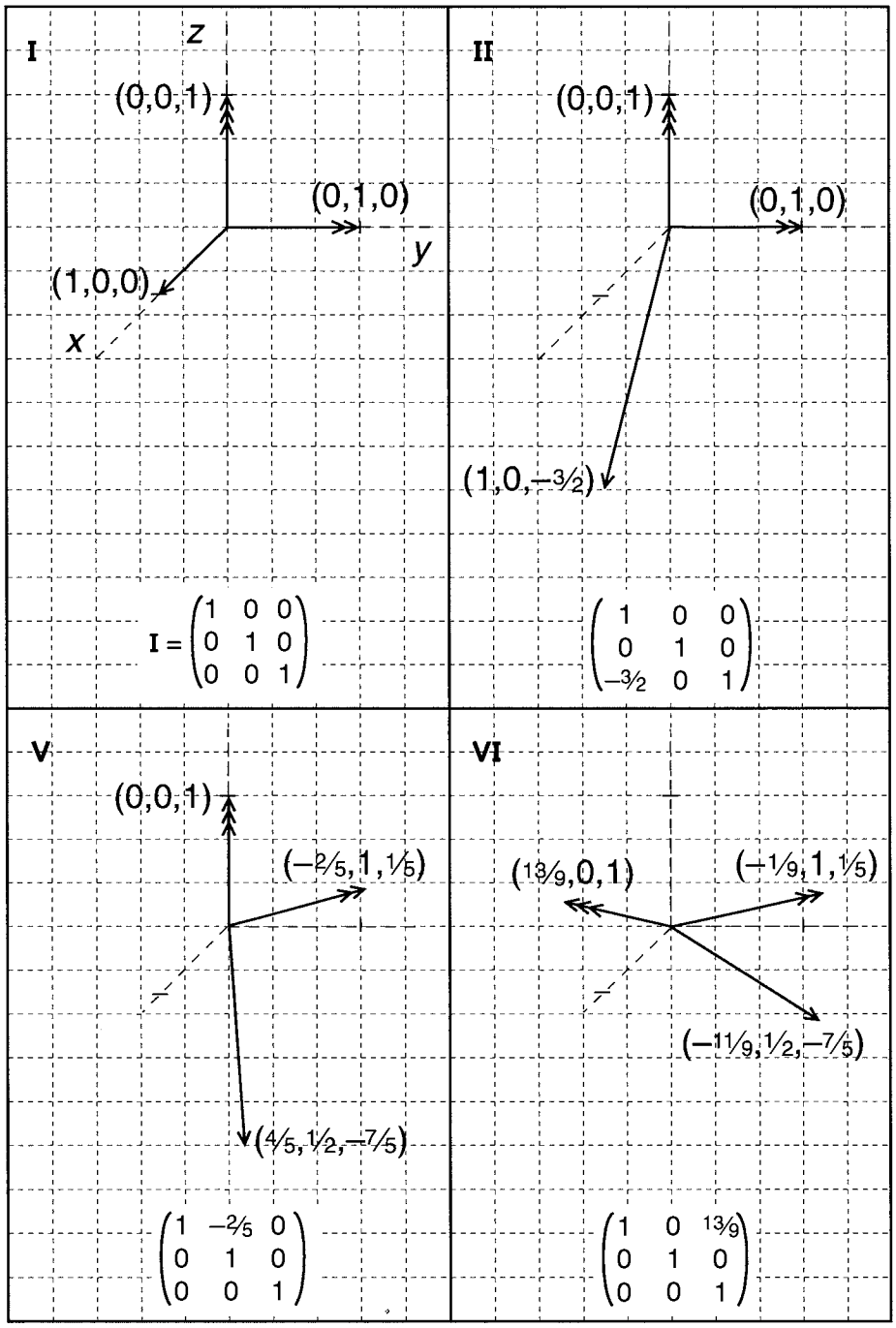
The vectors of panel VII are then transformed to those of panel I by a sequences of shears which are the inverses of those described on Master 6. For example, the shear that transforms the vectors from panel VII to VI has matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{9} \\ 0 & 0 & 1 \end{pmatrix}$.

Thus **T** can be written as a product of 6 shears and a scaling:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{5} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{2}{5} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\frac{13}{9} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{9} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{18}{5} \end{pmatrix}.$$

Actions

7. (Optional) Ask the students to find the determinant and the inverse of the matrix **T** of Action 6.



Comments

7. From Master 6, one sees that **T** does not change orientation. Also, since shears do not change volume, the parallelepiped determined by the vectors of panel I has the same volume as the rectangular parallelepiped of panel VII. Hence, $|\mathbf{T}| = 4(5)(^{18}/_{5}) = 72$.

Since, in Master 6, applying the transformations in panels II through VII to **T** yields the identity transformation, the composition of these transformations is $\mathbf{T}^{-1}$. The matrix of $\mathbf{T}^{-1}$ can be found by successively determining the effect of these transformations on the unit vectors as shown in Master 7.

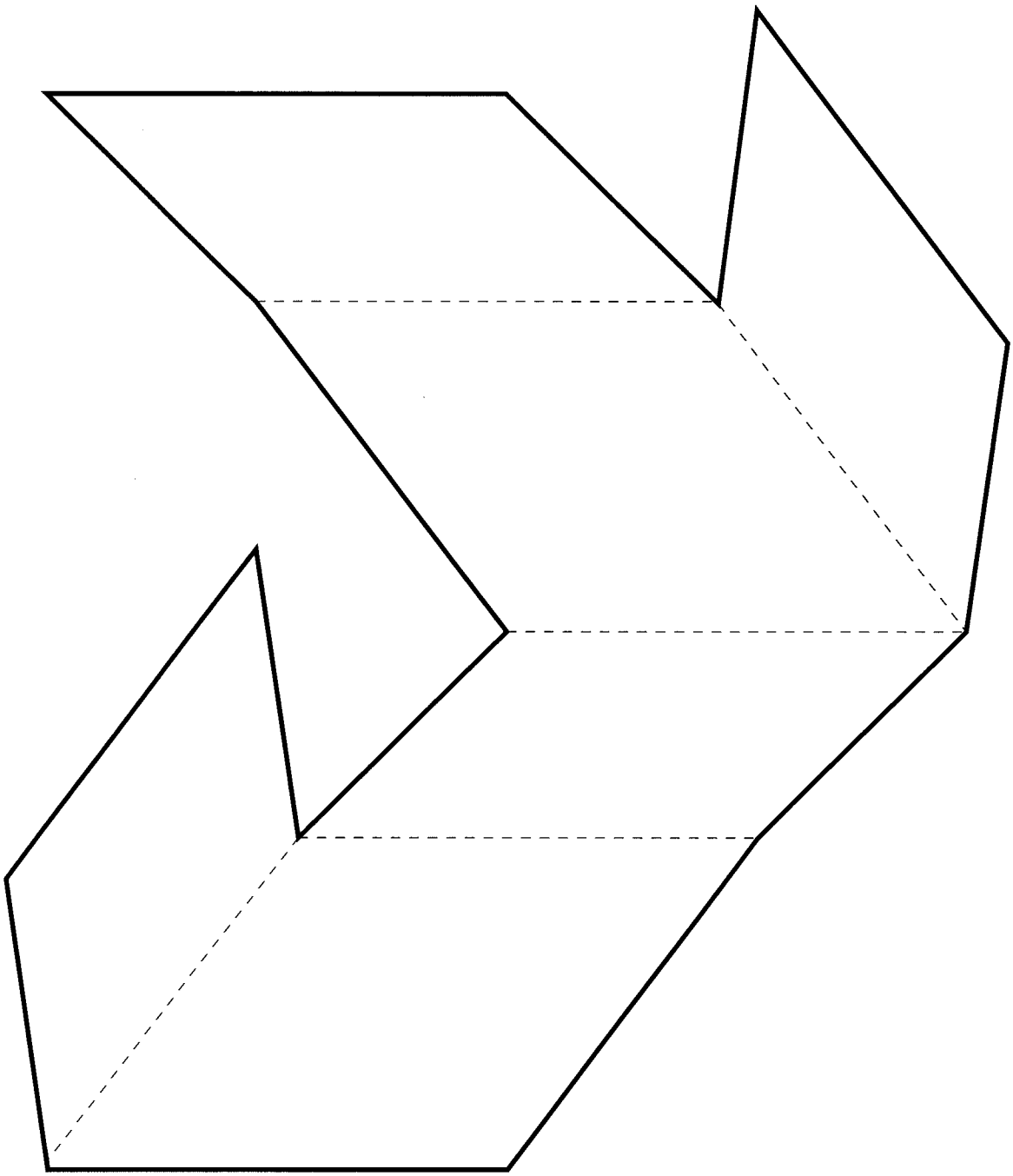
In Master 7, the matrix in each panel is the matrix of the transformation which is applied to the vectors in the previous panel. For example, the matrix in panel II is a shear parallel to the *yz*-plane which moves (1, 0, 0) to (1, 0, $-^{3}/_{2}$) and leaves (0, 0, 1) and (1, 0, 0) unchanged.

As a further example, the matrix in panel VI is a shear parallel to the *xy*-plane in the *x* direction. It moves (0, 0, 1) to ($^{13}/_{9}$, 0, 1). Since ($-^{2}/_{5}$, 1, $^{1}/_{5}$) is $^{1}/_{5}$ unit from the *yz*-plane, it is moved $^{1}/_{5}(^{13}/_{9})$ units in the *x* direction to ($-^{2}/_{5} + {}^{1}/_{5}(^{13}/_{9})$, 1, $^{1}/_{5}$), or ($-^{1}/_{9}$, 1, $^{1}/_{5}$). Similarly, ($^{4}/_{5}$, $^{1}/_{2}$, $-^{7}/_{5}$) is moved $-^{7}/_{5}(^{13}/_{9})$ units in the *x* direction to ($-^{11}/_{9}$, $^{1}/_{2}$, $-^{7}/_{5}$).

Note that the scale in panel VIII is one-fourth the scale in the other panels.

From the last panel, one sees that the original *x*, *y*, and *z* unit vectors are transformed, respectively, to ($-^{11}/_{36}$, $-^{2}/_{36}$, $-^{14}/_{36}$), ($-^{1}/_{36}$, $^{8}/_{36}$, $^{2}/_{36}$) and ($^{13}/_{36}$, $^{4}/_{36}$, $^{10}/_{36}$). Hence,

$$\mathbf{T}^{-1} = \frac{1}{36}\begin{pmatrix} -11 & -1 & 13 \\ -2 & 8 & 4 \\ -14 & 2 & 10 \end{pmatrix}.$$



Parallelepiped Net

Cut on solid lines. Fold on dashed lines.
Fold into parallelepiped. Tape edges.

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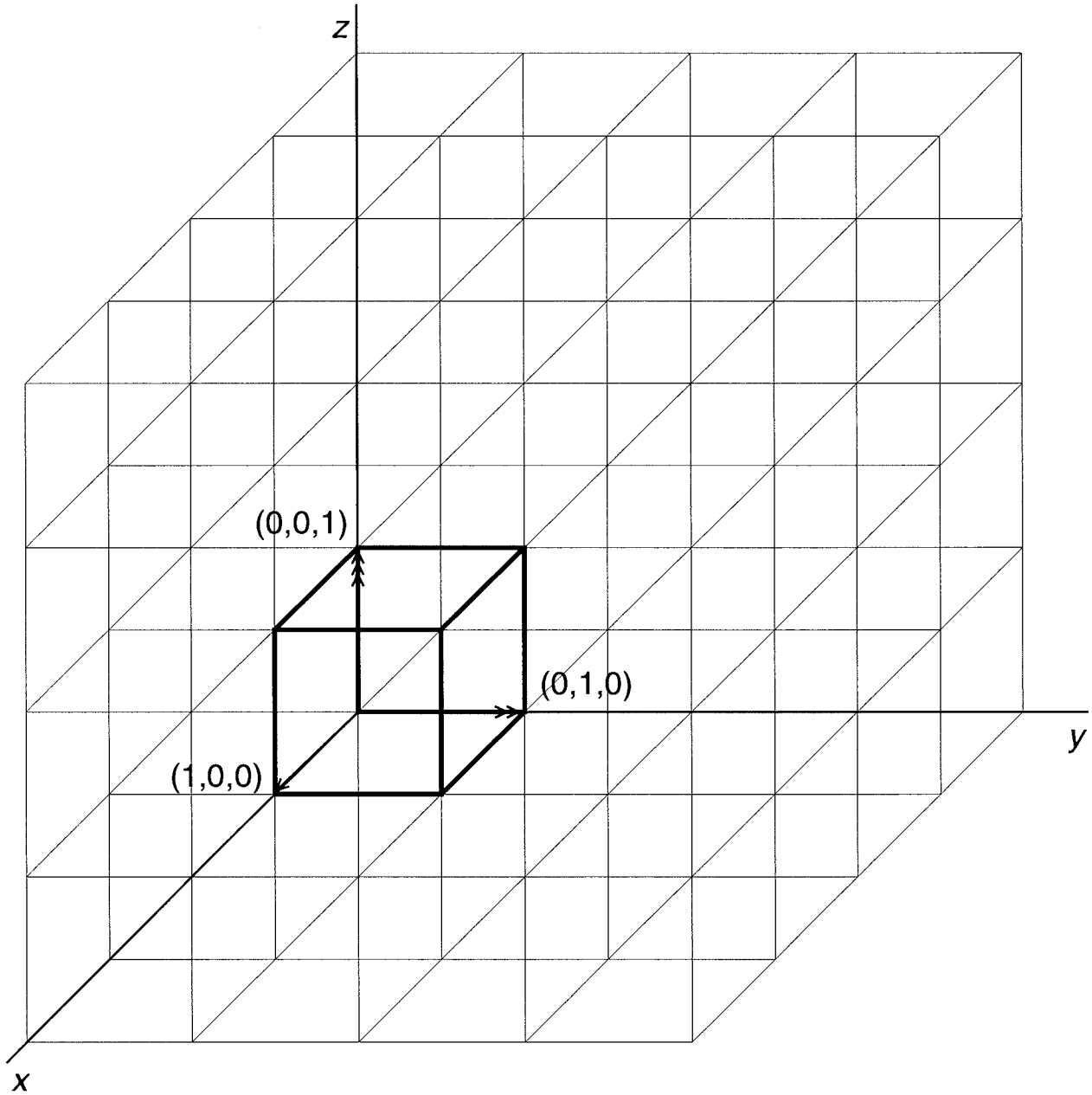

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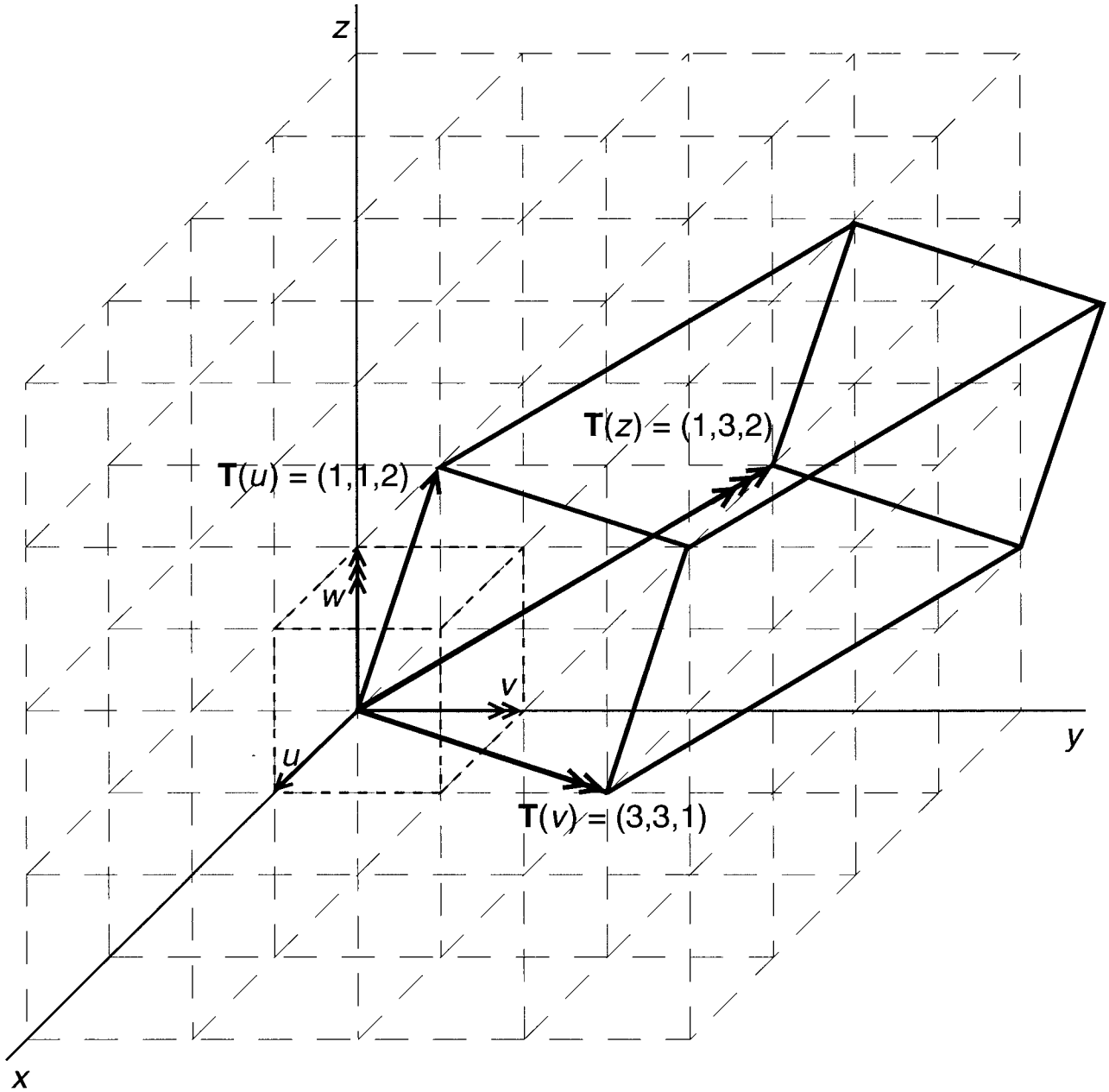

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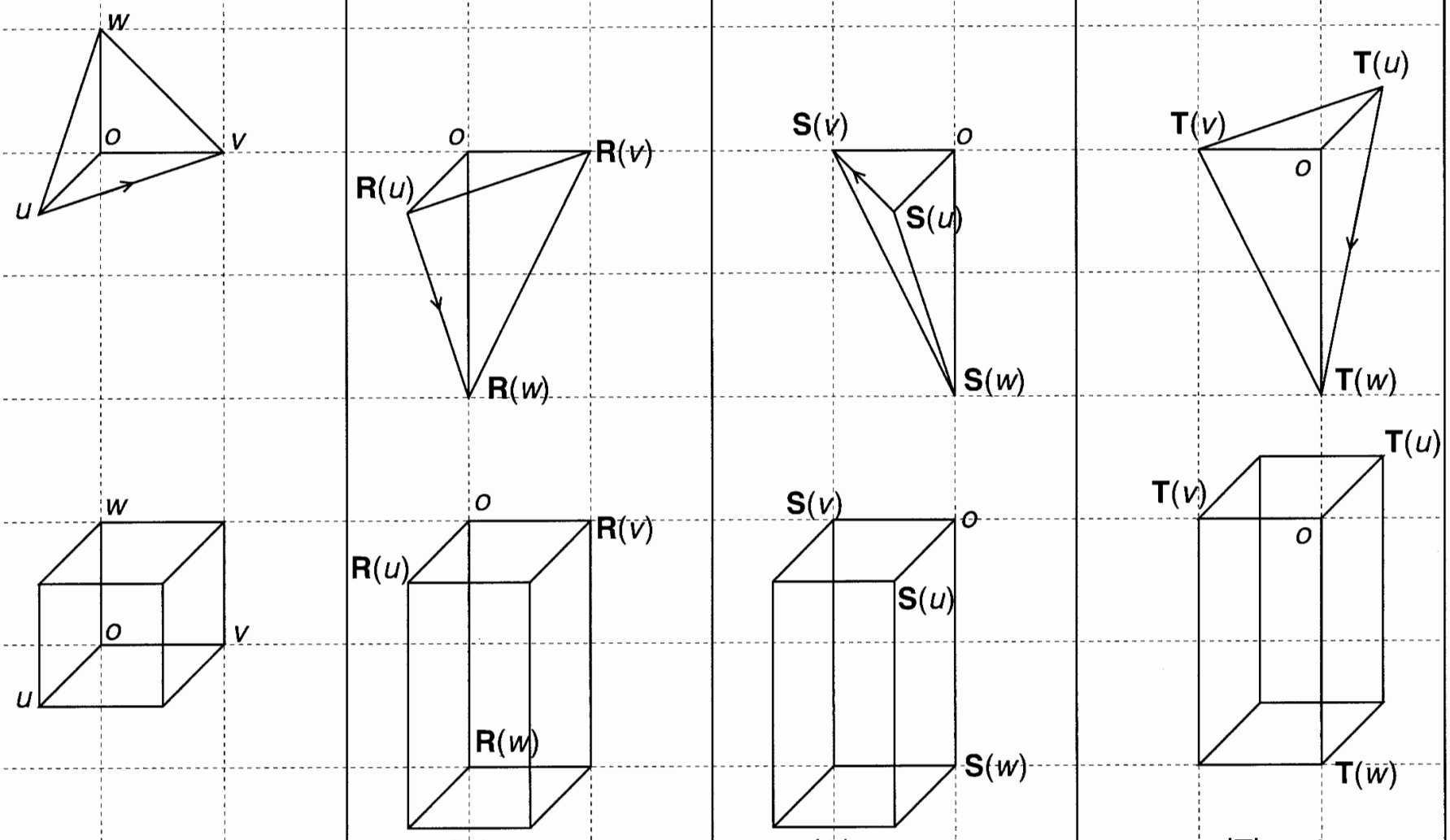
$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$



$|\mathbf{I}| = 1$ $\qquad$ $|\mathbf{R}| = -2$ $\qquad$ $|\mathbf{S}| = 2$ $\qquad$ $|\mathbf{T}| = -2$

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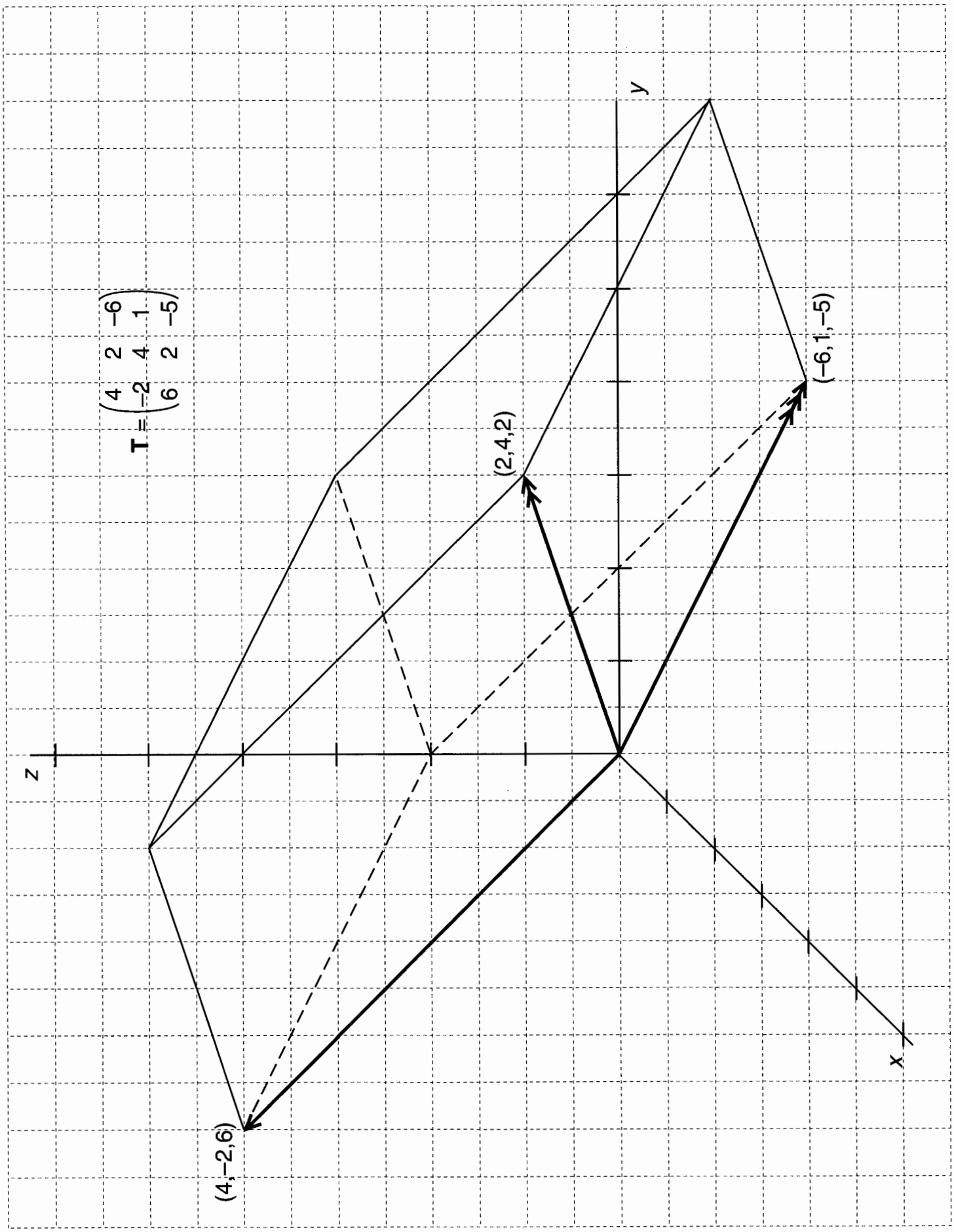

