

VISUAL ALGEBRA FOR COLLEGE STUDENTS

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WELCOME AND INTRODUCTION

VISUAL ALGEBRA FOR COLLEGE STUDENTS

WHAT IS VISUAL ALGEBRA?

Welcome to the *Visual Algebra for College Students* book. *Visual Algebra* is a powerful way to look at algebraic ideas using concrete models and to connect those models to symbolic work. *Visual Algebra* will take you through modeling integer operations with black and red tiles to modeling linear and quadratic patterns with black and red tiles, looking at the general forms of the patterns using algebra pieces and then connecting all of those ideas to symbolic manipulation, creating data sets, graphing, finding intercepts and points of intersection. Chapter Four extends these ideas to higher order polynomial functions (such as cubic polynomials) and ventures into modeling complex number operations with black, red, yellow and green tiles.

THE GOAL OF VISUAL ALGEBRA

The main goal of *Visual Algebra* is to help you gain a depth of understanding of basic algebraic skills. When you completely understand the algebra covered in this book, you should be able to show *visually* the algebra using a concrete model (algebra pieces), describe *verbally* the meaning of each step or move with the algebra pieces and connect this all *symbolically* to standard algebraic algorithms and procedures. In many cases, you will also be able to show the ideas from the visual model and symbolic work *graphically*. Overall, you will be able to think deeply about the topics and not rely on rote memorization or rules. You will understand these ideas so well that you can easily describe and effectively teach them to someone else.

THE STRUCTURE OF VISUAL ALGEBRA FOR COLLEGE STUDENTS

This book is designed as a hands-on book. Each section is dedicated to a small set of related topics and is presented as an *Activity Set* and a corresponding *Homework Set*. Each *Activity Set* starts with a description of the **Purpose** of the set, a list of the needed **Materials** for the set and an **Introduction** that gives definitions, examples and sometimes technology tips. Each *Activity Set* then moves to a set of exploration based activity questions (referred to as “activities”) presented with space to write in your exploratory work and solutions. These *Activity Sets* are entirely self-contained with graphing grids and other diagrams embedded into your workspaces. Each *Homework Set* gives a set of homework questions related to the *Activity Set*.

The end of each chapter of *Visual Algebra for College Students* has an itemized and referenced list of vocabulary and review topics for that chapter and a chapter practice exam.

The “back of the book” material for *Visual Algebra for College Students* contains selected answers to *Activity Set* activities (marked with an asterisk (*)) and complete solutions to each end of chapter practice exam.

CLASS APPROACH FOR VISUAL ALGEBRA

Although many of the ideas in this book can be used for self study, the ideal situation is that you will work in small groups of three or four students in an interactive classroom environment as you explore the *Visual Algebra* topics.

EFFECTIVE GROUP WORK IDEAS FOR VISUAL ALGEBRA

Here are some effective ideas to think about while working in a group:

- ✓ Equal and friendly sharing is the key to a good group; no one (or two) persons should ever tell others in their group answers and correspondingly, no group member should always ask others for help. To study mathematics, each person must gain and earn their own knowledge. This means that each person will usually have to think, struggle, explore, make conjectures, make false starts, make errors, correct errors and working together with their group, find correct and valid solution paths.
- ✓ Everyone should write out their own work and complete their own *Activity Sets*. No group member should write on another group member's pages.
- ✓ When a group starts a new activity, you may wish to read the question individually or take turns reading questions out loud. After the question is understood, for this class, it is ideal to briefly share and discuss ideas about the question (see Group Protocol's below) and as a group, work out the question with a single set of algebra pieces. When graphing calculators are used, make sure each group member can work with their own calculator.
- ✓ When a member of the group has a question, try to ask leading and helpful questions back that will help the group member answer their original question on their own. In general, if you know an answer and tell it to someone else, then you will still know the answer and your friend may briefly retain the information, but, in the long run, because they have not gained and earned this knowledge for themselves, they are unlikely to remember it.
- ✓ Your instructor may choose to let you pick your own group or may assign you to a group. Although it often seems easy to work with people that are "just like you," it is also often more effective to work with people with a variety of different learning styles and approaches. In a discussion environment, it is ideal to have a variety of perspectives. For future teachers, working with people who think differently than you is excellent practice for working in your future classroom.

GO AROUND PROTOCOL

"The Power of Protocols," J.P. McDonald, N. Mohr, A. Dichter and E. C. McDonald, Teachers College Press, 2003 has many useful ideas for effective group work.

One of the ideas from "The Power of Protocols," is the *Go Around Protocol*, can be especially useful in the *Visual Algebra for College Students* classroom. The *Go Around Protocol* is a very simple idea; when a group is working on an idea, each member in the group gets a specified amount of time (usually around 30 seconds) to discuss and introduce their ideas (while other group members listen attentively). Then the role of speaker rotates to the next person in the group. You can see this protocol exactly matches the effective group work goals set forth in the preceding paragraphs. Each group member is given an equal and friendly share in the group's discussion. Your course instructor may choose to make the *Go Around Protocol* formal by setting the speaking time (such as 30 seconds) and the direction the role of speaking rotates (such as counterclockwise). On the other hand, your instructor may simply allow you to informally manage effective sharing within your own group.

SHARING WITH THE WHOLE CLASS

Sharing among class groups can be very powerful. When your instructor asks you to share, be sure to ask questions, take your turn and volunteer frequently. Remember, when a class discusses ideas and possible errors, everyone benefits.

REQUIRED MATERIALS FOR VISUAL ALGEBRA

The concrete models used in *Visual Algebra* are sets of *Algebra Pieces* available from the *Math Learning Center* and are linked to the *Visual Algebra for College Students* Product Page (www.mathlearningcenter.org/store/product-31698032.htm). These sets include black and red tiles, black and red n -strips, white x and opposite x -strips and black and red x -squares. In Chapter Four we also use green and yellow tiles which are also available from the *Math Learning Center*. A graphing calculator is also required in Chapter Three.

STUDENT ELECTRONIC RESOURCES

The following files are available at the Math Learning Center *Visual Algebra for College Students* Product Page (www.mathlearningcenter.org/store/product-31698032.htm) where the complete *Visual Algebra for College Students* book is also available for a free download.

Algebra Pieces.doc

This file contains