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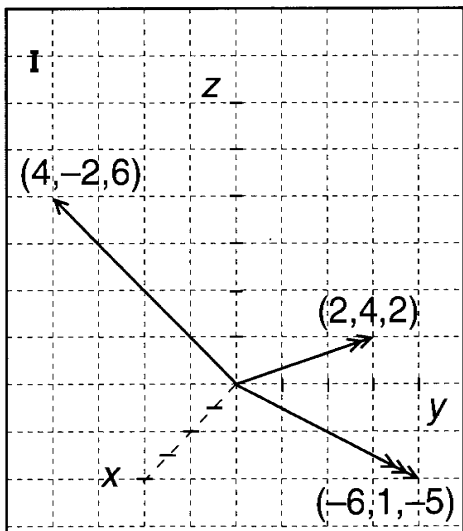
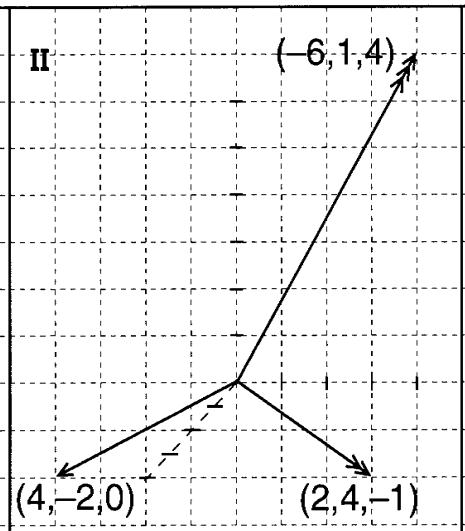
I



$$\mathbf{T} = \begin{pmatrix} 4 & 2 & -6 \\ -2 & 4 & 1 \\ 6 & 2 & -5 \end{pmatrix}$$

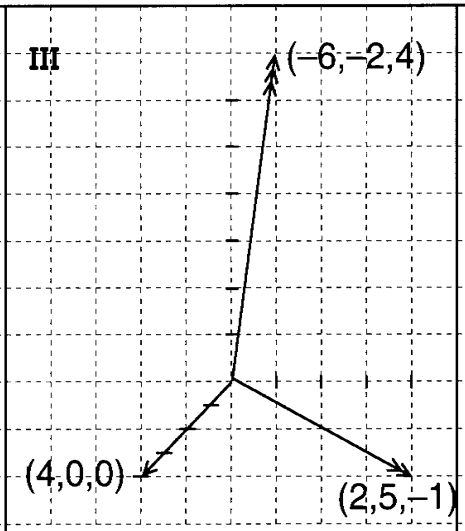
II



Shear parallel to *yz*-plane in negative *z* direction to move (4,–2,6) to (4,–2,0). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix}$$

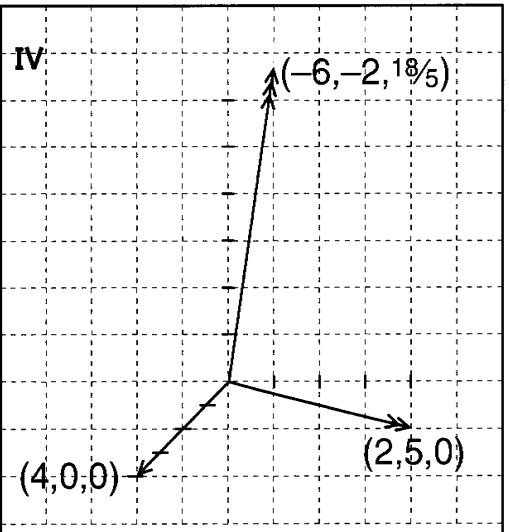
III



Shear parallel to *yz*-plane in positive *y* direction to move (4,–2,0) to (4,0,0). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

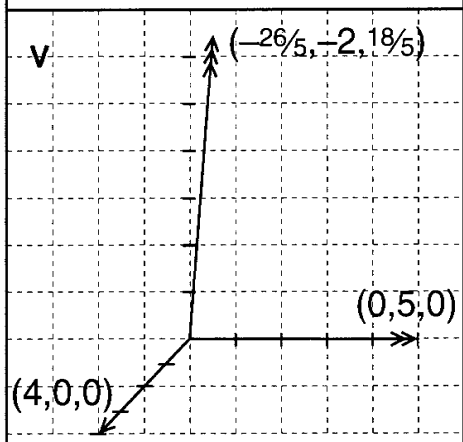
IV



Shear parallel to *xz*-plane in positive *z* direction to move (2,5,–1) to (2,5,0). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/5 & 1 \end{pmatrix}$$

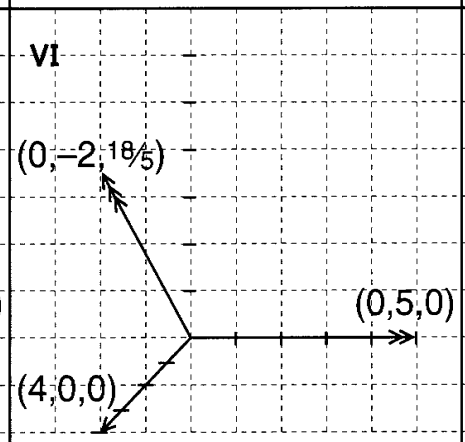
V



Shear parallel to *xz*-plane in negative *z* direction to move (2,5,0) to (0,5,0). The matrix of the transformation is

$$\begin{pmatrix} 1 & -2/5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

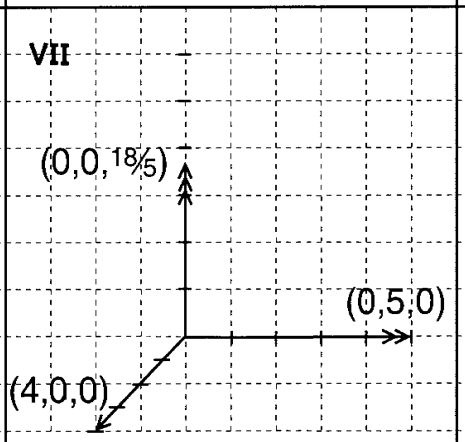
VI



Shear parallel to *xy*-plane in positive *x* direction to move (–26/5,–2,18/5) to (0,–2,18/5). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 13/9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

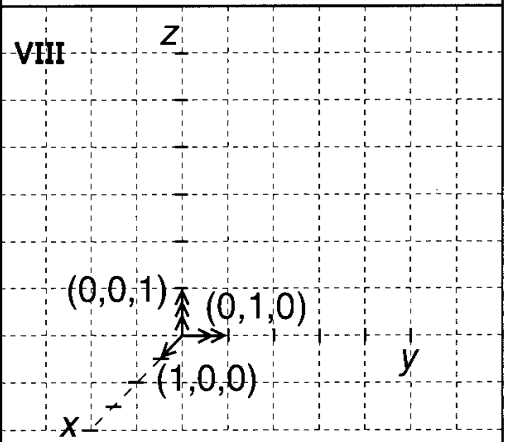
VII



Shear parallel to *xy*-plane in positive *y* direction to move (0,–2,18/5) to (0,0,18/5). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5/9 \\ 0 & 0 & 1 \end{pmatrix}$$

VIII



Scale vectors to unit length. The matrix of the transformation is

$$\begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 5/18 \end{pmatrix}$$

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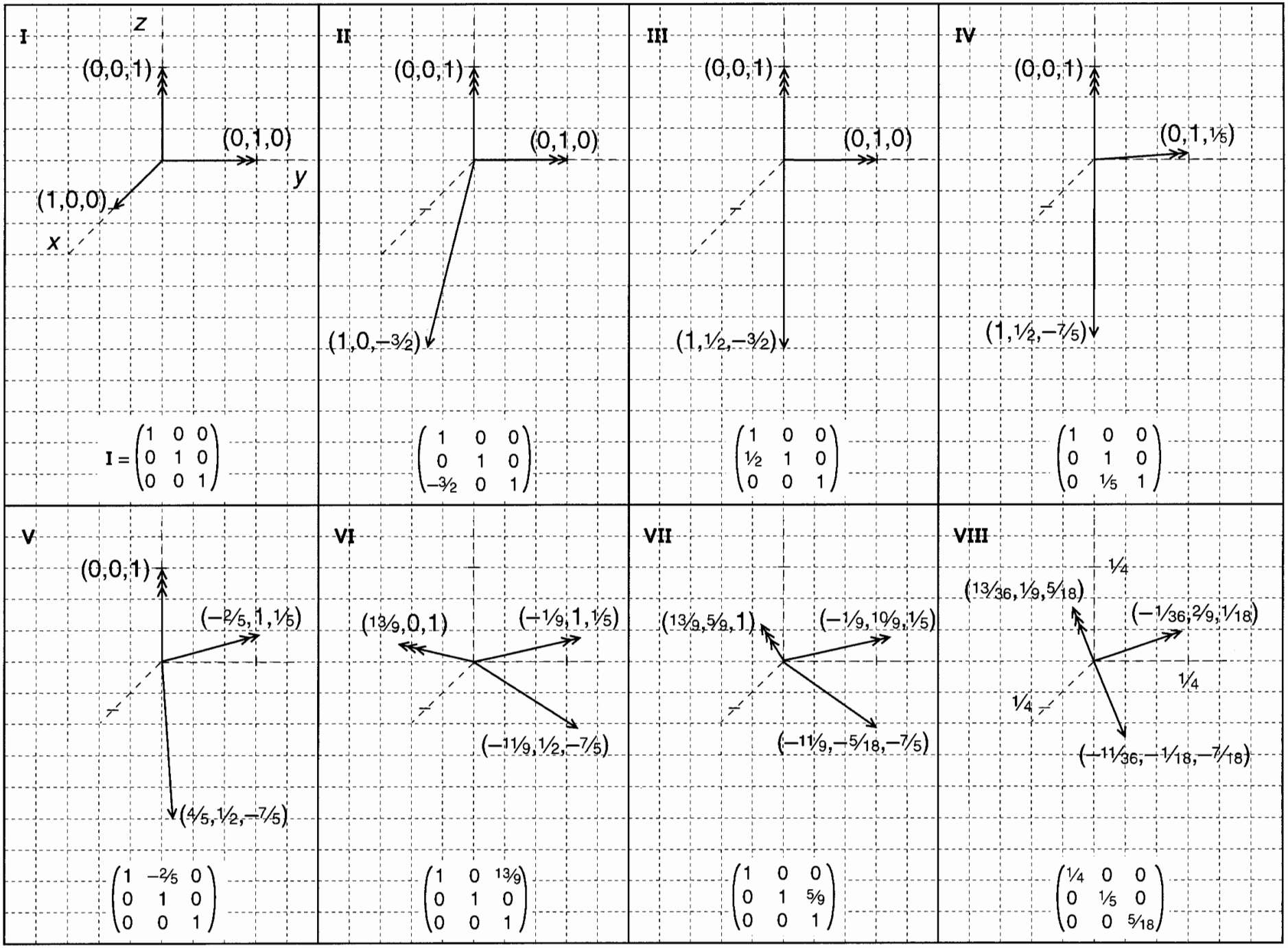

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2/5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 13/9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5/9 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 5/18 \end{pmatrix}$$

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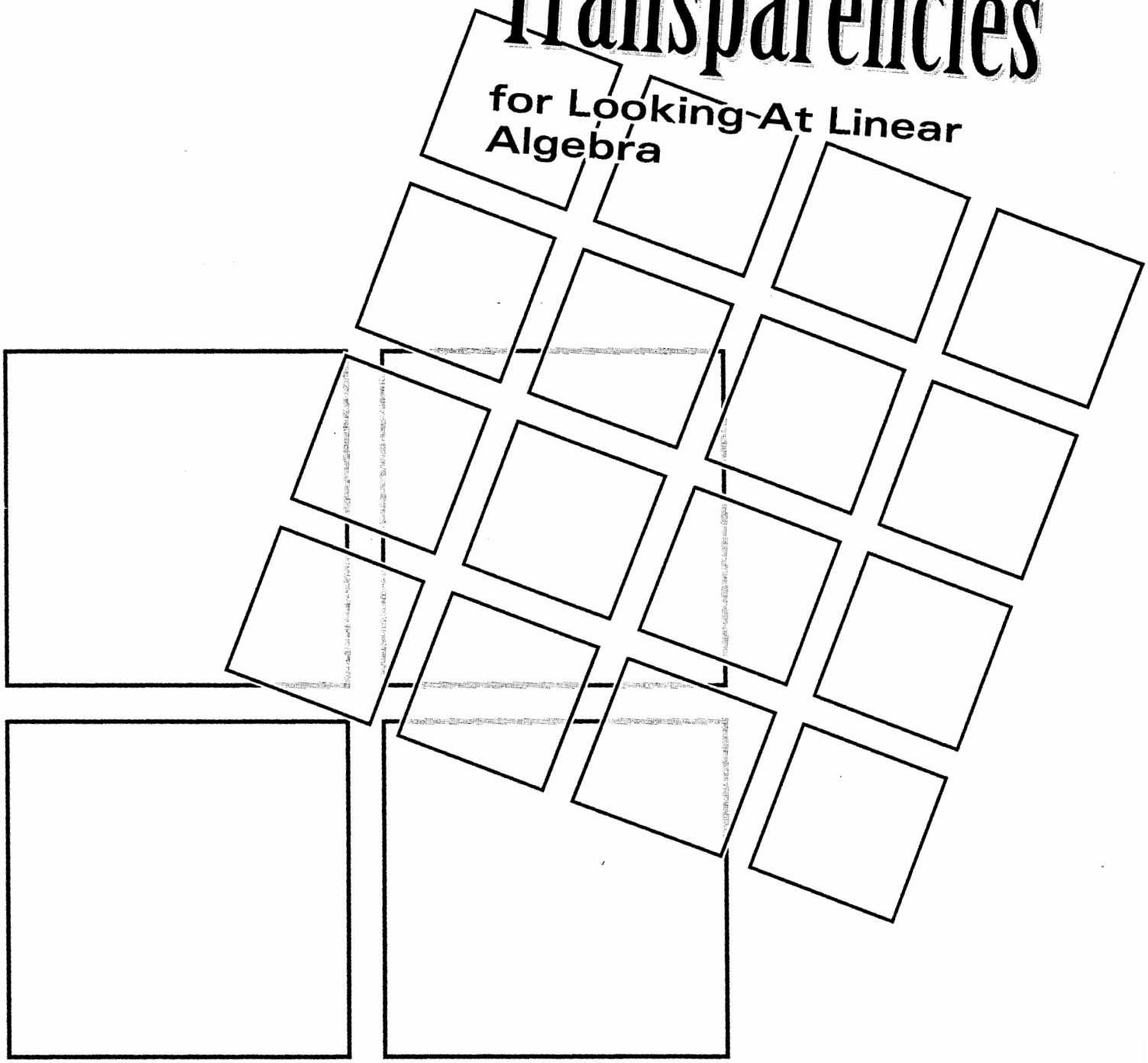
Unit XIV / Math and the Mind's Eye

Transparencies

for Looking At Linear Algebra



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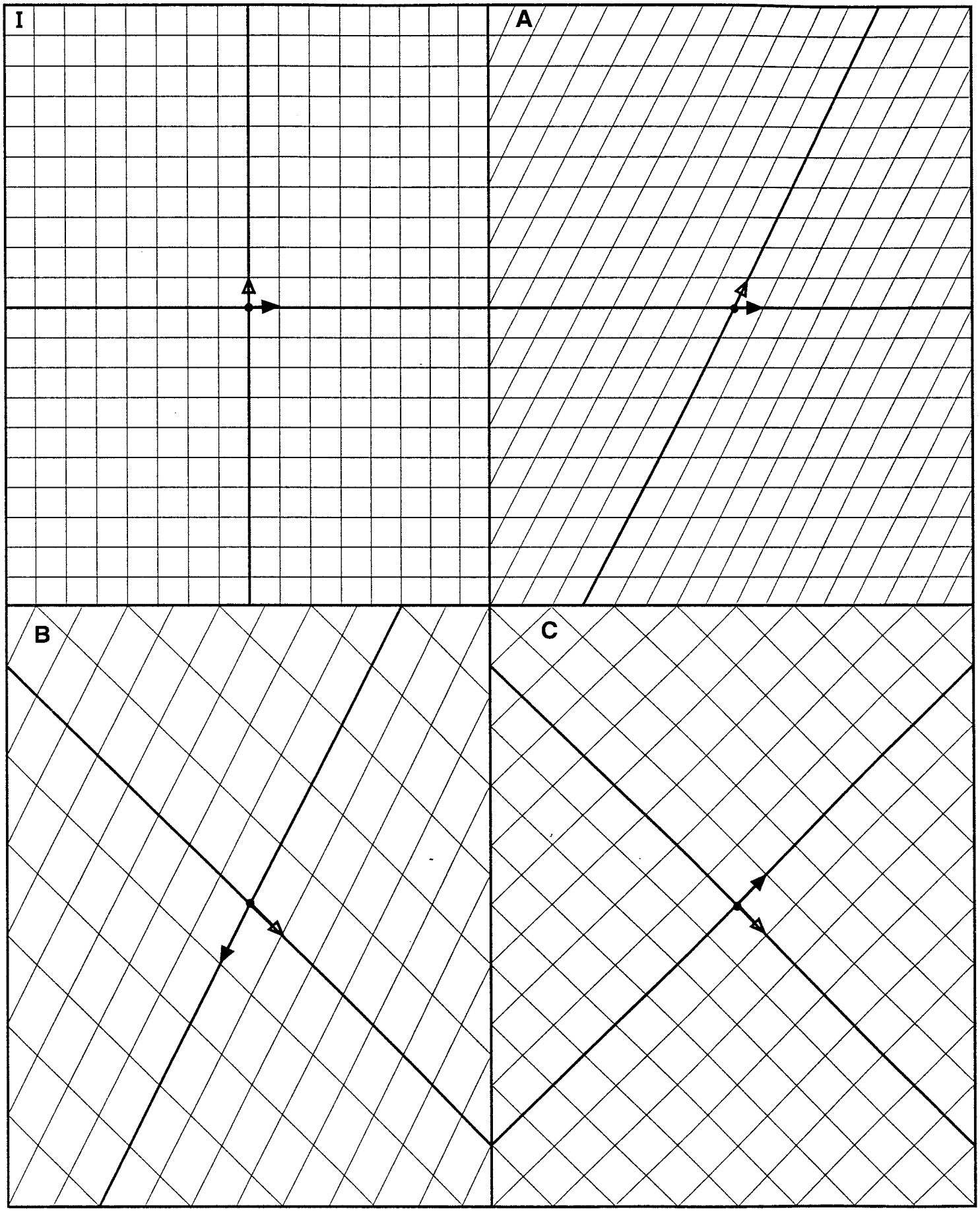

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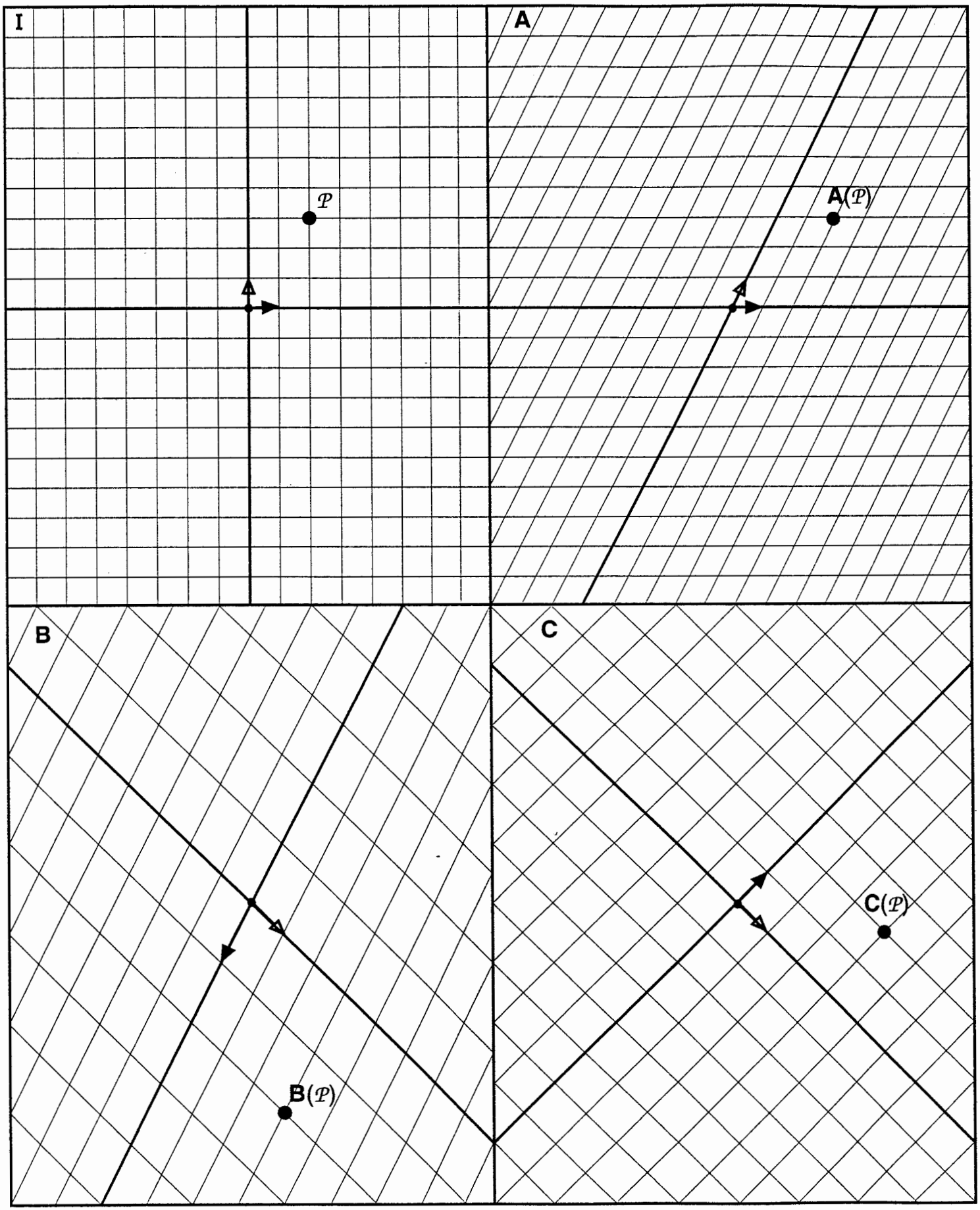

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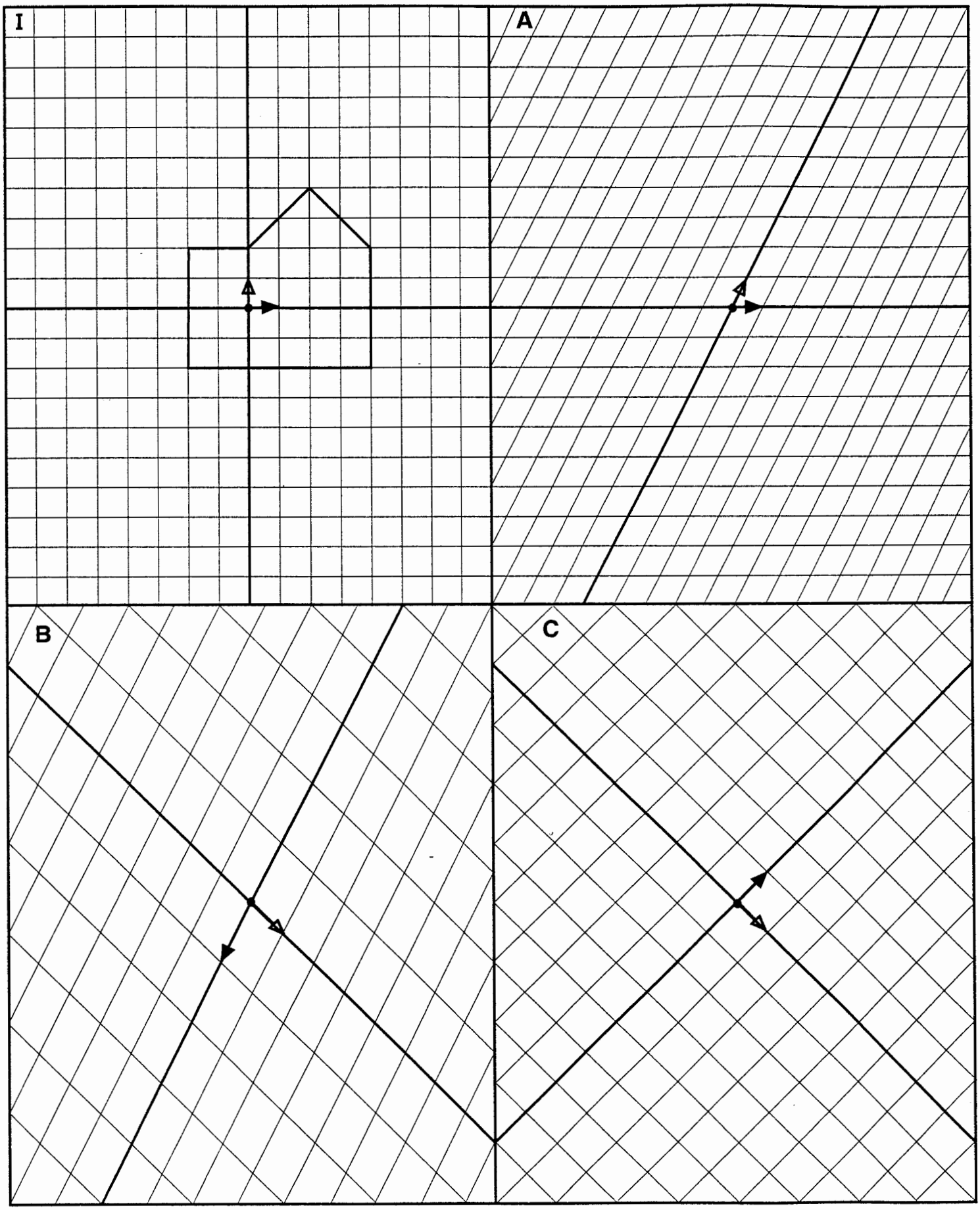

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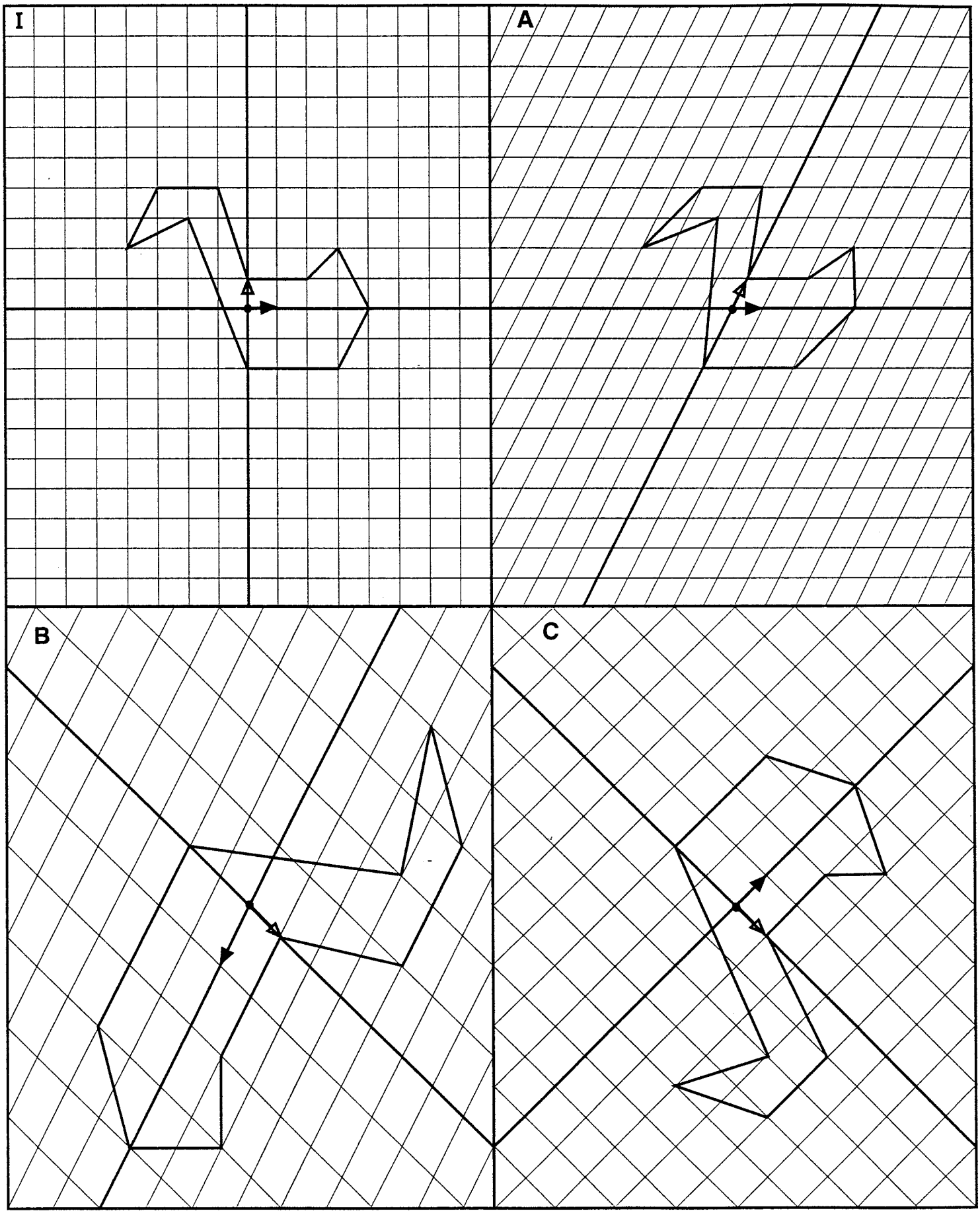

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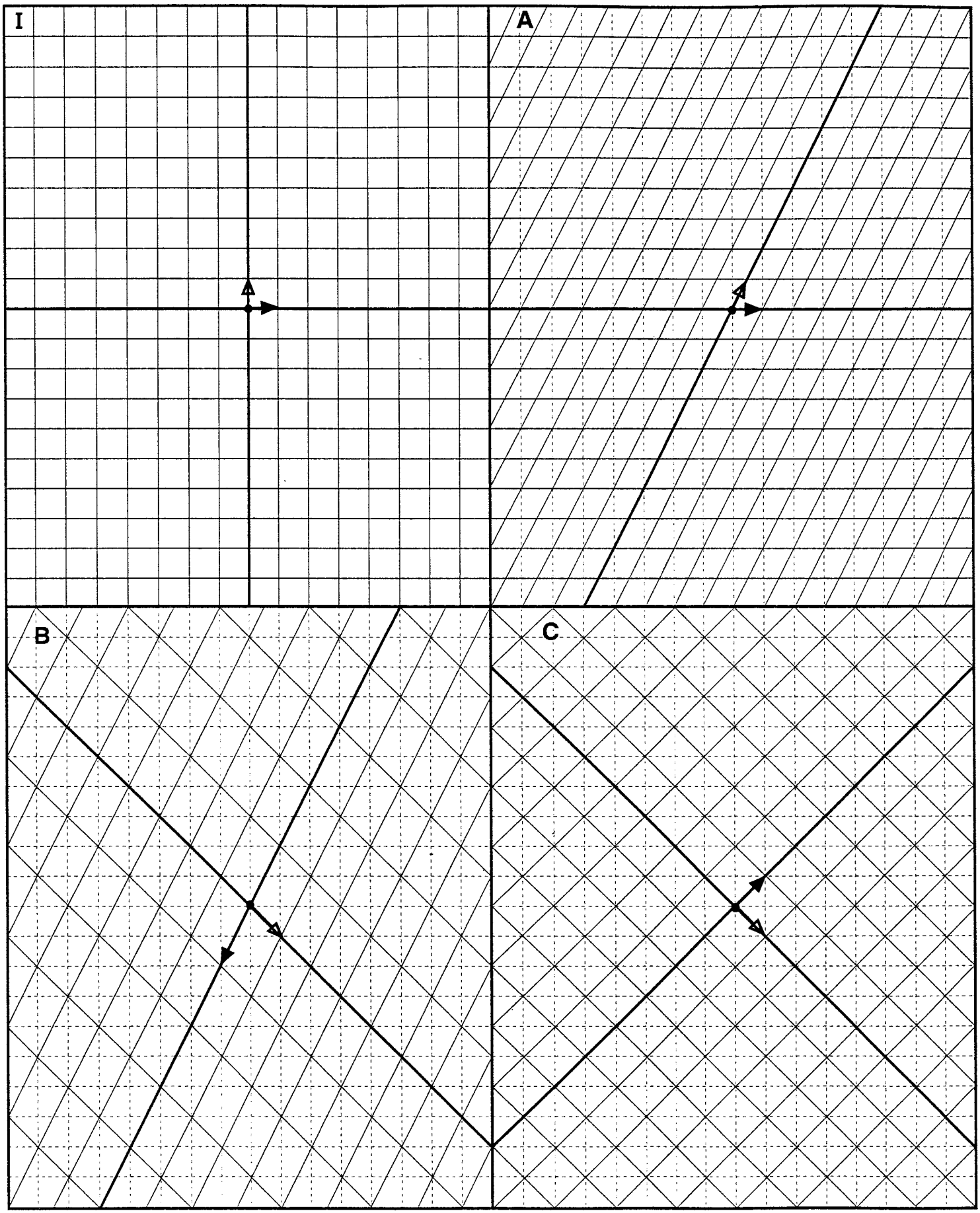

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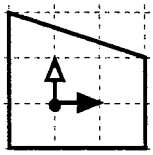

$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

$\mathbf{R}(\mathcal{F})$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

$\mathbf{S}(\mathcal{F})$

$\mathbf{T} = \begin{pmatrix} -2 & -1 \\ 2 & -1 \end{pmatrix}$

$\mathbf{T}(\mathcal{F})$

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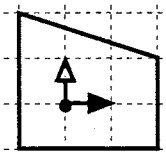
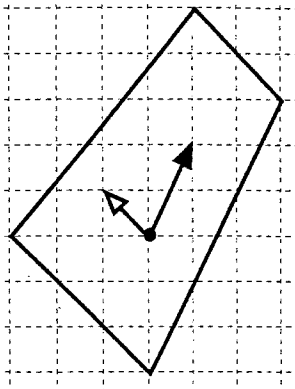
$\mathbf{I}$



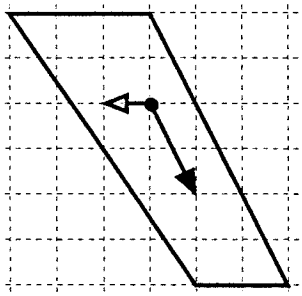
$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$



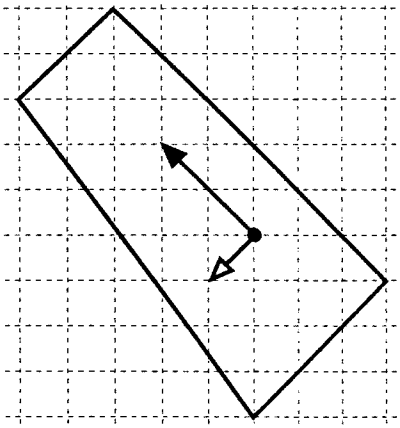
$\mathbf{R}(\mathcal{F})$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$



$\mathbf{S}(\mathcal{F})$

$\mathbf{T} = \begin{pmatrix} -2 & -1 \\ 2 & -1 \end{pmatrix}$



$\mathbf{T}(\mathcal{F})$

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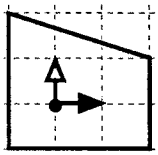
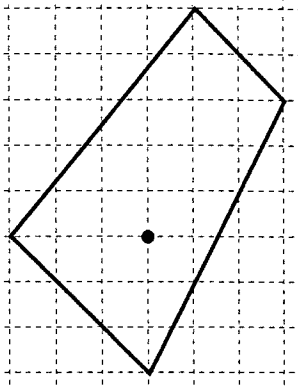
$\mathbf{I}$



$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$



$\mathbf{R}(\mathcal{F})$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

$\mathbf{S}(\mathbf{R}(\mathcal{F}))$

$\mathbf{U} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

$\mathbf{U}(\mathcal{F})$

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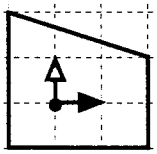
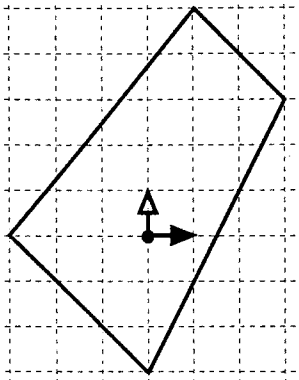


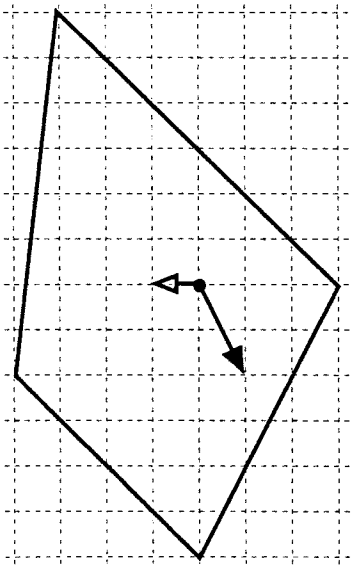
$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$



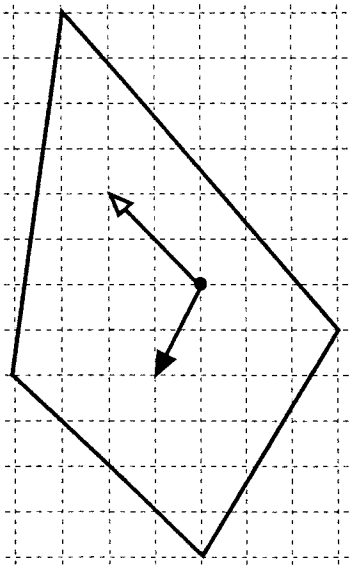
$\mathbf{R}(\mathcal{F})$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$



$\mathbf{S}(\mathbf{R}(\mathcal{F}))$

$\mathbf{U} = \begin{pmatrix} -1 & -2 \\ -2 & 2 \end{pmatrix}$



$\mathbf{U}(\mathcal{F})$

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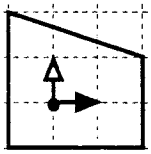
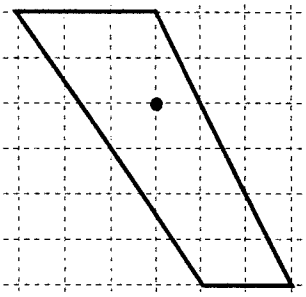


$\mathcal{F}$

$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

$\mathbf{R}(\mathbf{S}(\mathcal{F}))$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$



$\mathbf{S}(\mathcal{F})$

$\mathbf{V} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

$\mathbf{V}(\mathcal{F})$

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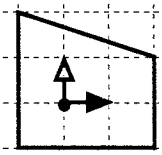

$\mathcal{F}$

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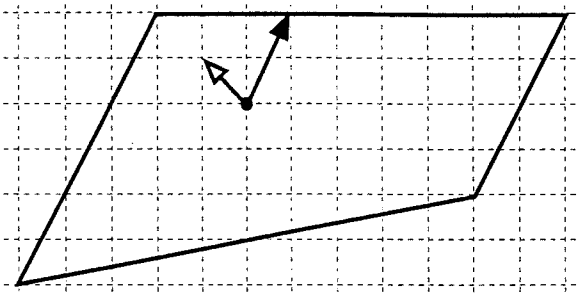
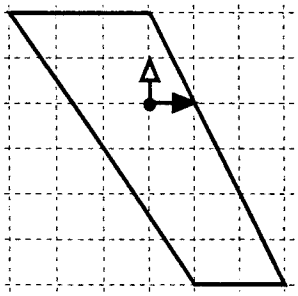


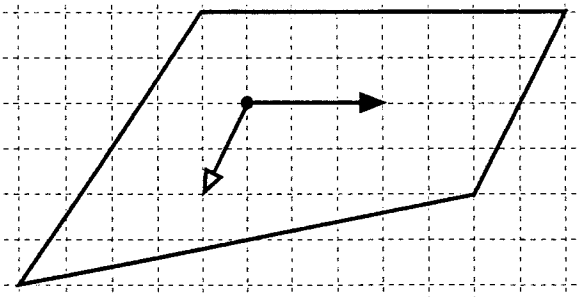
$\mathbf{R}(\mathbf{S}(\mathcal{F}))$

$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$



$\mathbf{S}(\mathcal{F})$

$\mathbf{V} = \begin{pmatrix} 3 & -1 \\ 0 & -2 \end{pmatrix}$



$\mathbf{V}(\mathcal{F})$

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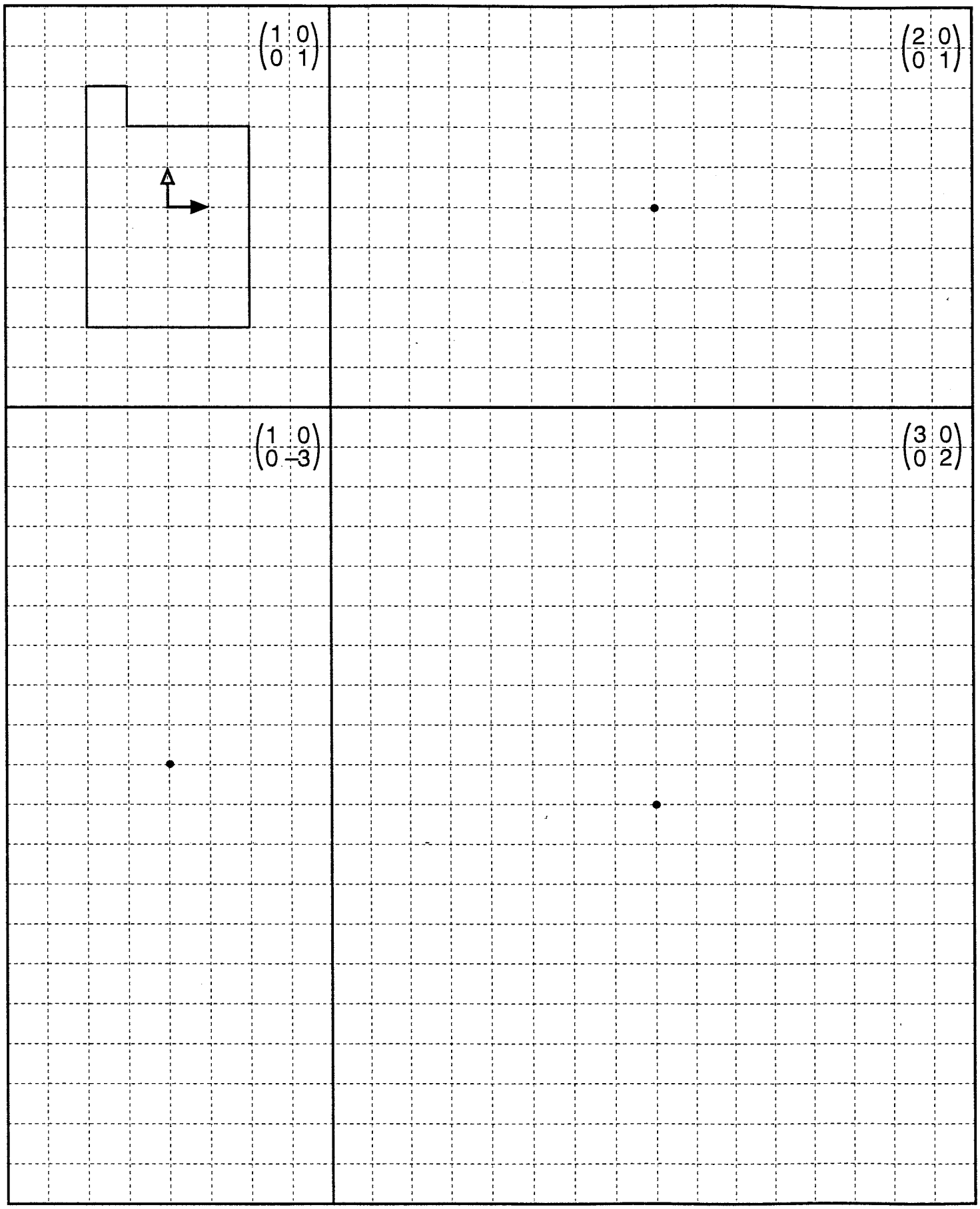

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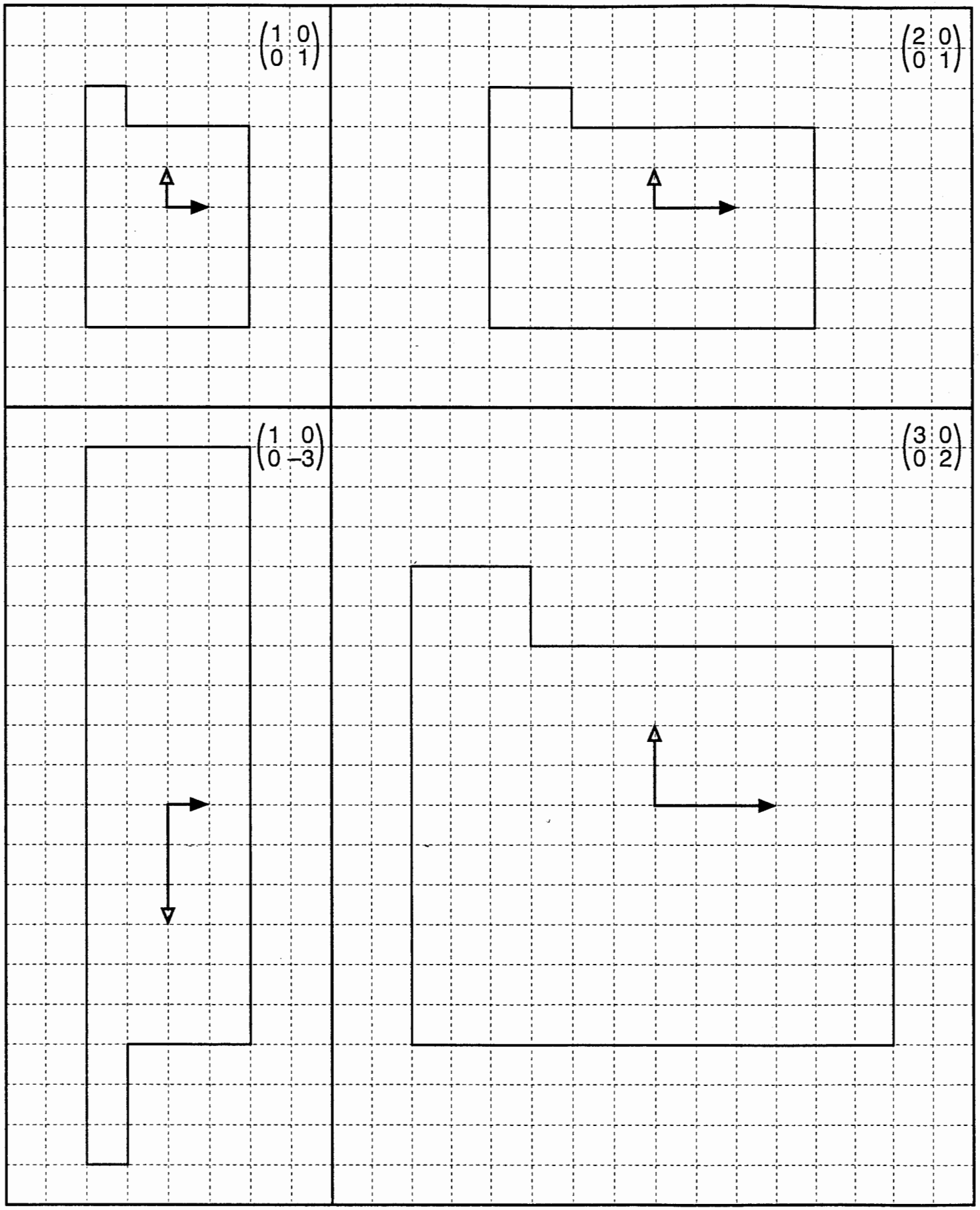

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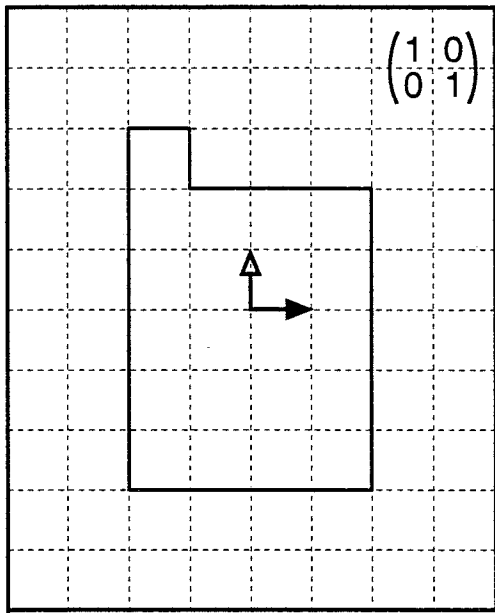

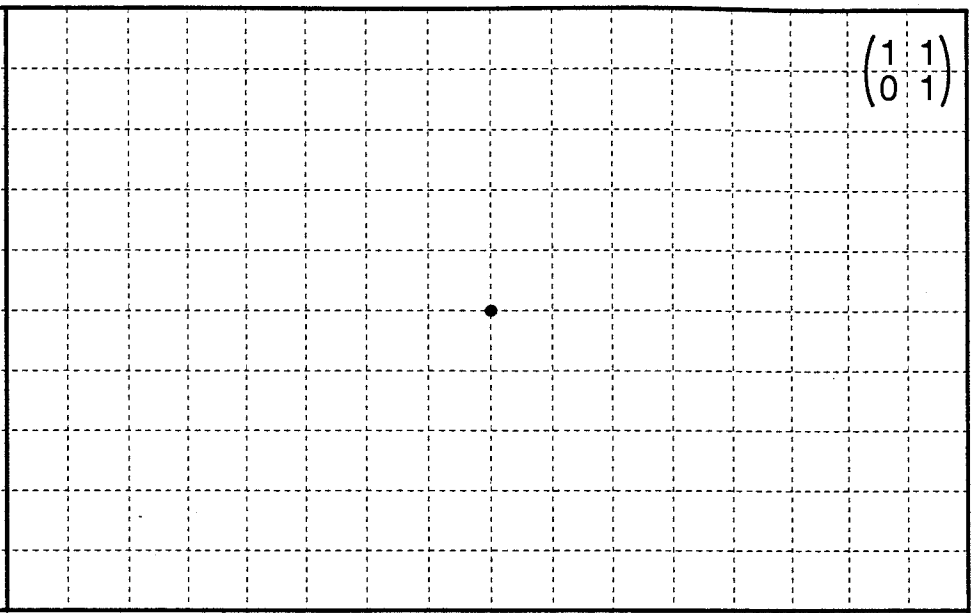

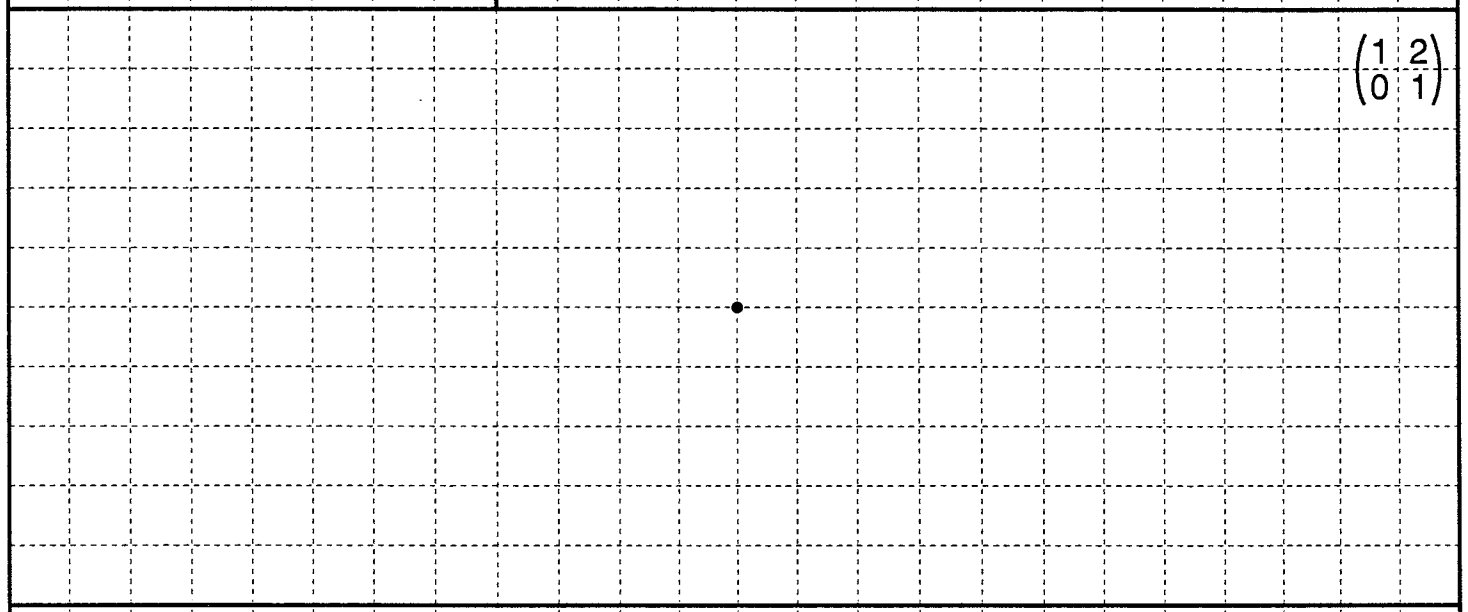

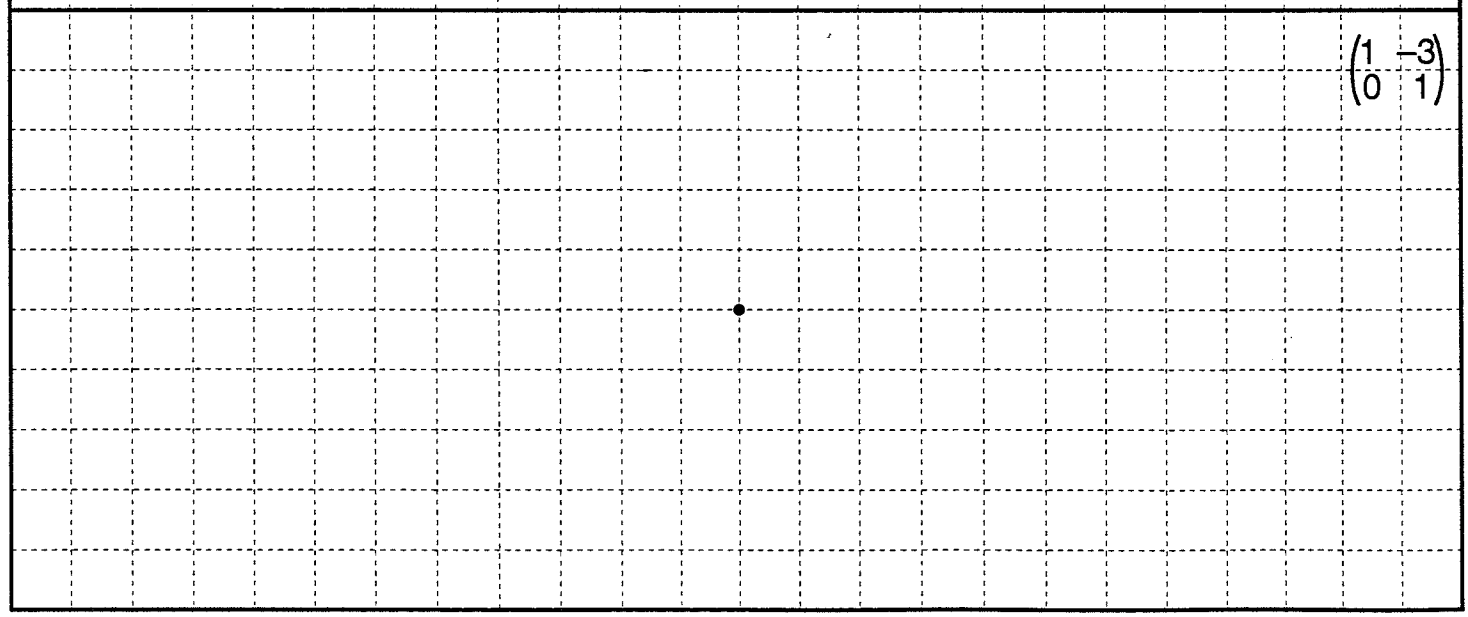

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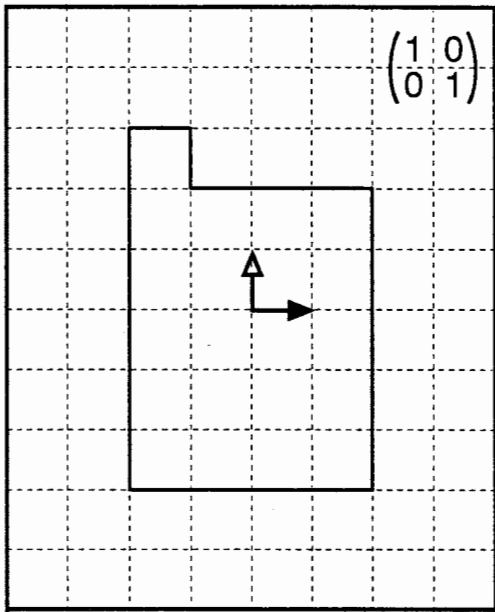

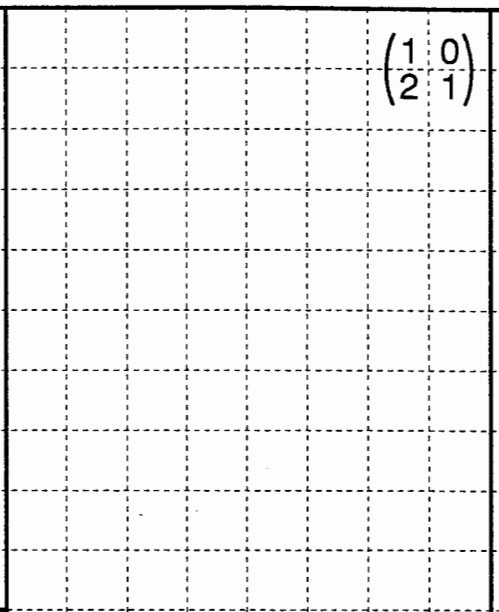

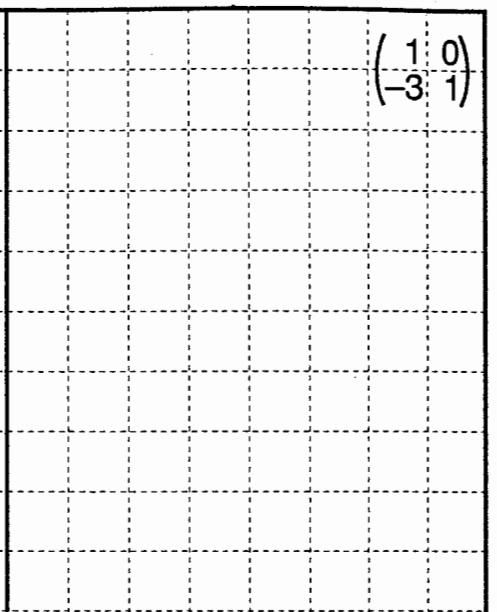

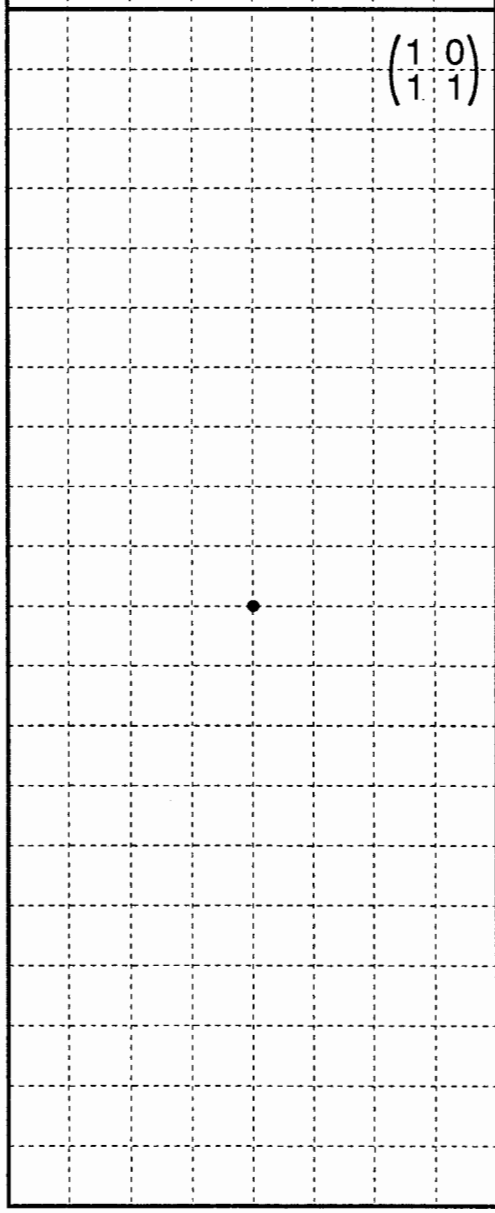

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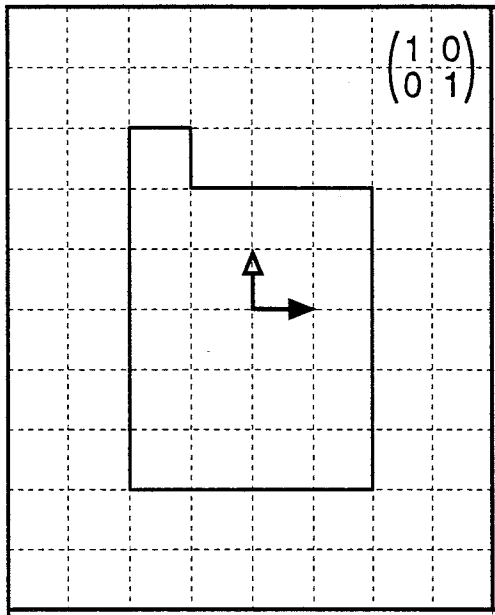

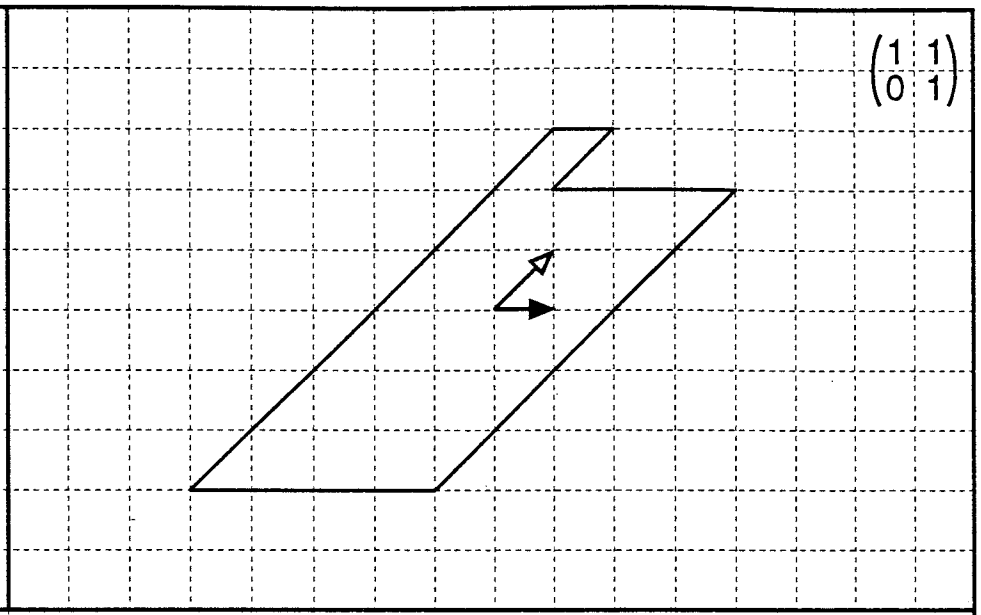

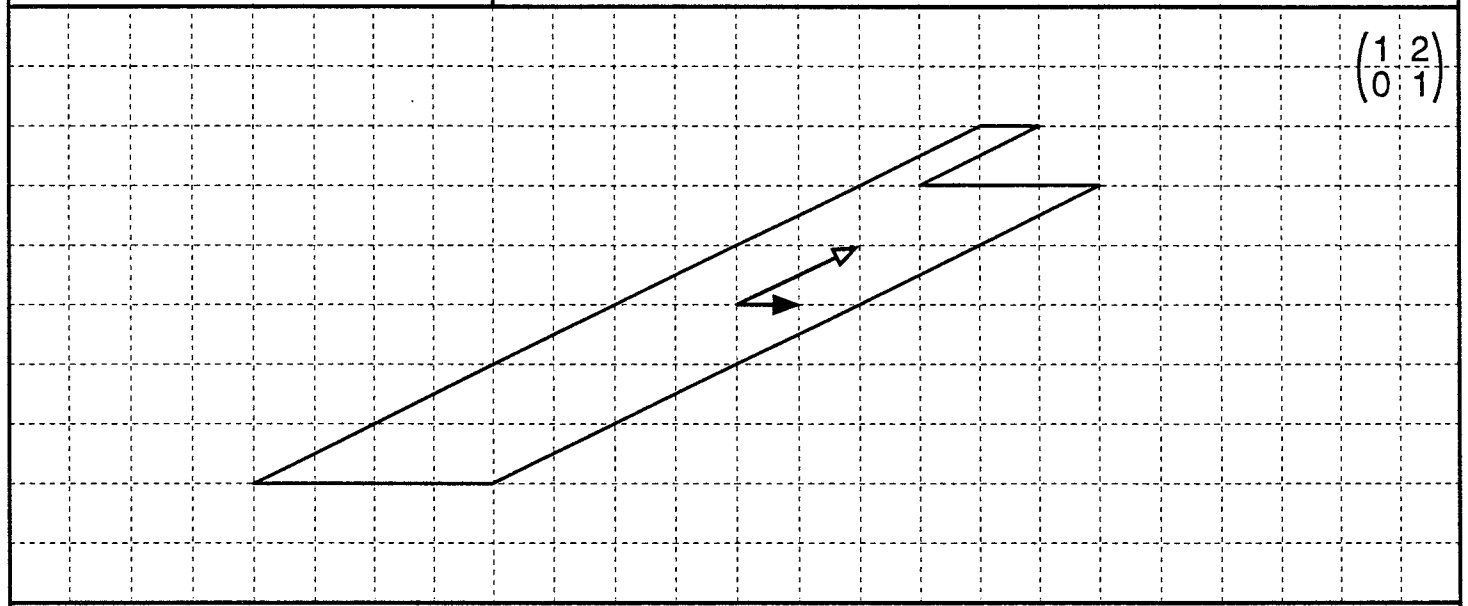

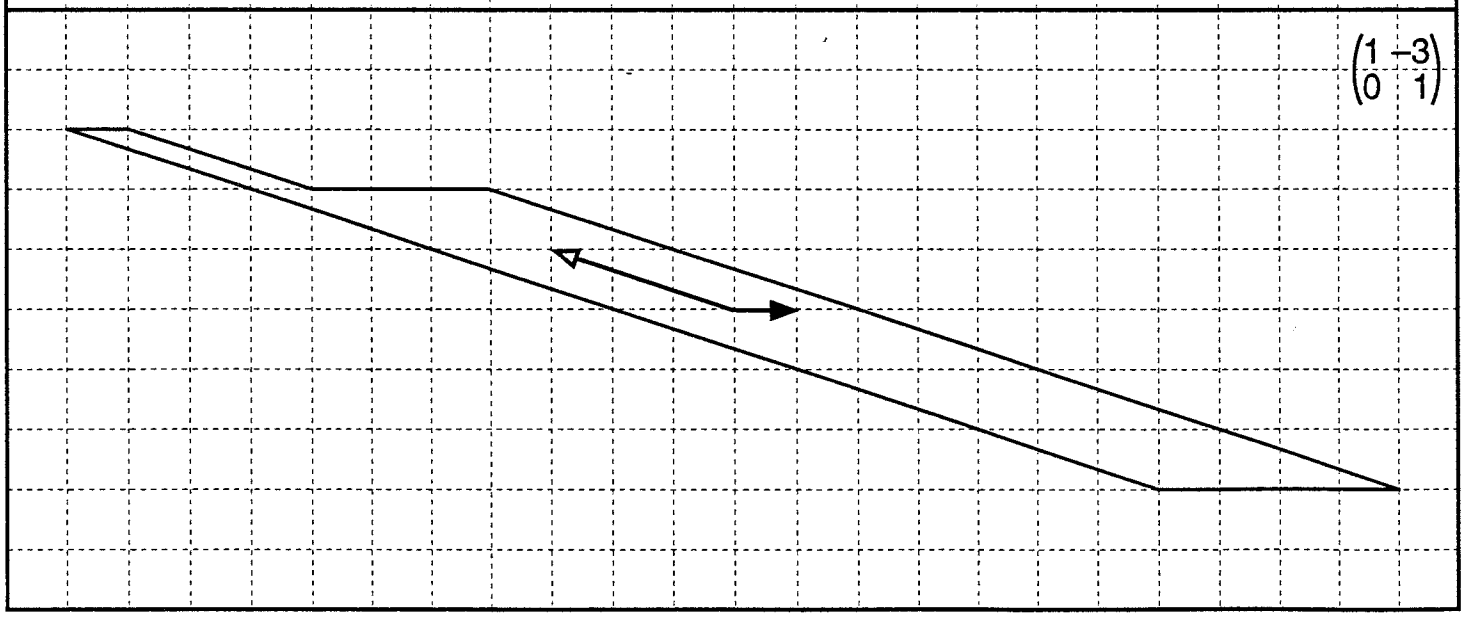

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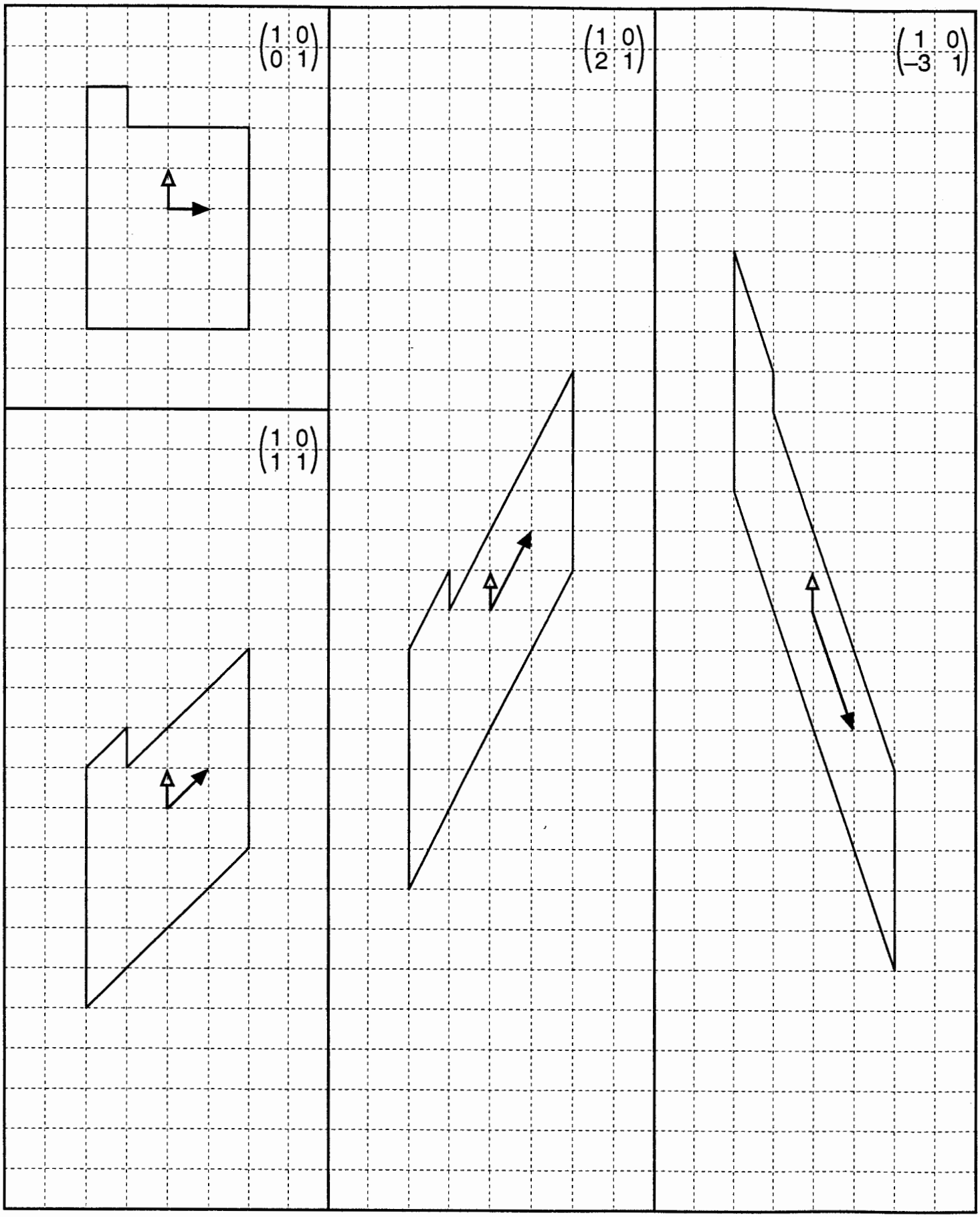

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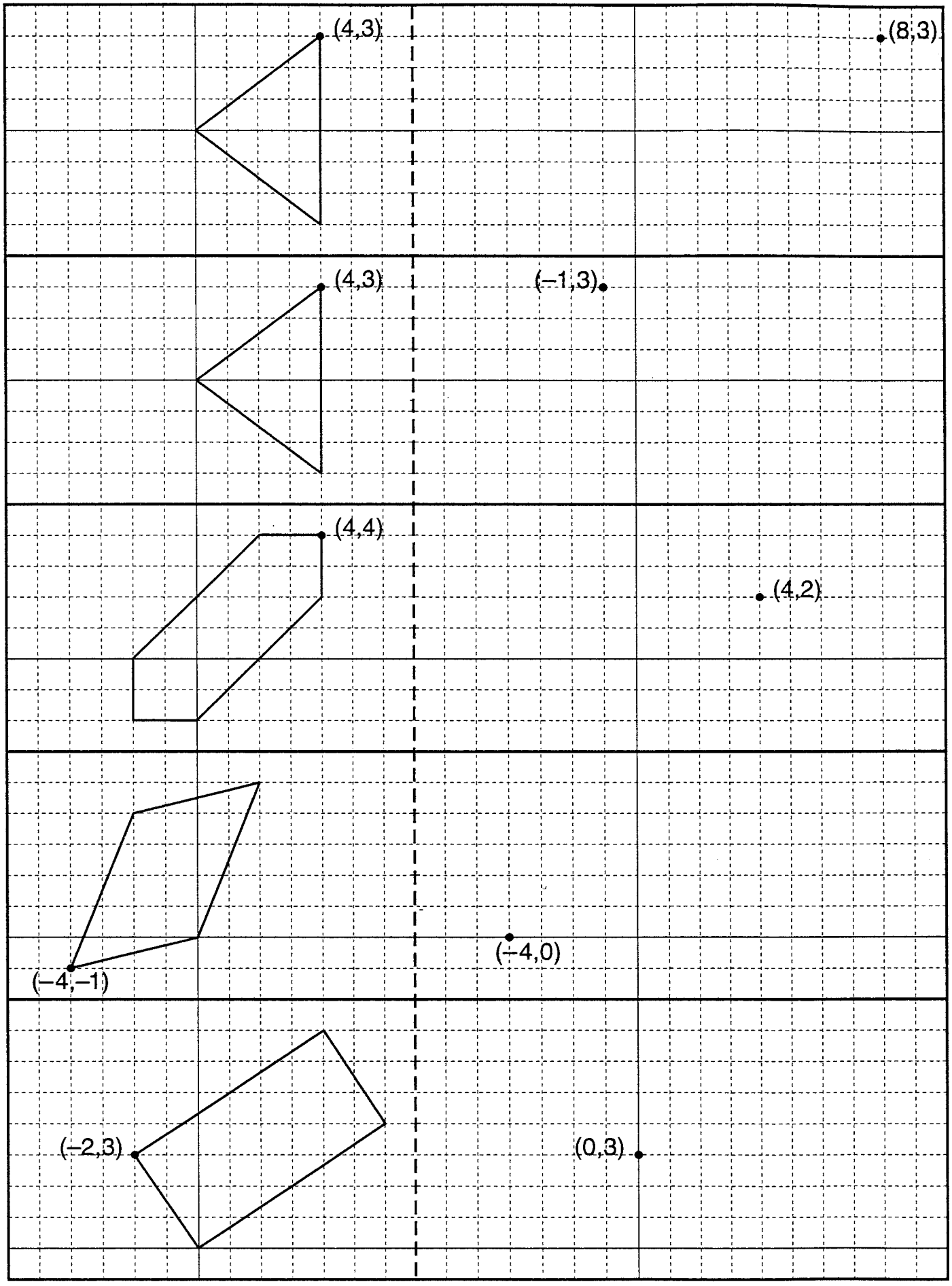

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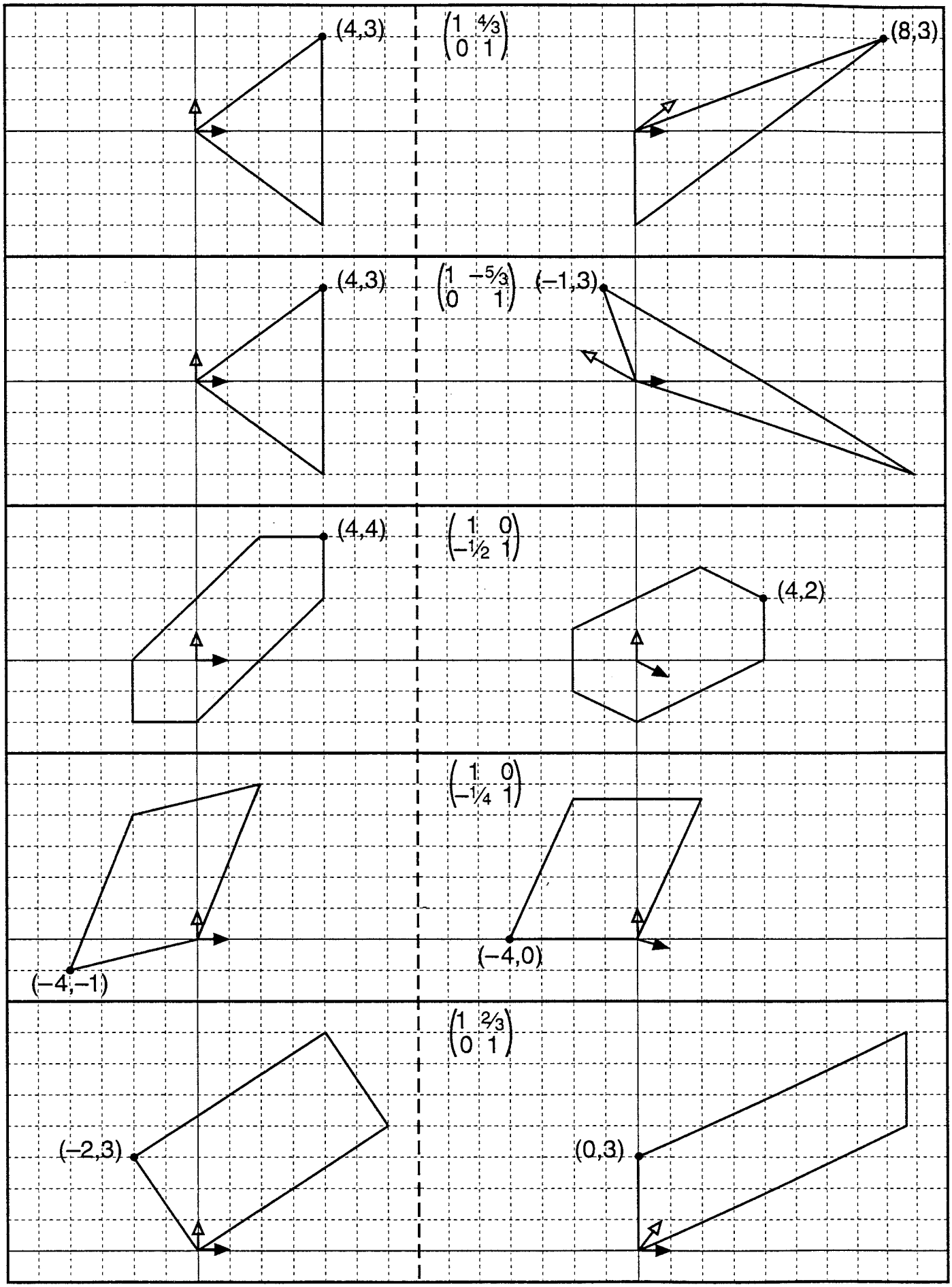

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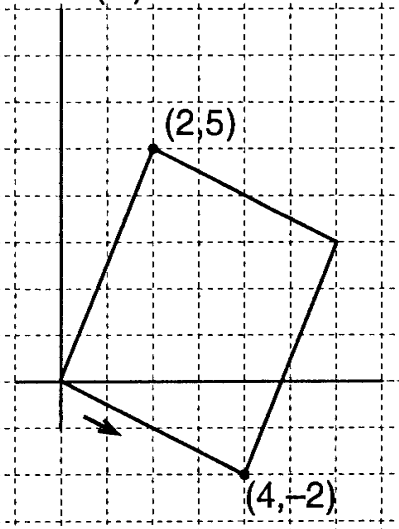
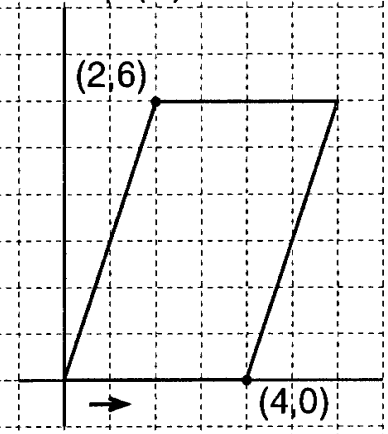
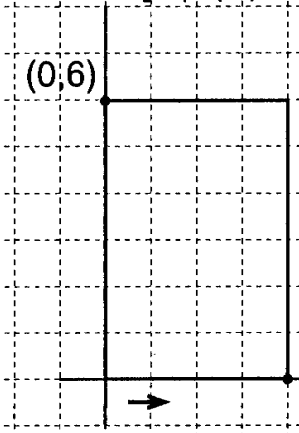


$$\mathbf{T} = \begin{pmatrix} 4 & 2 \\ -2 & 5 \end{pmatrix}$$



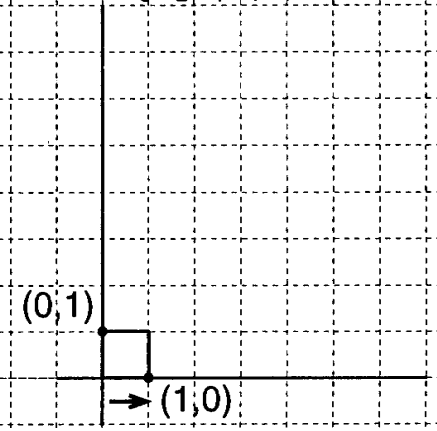
$\mathbf{R}_1$ moves (4,–2) to (4,0)

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$$



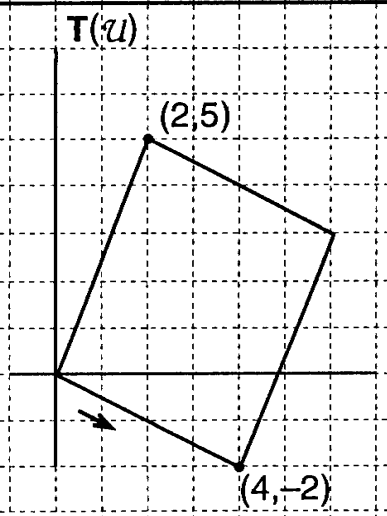
$\mathbf{R}_2$ moves (2,6) to (0,6)

$$\mathbf{R}_2 = \begin{pmatrix} 1 & -1/3 \\ 0 & 1 \end{pmatrix}$$

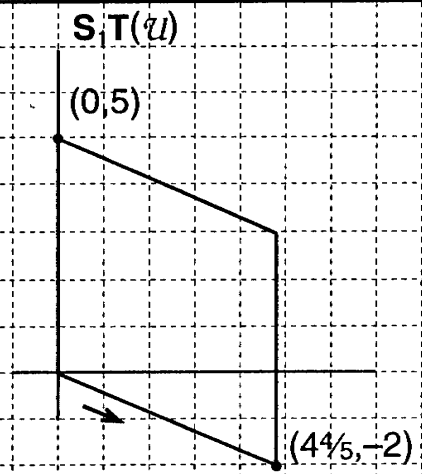


$\mathbf{R}_3$ reduces figure to u

$$\mathbf{R}_3 = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/6 \end{pmatrix}$$

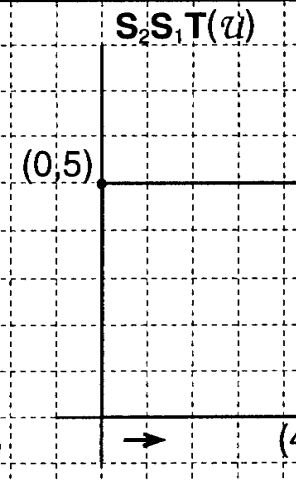


$$\mathbf{T} = \begin{pmatrix} 4 & 2 \\ -2 & 5 \end{pmatrix}$$



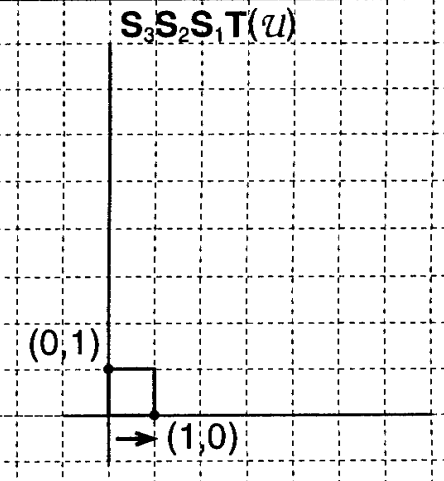
$\mathbf{S}_1$ moves (2,5) to (0,5)

$$\mathbf{S}_1 = \begin{pmatrix} 1 & -2/5 \\ 0 & 1 \end{pmatrix}$$



$\mathbf{S}_2$ moves (4⅘,–2) to (4⅘,0)

$$\mathbf{S}_2 = \begin{pmatrix} 1 & 0 \\ 5/12 & 1 \end{pmatrix}$$



$\mathbf{S}_3$ reduces figure to u

$$\mathbf{S}_3 = \begin{pmatrix} 5/24 & 0 \\ 0 & 1/5 \end{pmatrix}$$

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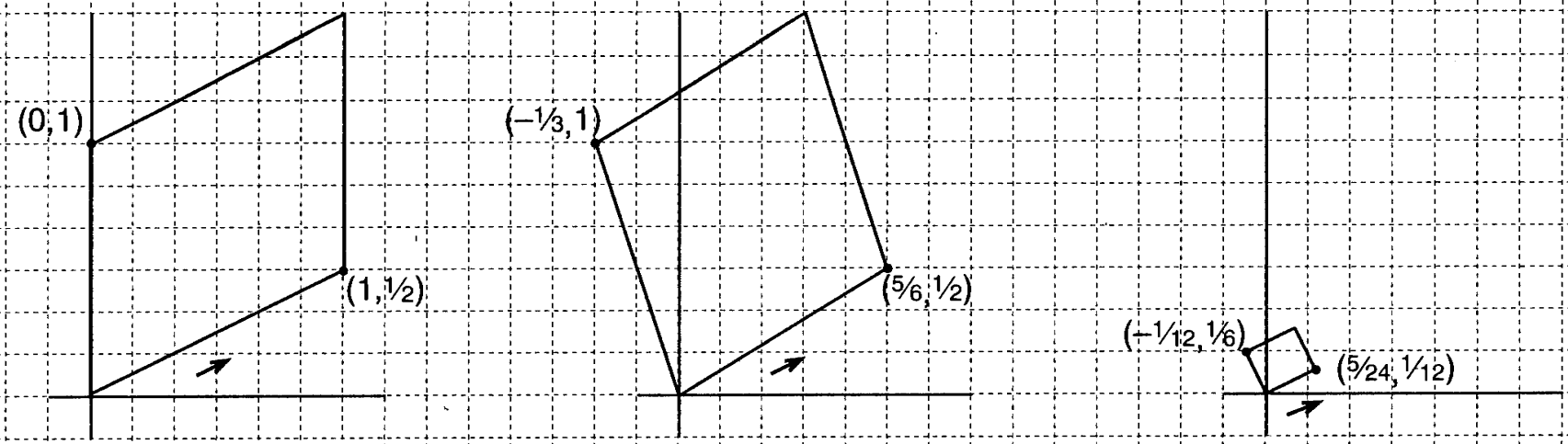
$$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} \qquad \mathbf{R}_2 = \begin{pmatrix} 1 & -1/3 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{R}_3 = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/6 \end{pmatrix}$$



$\mathbf{R}_1 = (\mathcal{U})$ $\qquad$ $\mathbf{R}_2\mathbf{R}_1 = (\mathcal{U})$ $\qquad$ $\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1 = (\mathcal{U})$

$$\mathbf{T}^{-1} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1 = \begin{pmatrix} 5/24 & -1/12 \\ 1/12 & 1/6 \end{pmatrix}$$

6 intervals = 1

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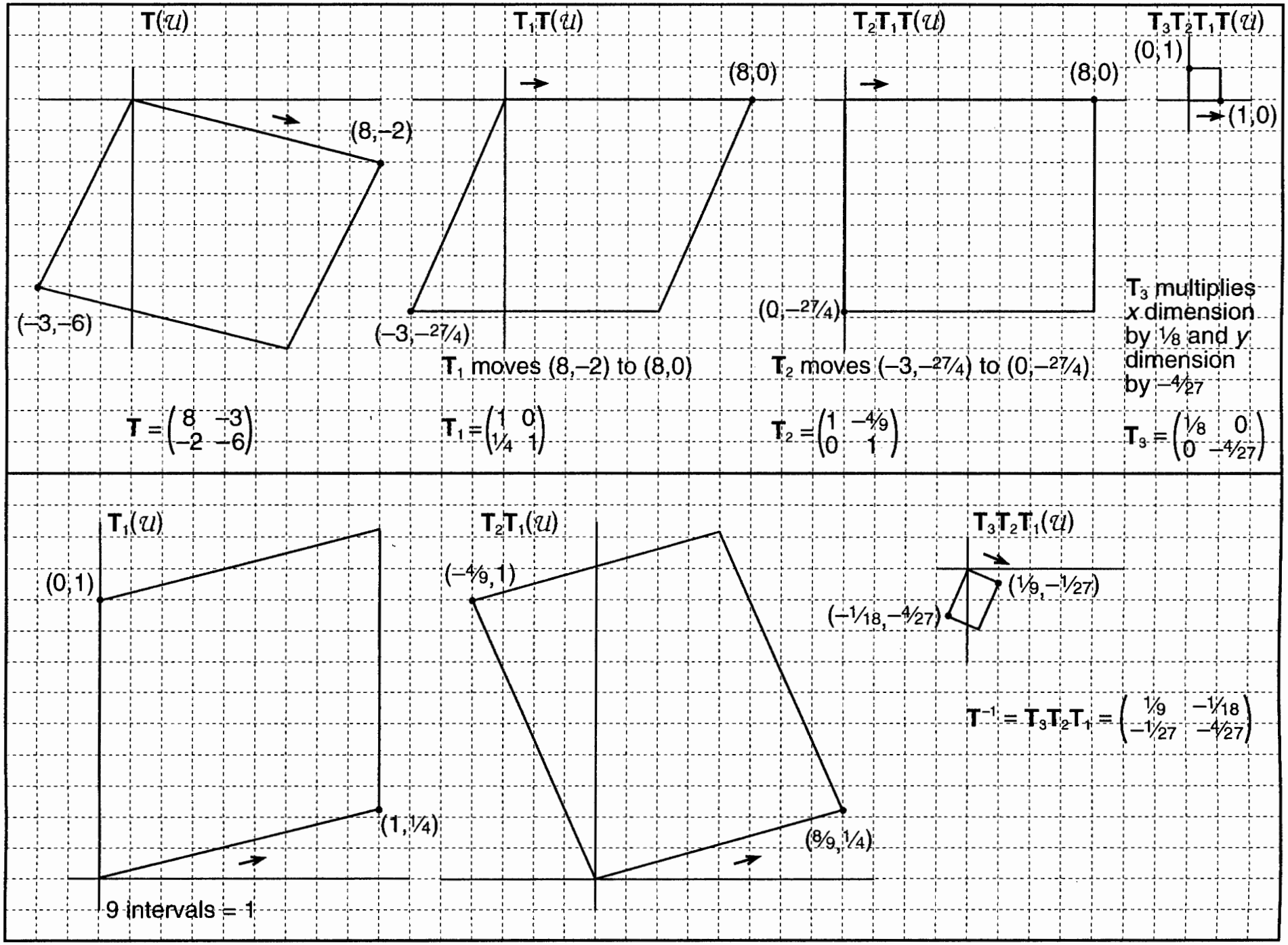

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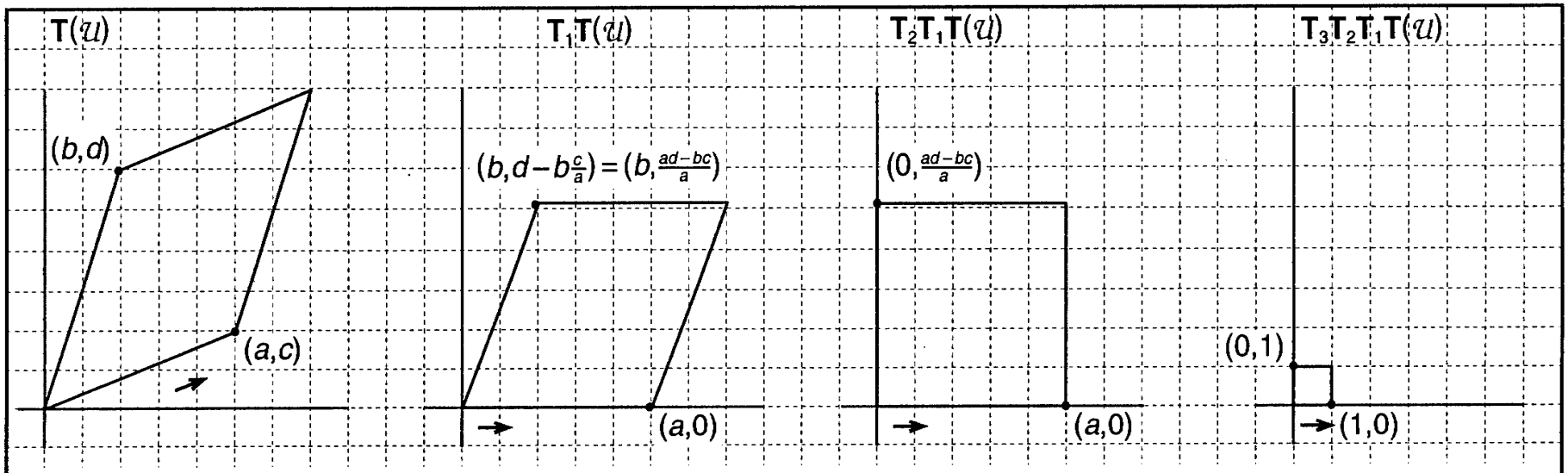
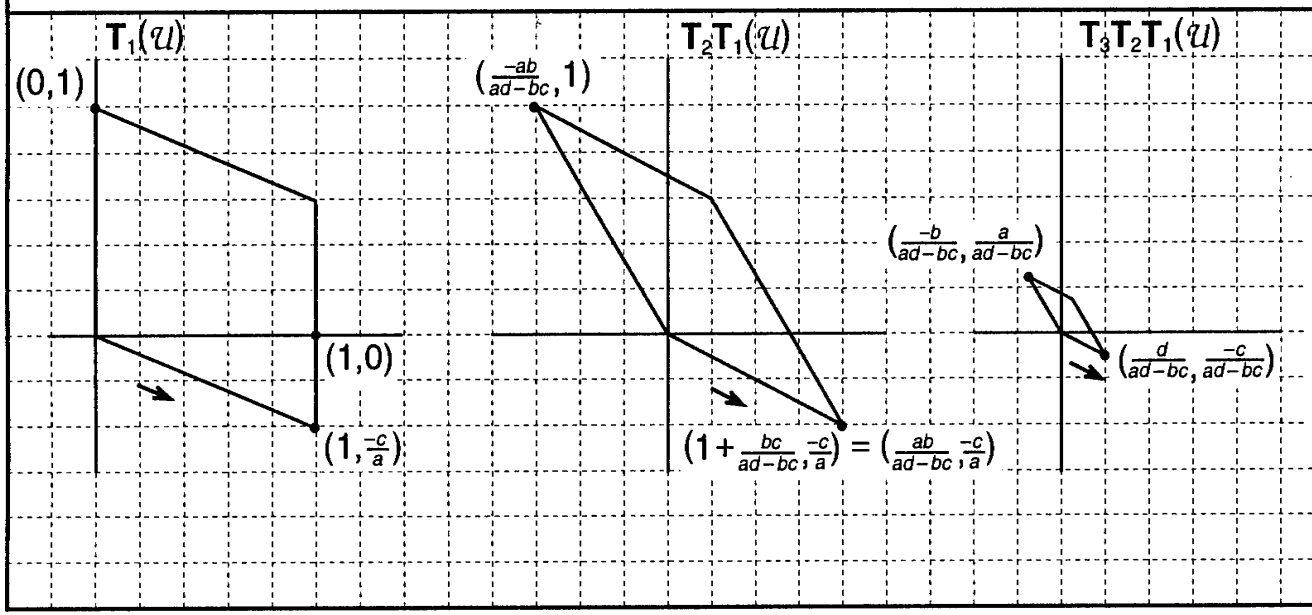


T_1 moves (a,c) to $(a,0)$

T_2 moves $(b,\frac{ad-bc}{a})$ to $(0,\frac{ad-bc}{a})$

T_3 reduces x and y dimensions by factors of a and $\frac{ad-bc}{a}$, respectively

$$T=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad T_1=\begin{pmatrix} 1 & 0 \\ \frac{-c}{a} & 1 \end{pmatrix} \qquad T_2=\begin{pmatrix} 1 & \frac{-ab}{ad-bc} \\ 0 & 1 \end{pmatrix} \qquad T_3=\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{a}{ad-bc} \end{pmatrix}$$



$$T^{-1}=T_3T_2T_1=\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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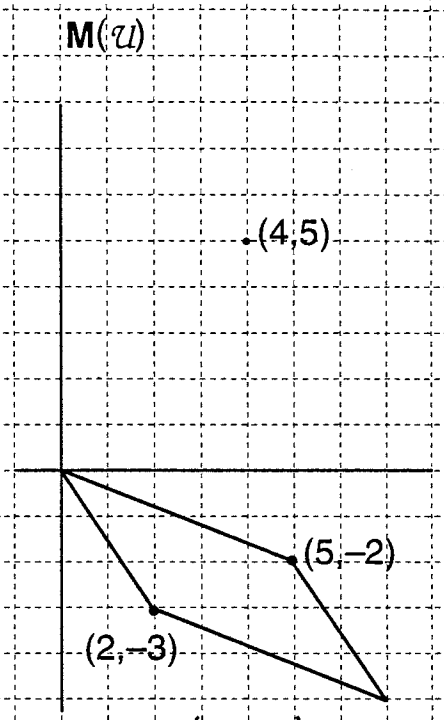
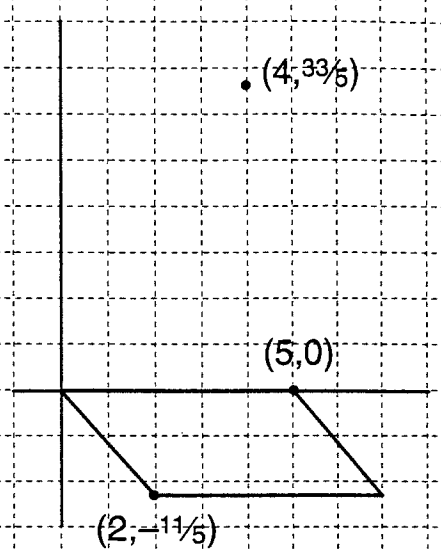
1.



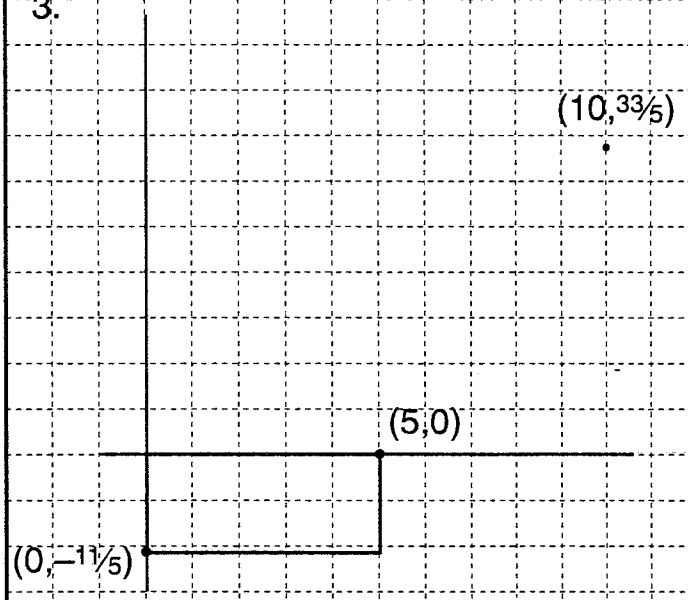
$$\mathbf{M} = \begin{pmatrix} 5 & 2 \\ -2 & -3 \end{pmatrix}$$

2.



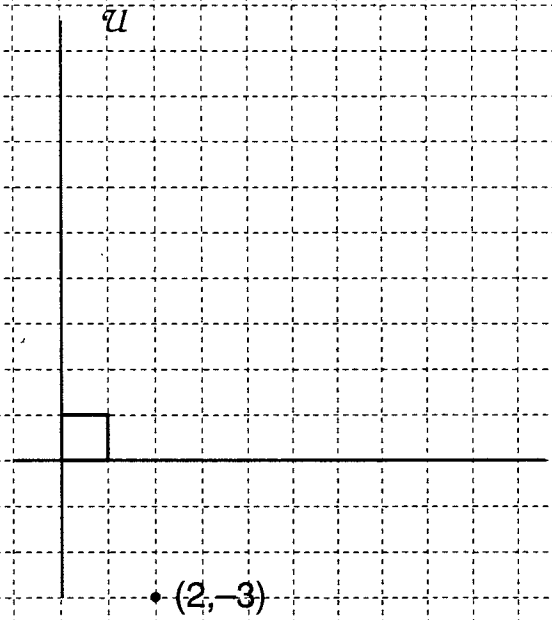
This shear moves (5,–2) up to (5,0), and thus moves (2,–3) up $\frac{2}{5}$ x 2 units to (2,$-\frac{11}{5}$) and moves (4,5) up $\frac{4}{5}$ x 2 units to (4,$\frac{33}{5}$).

3.



This shear moves (2,$-\frac{11}{5}$) to the left 2 units to (0,$-\frac{11}{5}$)) and moves (4,$\frac{33}{5}$) to the right 3 x 2 units to (10,$\frac{33}{5}$).

4.



This scaling divides *x* coordinates by 5 and *y* coordinates by $-\frac{11}{5}$. In particular, it moves (10,$\frac{33}{5}$) to (2, –3).

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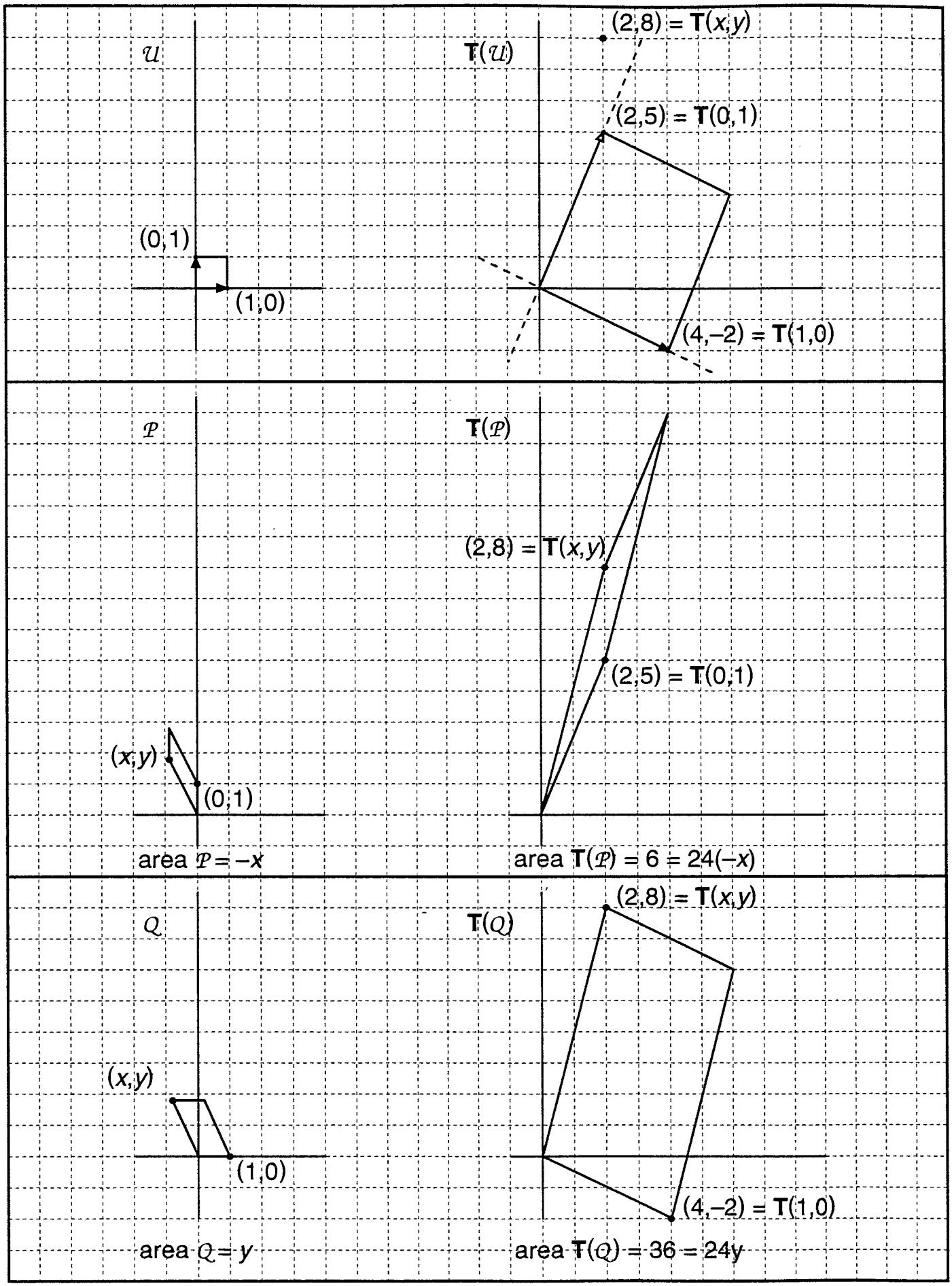

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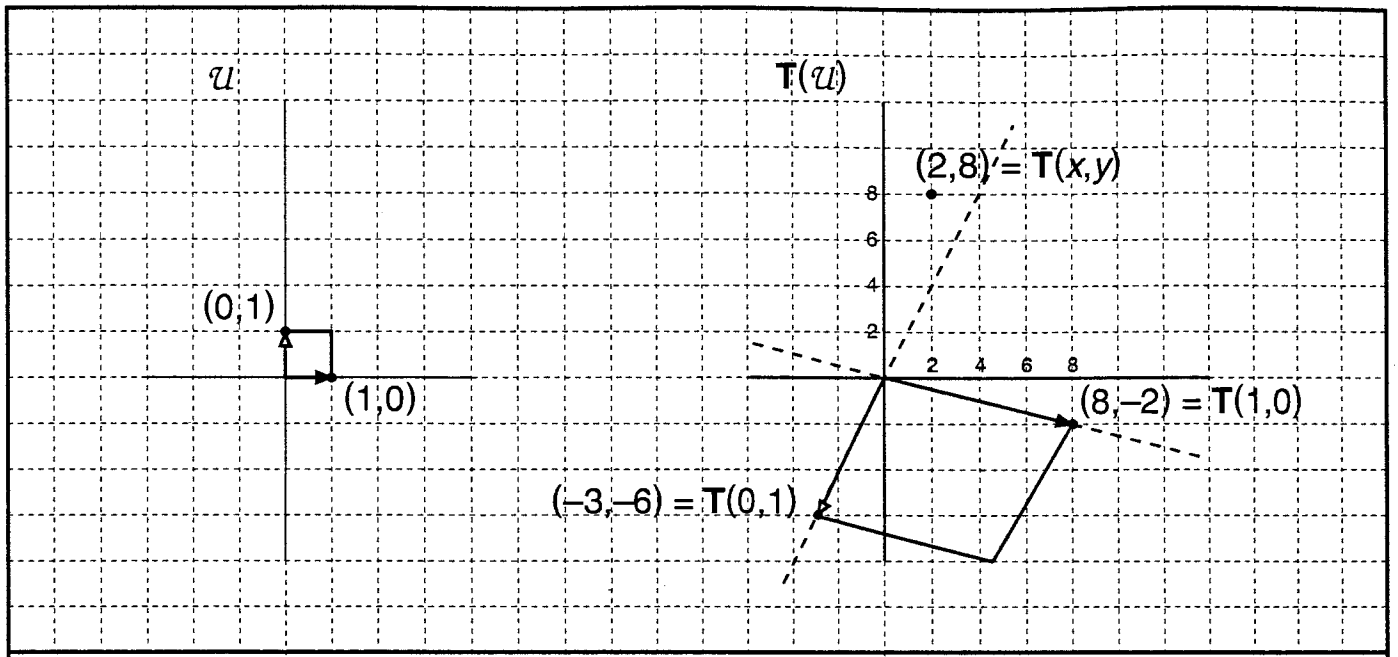

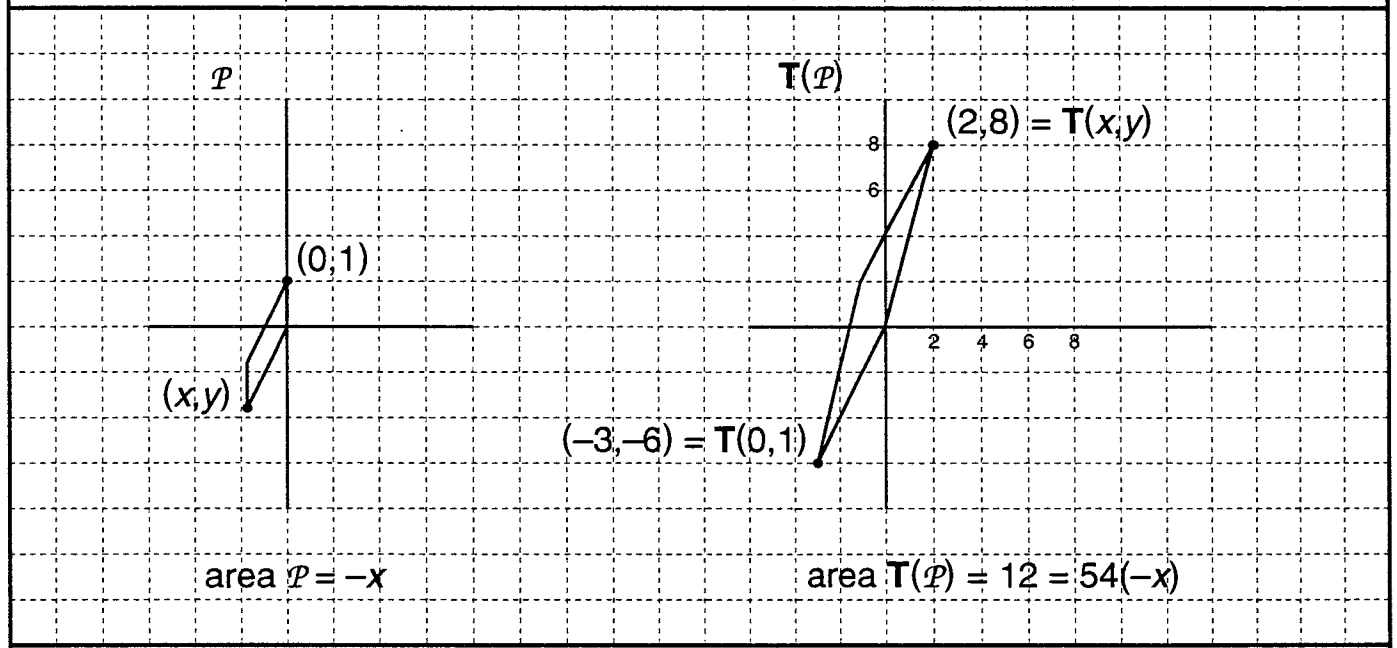

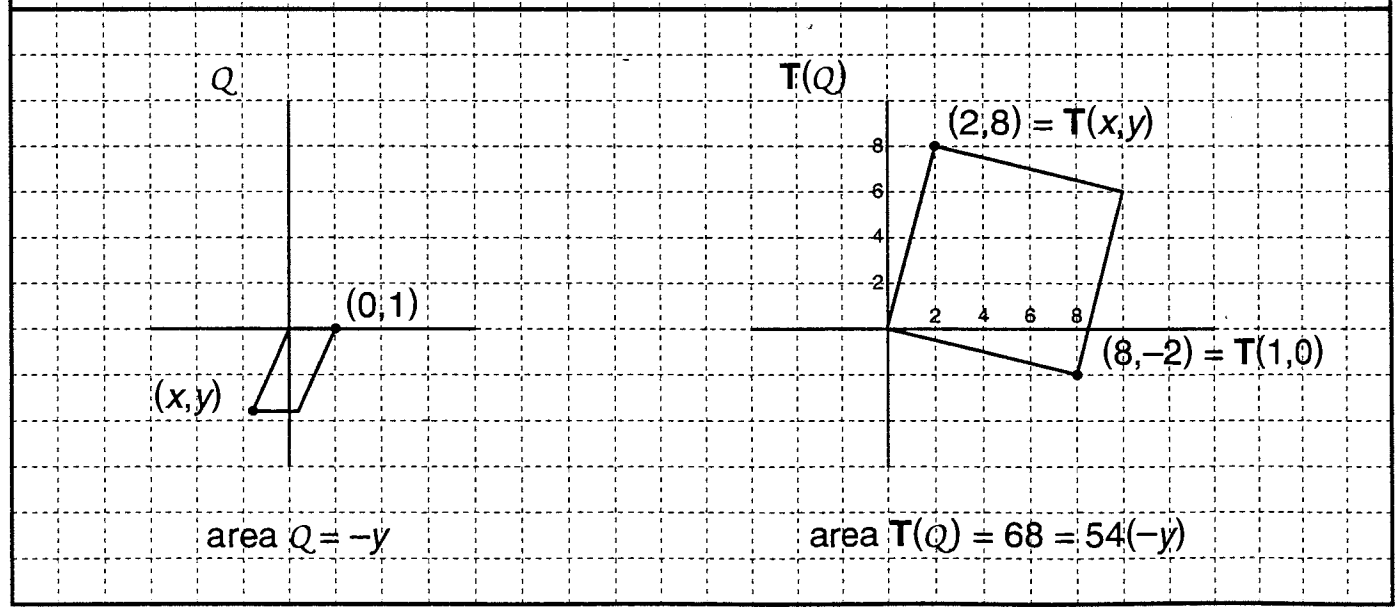

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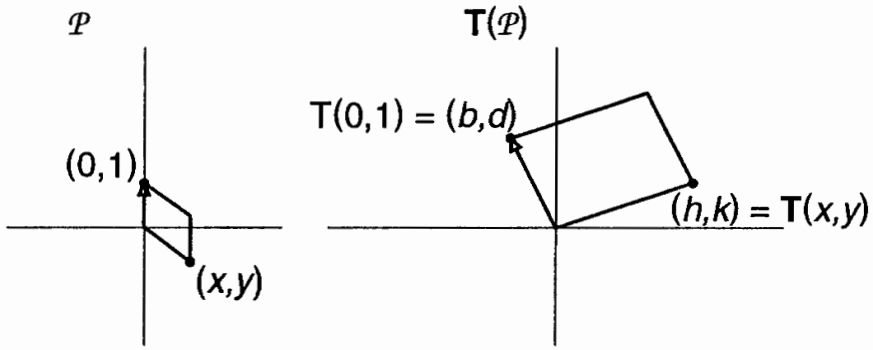
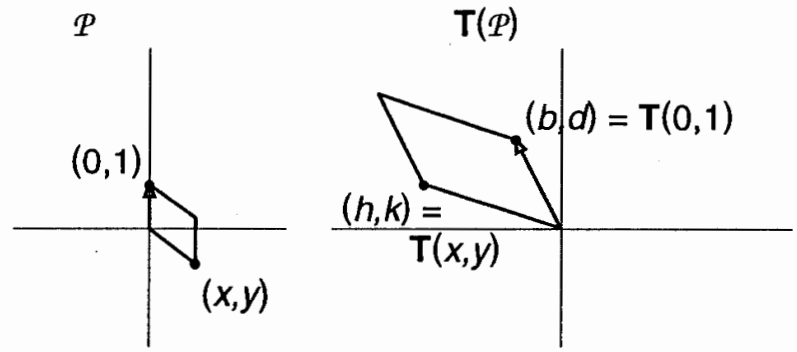
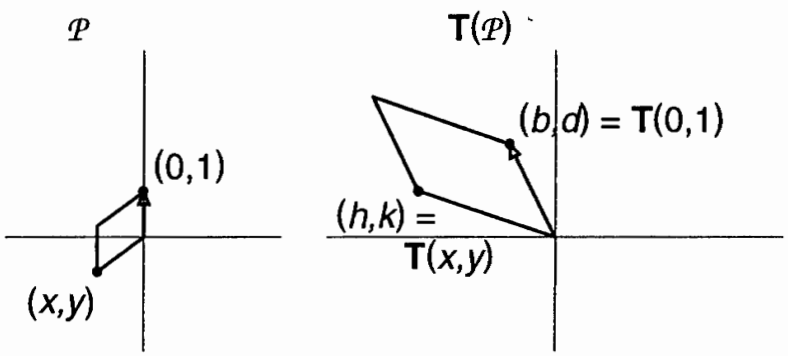
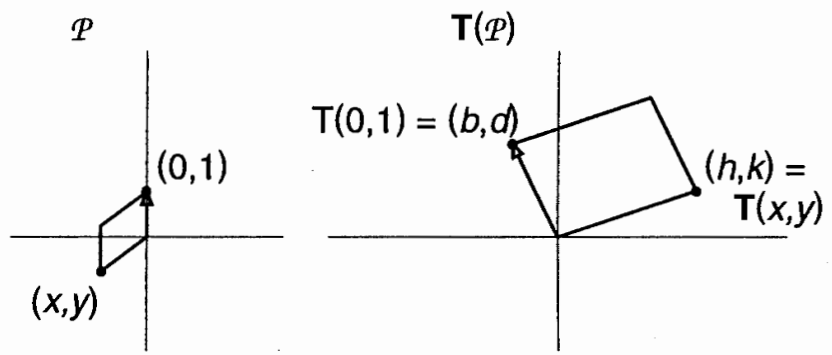
$\lvert\mathbf{T}\rvert > 0$; i.e., $\mathbf{T}$ does not change orientation	$\lvert\mathbf{T}\rvert < 0$; i.e., $\mathbf{T}$ changes orientation
Case 1. $x > 0$  area $\mathcal{P} = x$ area $\mathbf{T}(\mathcal{P}) = hd - kb = x\lvert\mathbf{T}\rvert$ $x = \frac{hd - kb}{\lvert\mathbf{T}\rvert}$	Case 1. $x > 0$  area $\mathcal{P} = x$ area $\mathbf{T}(\mathcal{P}) = bk - dh = x(-\lvert\mathbf{T}\rvert)$ $x = \frac{hd - bk}{\lvert\mathbf{T}\rvert}$
Case 2. $x < 0$  area $\mathcal{P} = -x$ area $\mathbf{T}(\mathcal{P}) = bk - dh = -x\lvert\mathbf{T}\rvert$ $x = \frac{hd - bk}{\lvert\mathbf{T}\rvert}$	Case 2. $x < 0$  area $\mathcal{P} = -x$ area $\mathbf{T}(\mathcal{P}) = hd - kb = (-x)(-\lvert\mathbf{T}\rvert)$ $x = \frac{hd - bk}{\lvert\mathbf{T}\rvert}$

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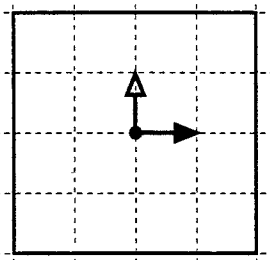
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity



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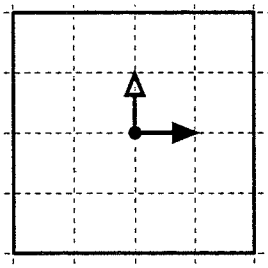
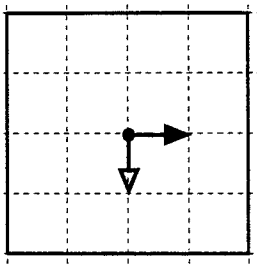
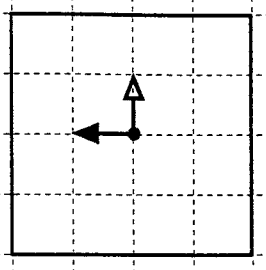
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity



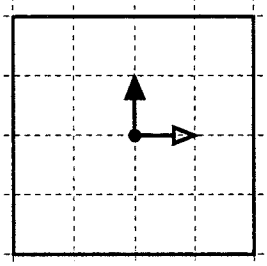
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection about x-axis



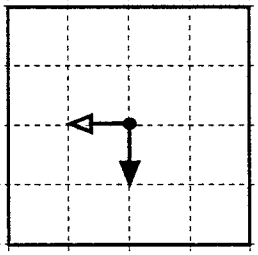
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection about y-axis



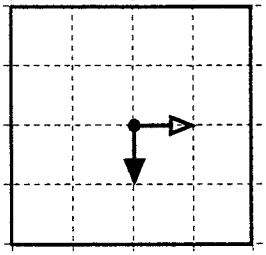
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Reflection about $x = y$



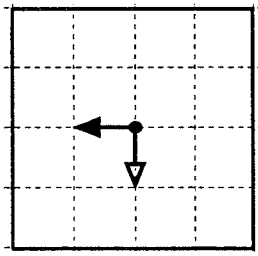
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ Reflection about $x = -y$



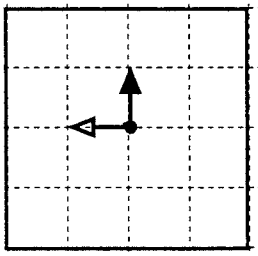
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Rotation 90° clockwise about origin



$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Rotation 180° clockwise about origin



$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Rotation 270° clockwise about origin



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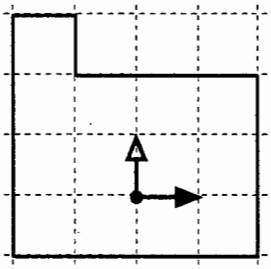
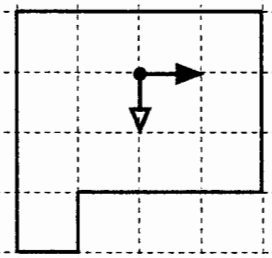
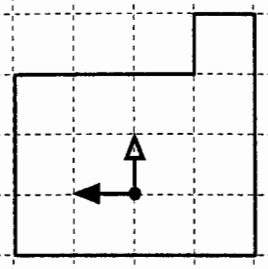
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity



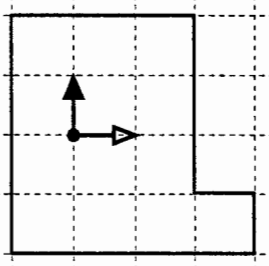
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection about x-axis



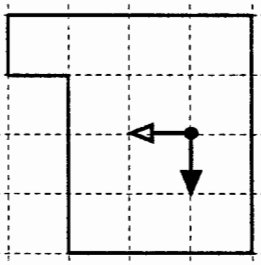
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Reflection about y-axis



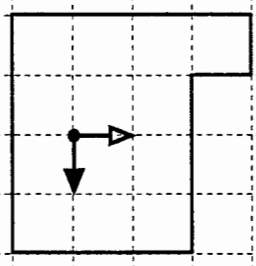
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Reflection about $x = y$



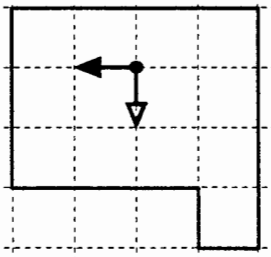
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ Reflection about $x = -y$



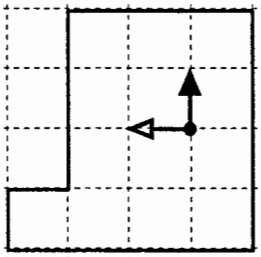
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Rotation 90° clockwise about origin



$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Rotation 180° clockwise about origin



$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Rotation 270° clockwise about origin



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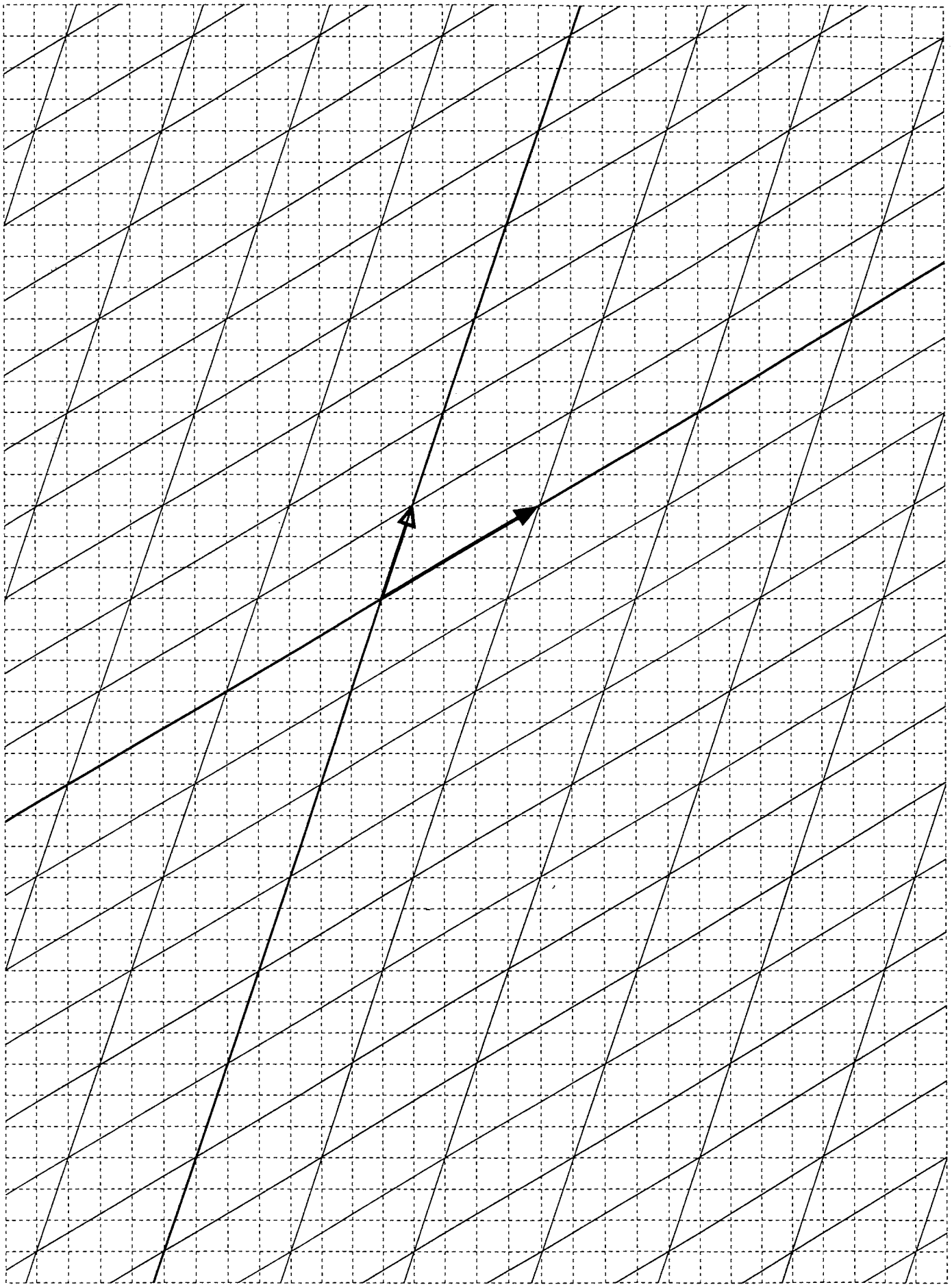
S-Grid



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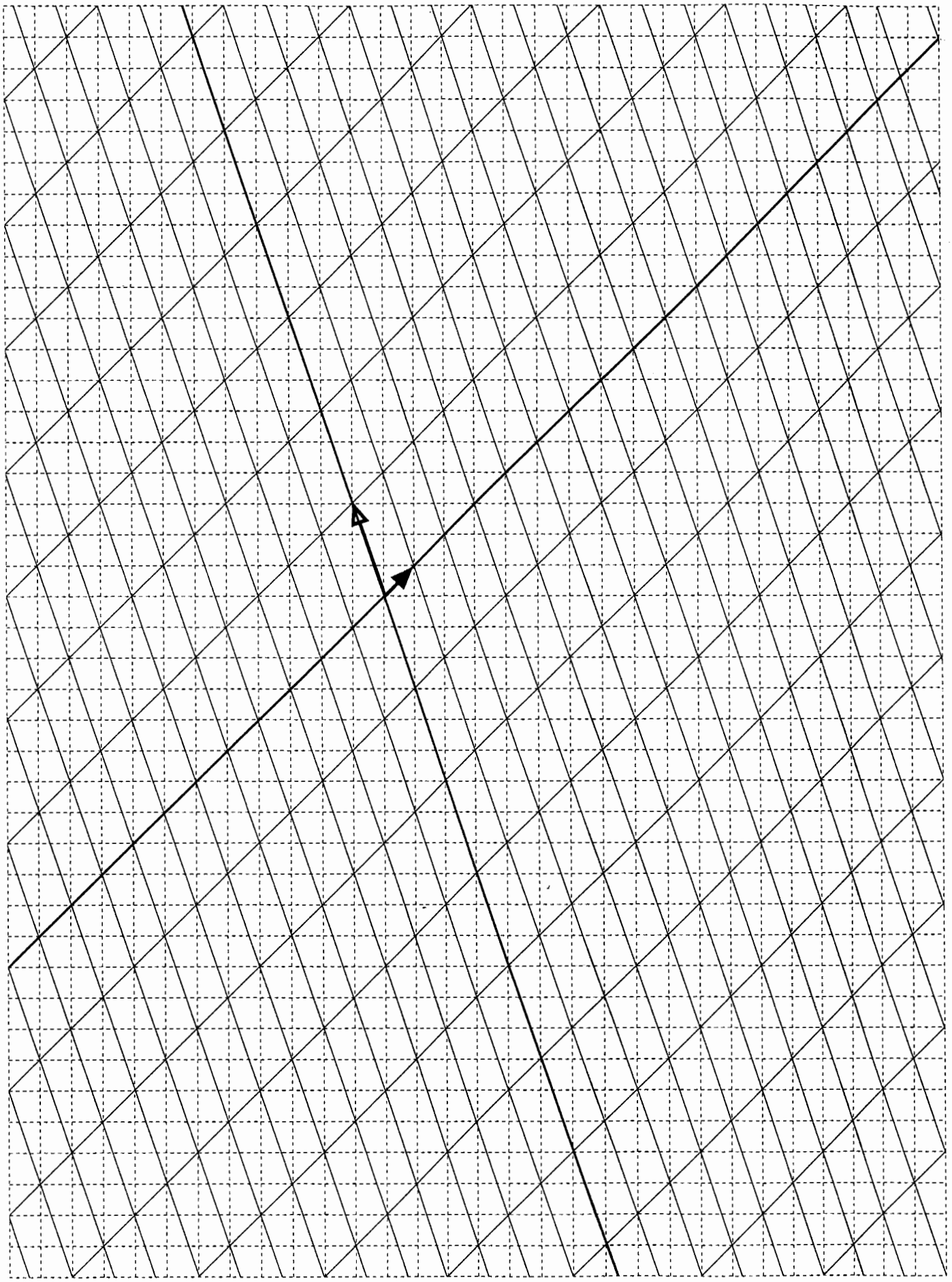
E-Grid



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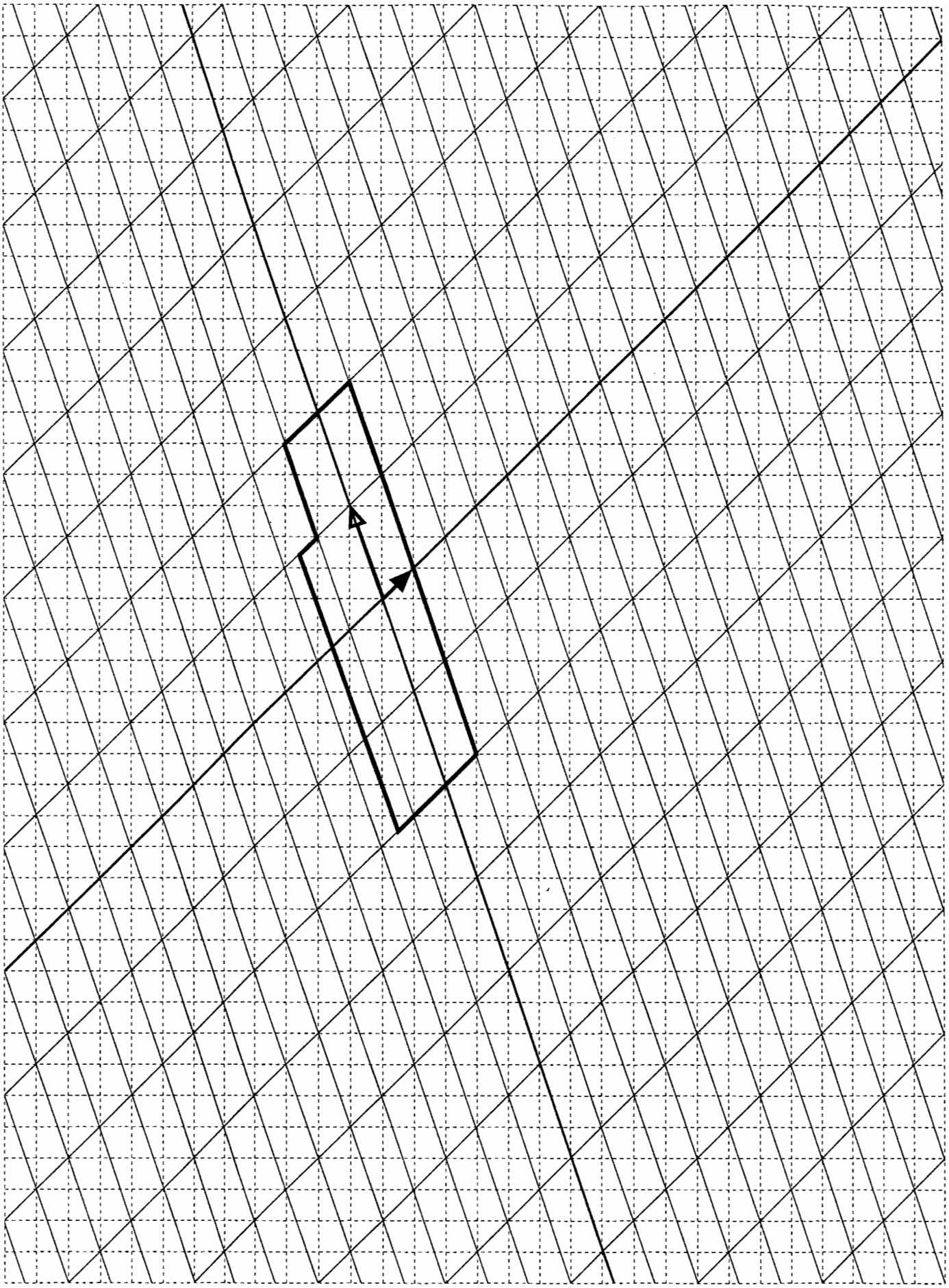
E-Grid



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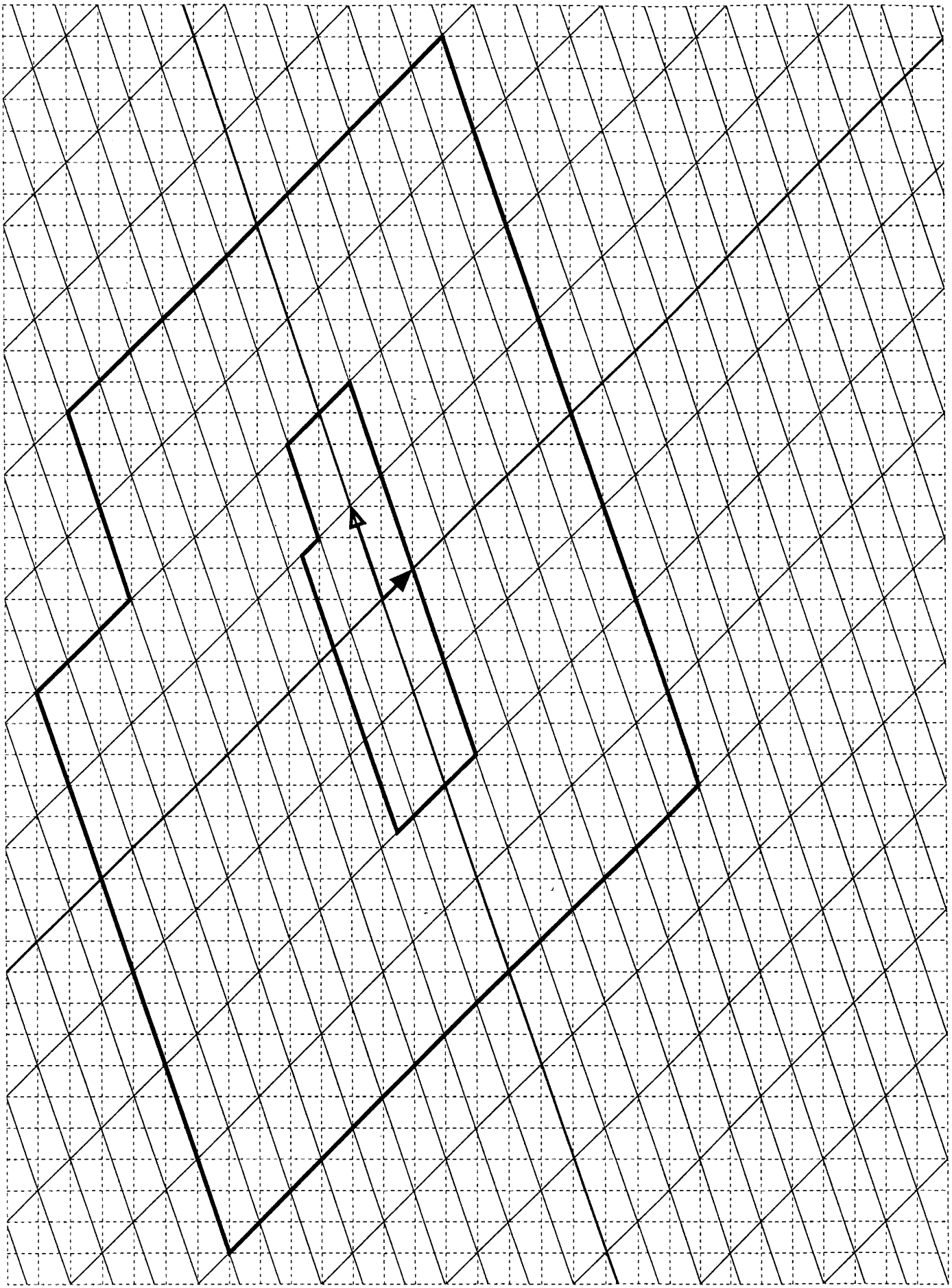
E-Grid



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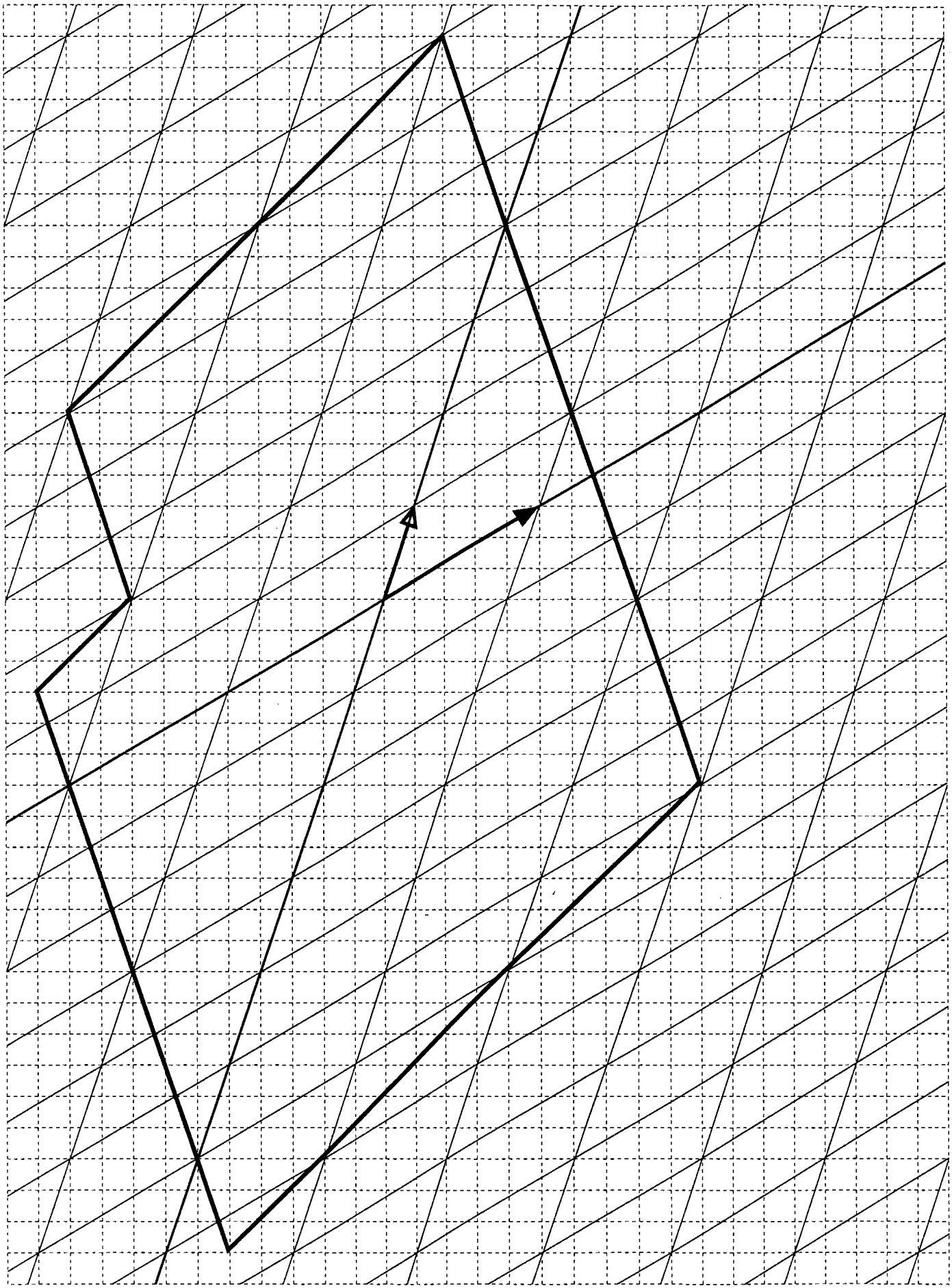
S-Grid



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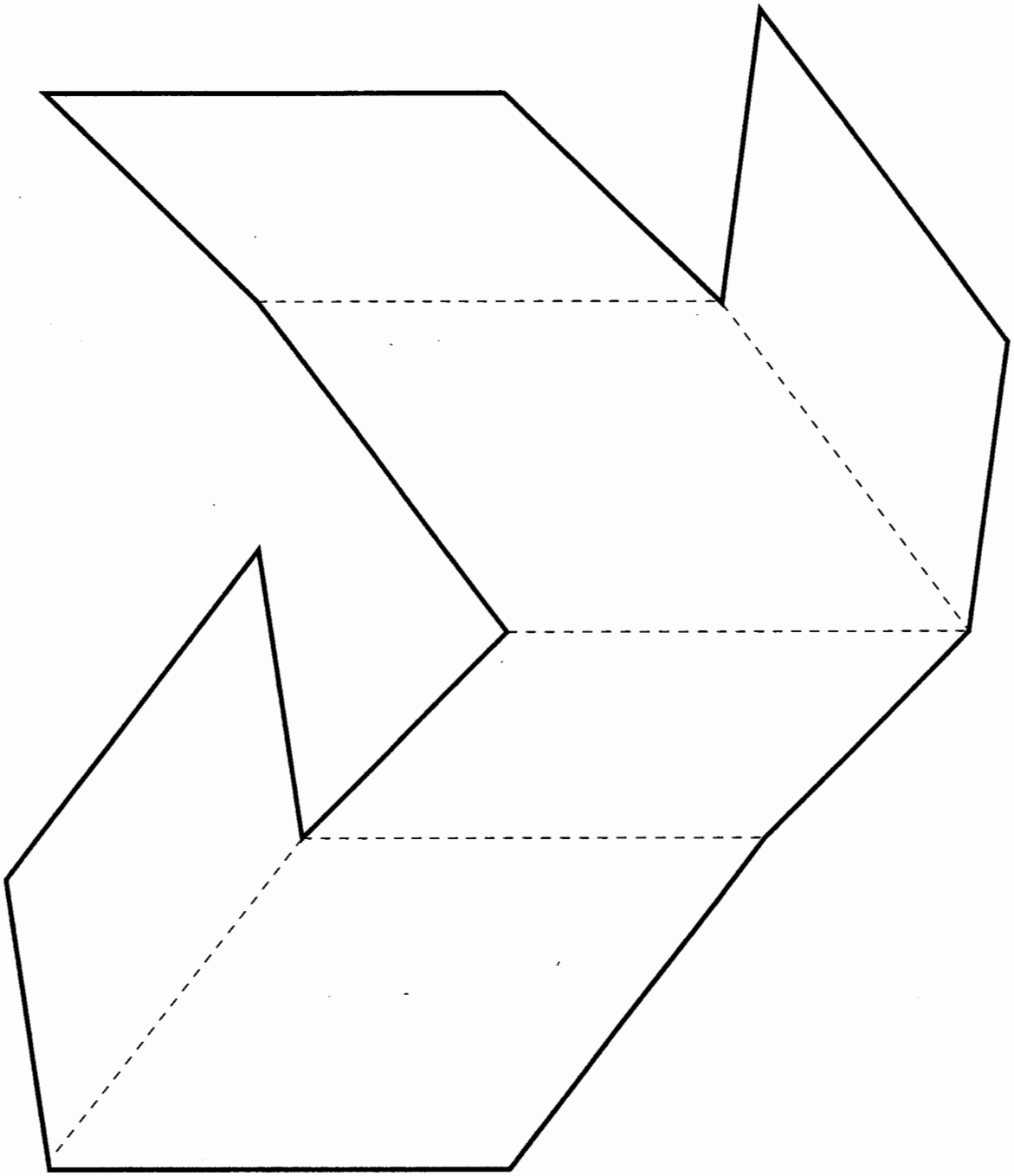

Parallelepiped Net

Cut on solid lines. Fold on dashed lines.
Fold into parallelepiped. Tape edges.

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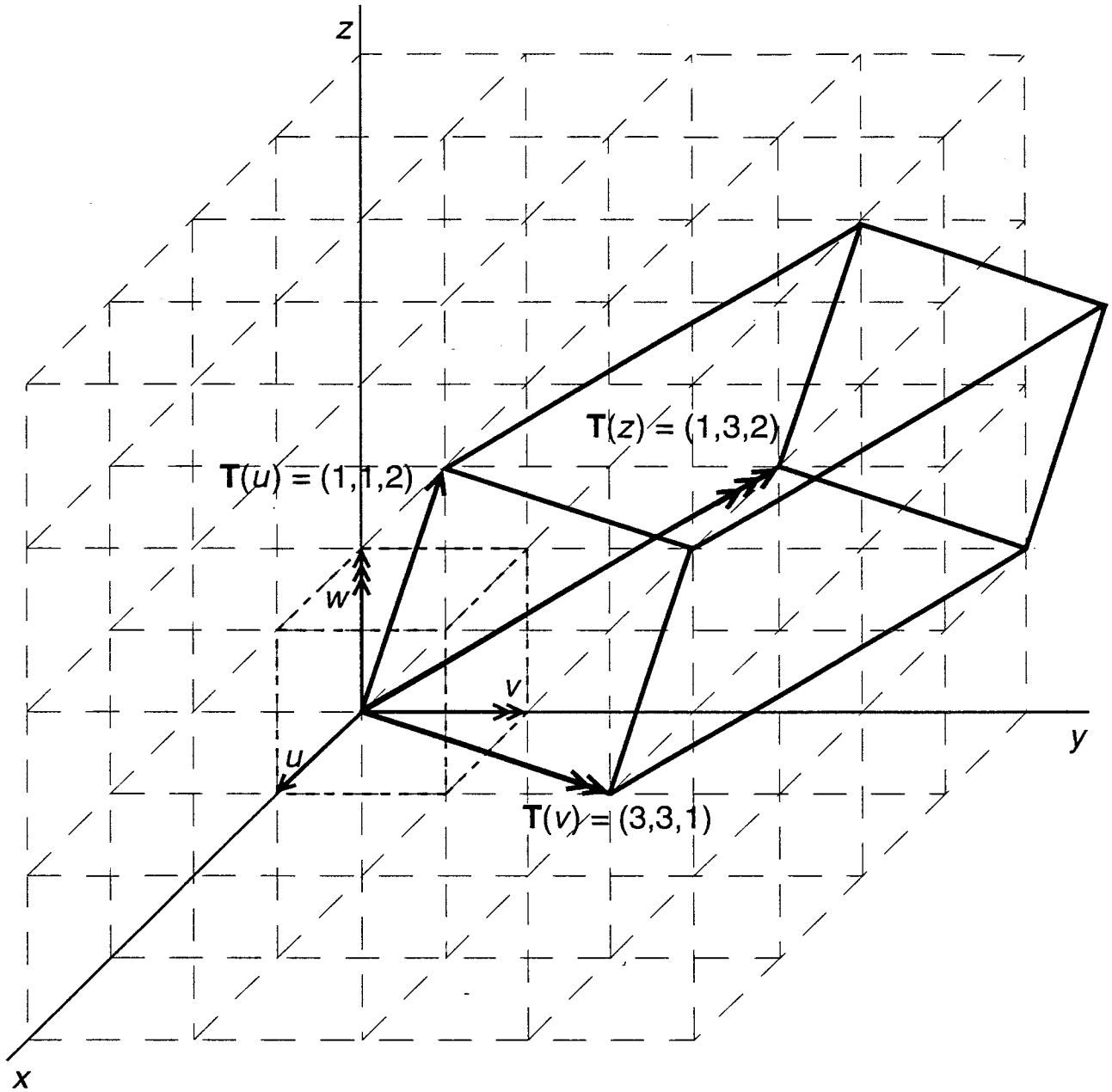

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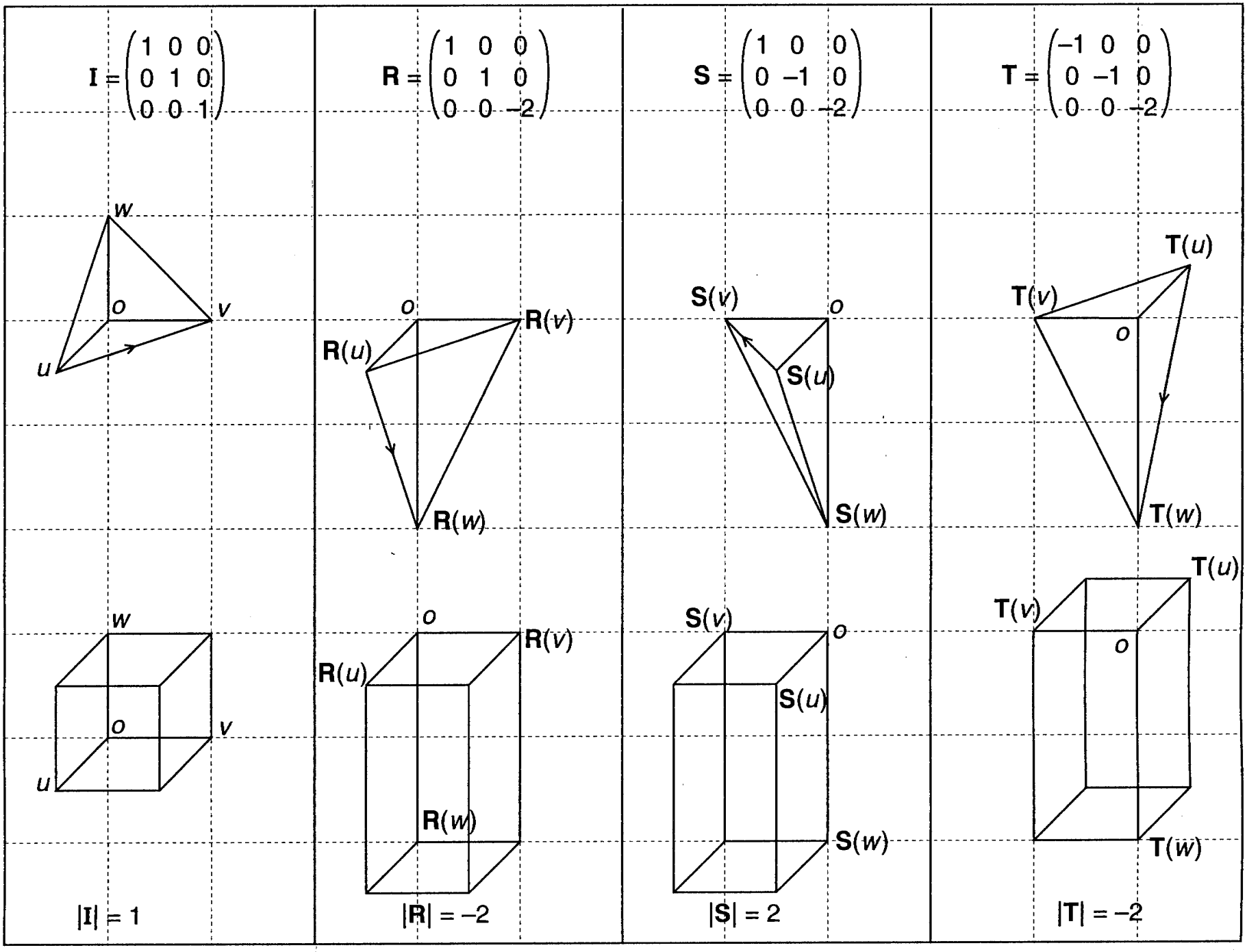
$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$



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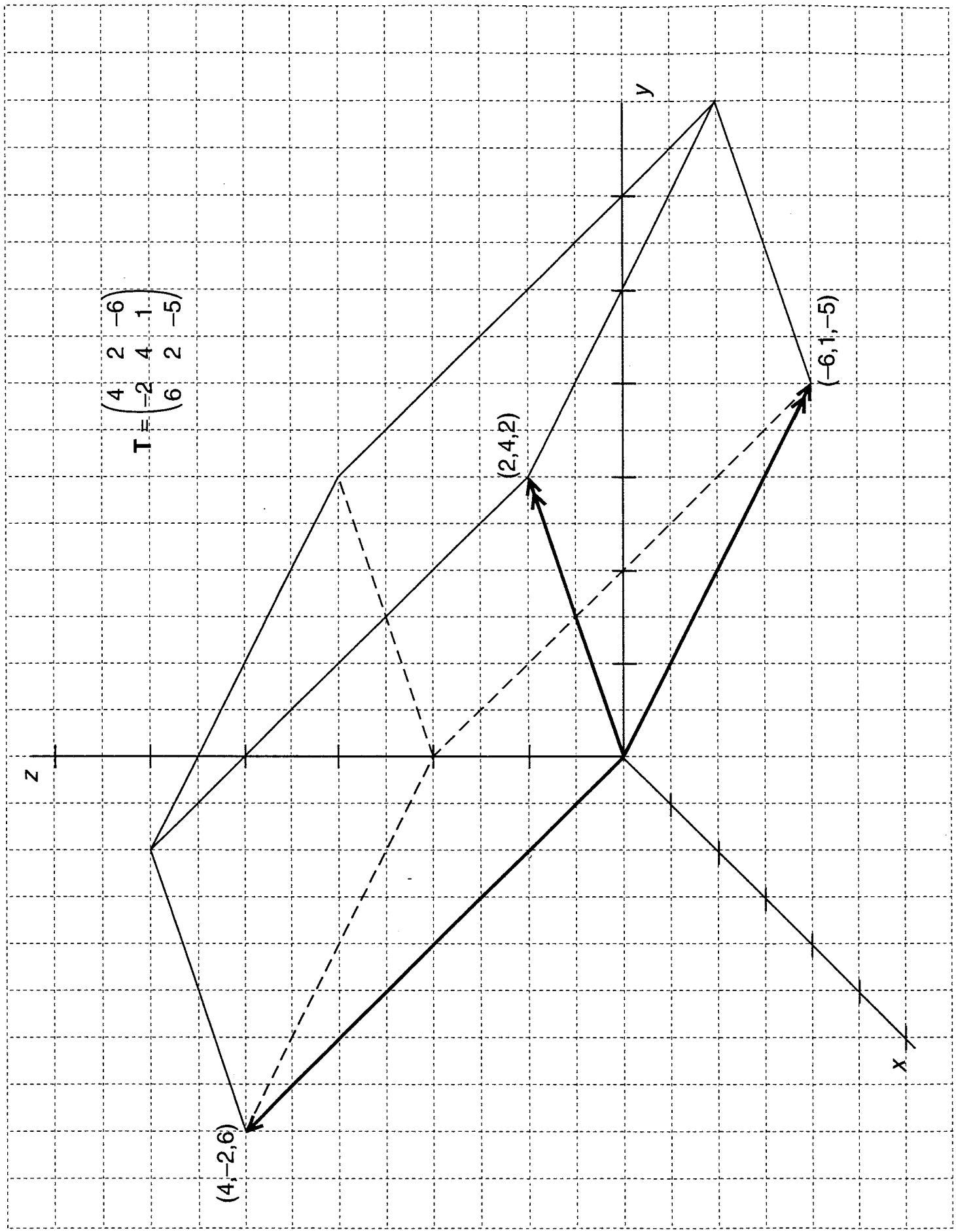

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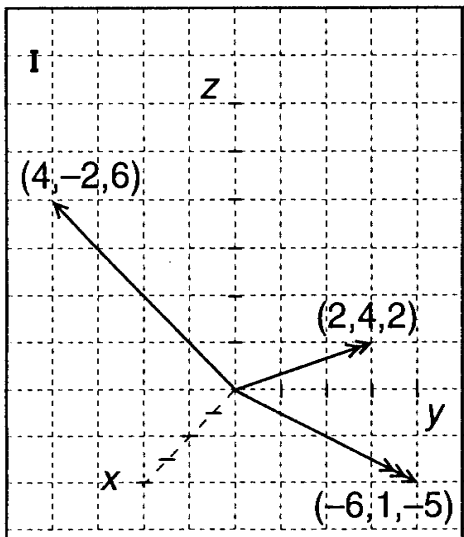
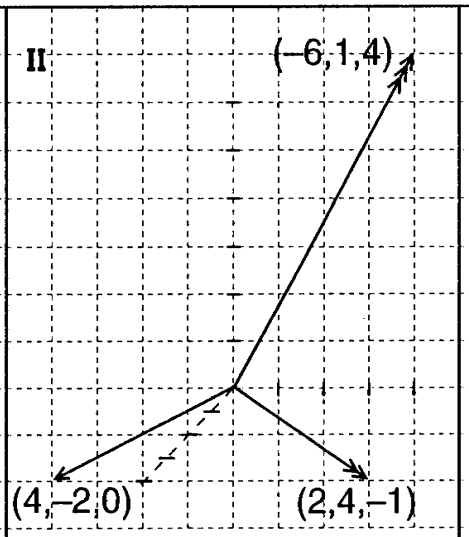
I



$$T = \begin{pmatrix} 4 & 2 & -6 \\ -2 & 4 & 1 \\ 6 & 2 & -5 \end{pmatrix}$$

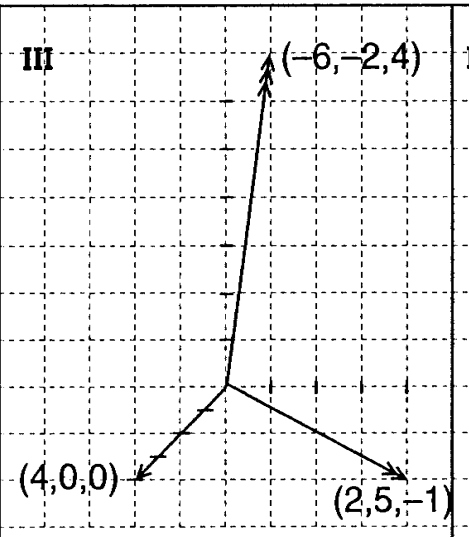
II



Shear parallel to *yz*-plane in negative *z* direction to move (4,–2,6) to (4,–2,0). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix}$$

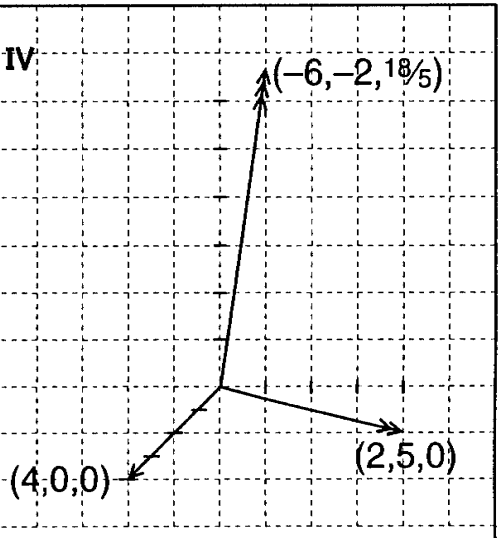
III



Shear parallel to *yz*-plane in positive *y* direction to move (4,–2,0) to (4,0,0). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

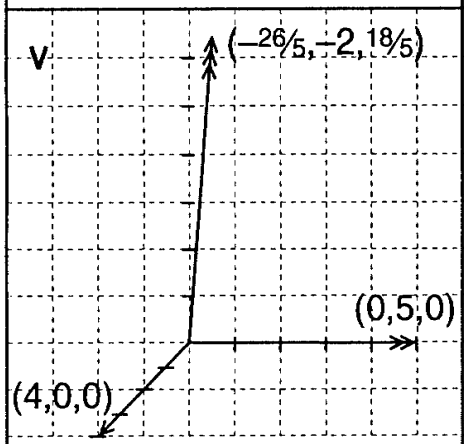
IV



Shear parallel to *xz*-plane in positive *z* direction to move (2,5,–1) to (2,5,0). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/5 & 1 \end{pmatrix}$$

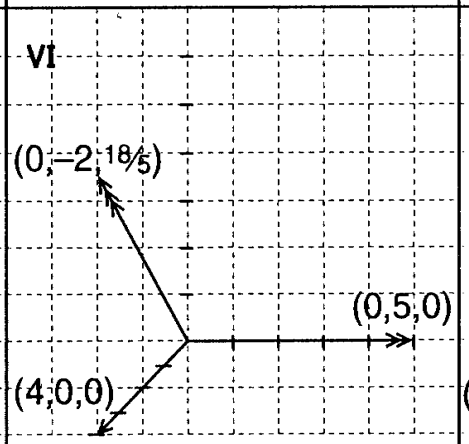
V



Shear parallel to *xz*-plane in negative *z* direction to move (2,5,0) to (0,5,0). The matrix of the transformation is

$$\begin{pmatrix} 1 & -2/5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

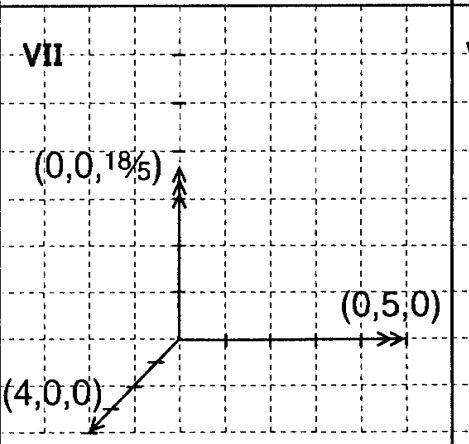
VI



Shear parallel to *xy*-plane in positive *x* direction to move (–26/5,–2,18/5) to (0,–2,18/5). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 13/9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

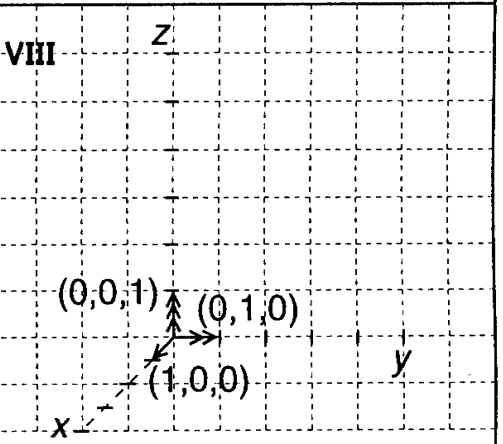
VII



Shear parallel to *xy*-plane in positive *y* direction to move (0,–2,18/5) to (0,0,18/5). The matrix of the transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5/9 \\ 0 & 0 & 1 \end{pmatrix}$$

VIII



Scale vectors to unit length. The matrix of the transformation is

$$\begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 5/18 \end{pmatrix}$$

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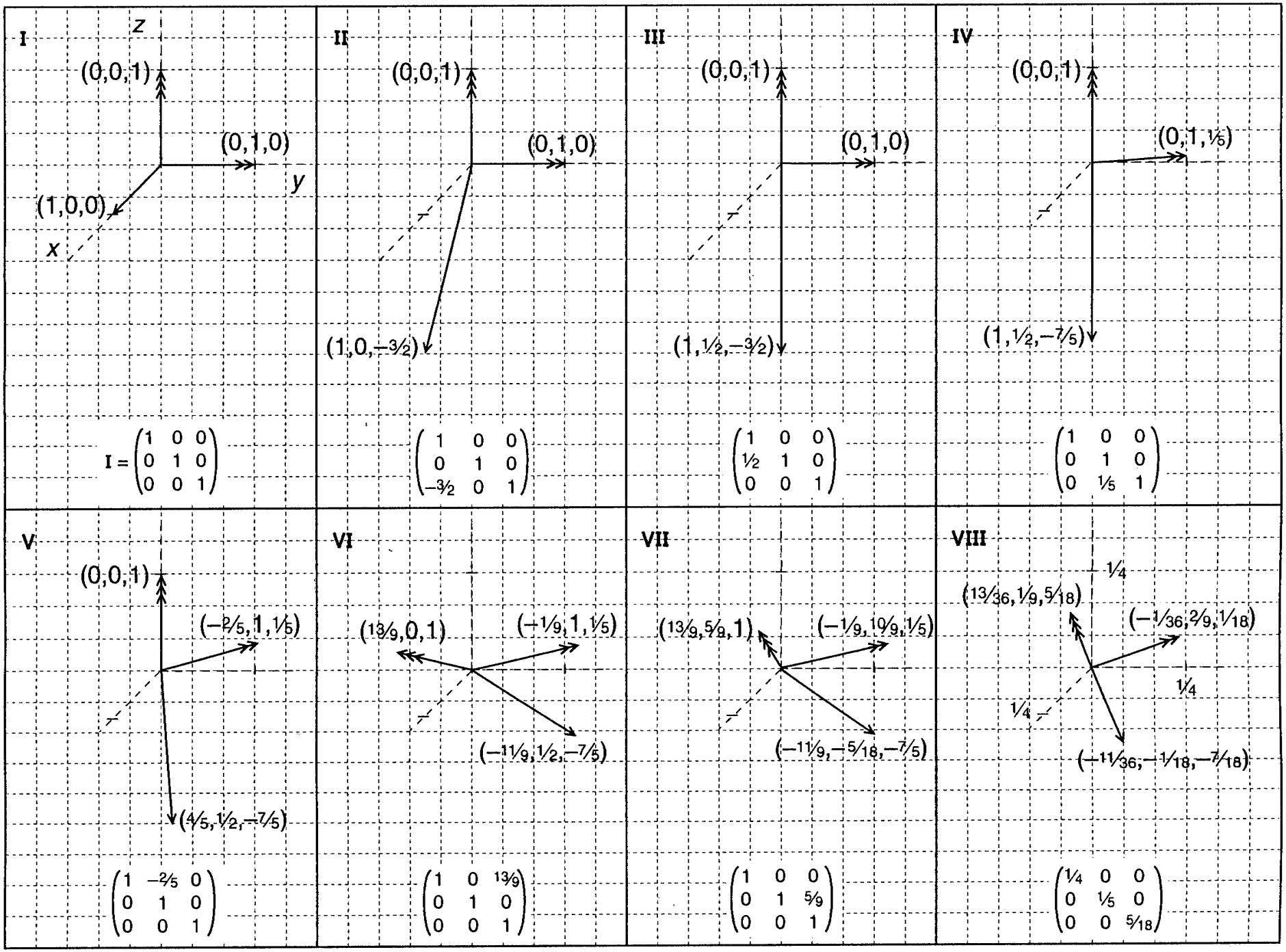

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2/5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 13/9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5/9 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 5/18 \end{pmatrix}$$

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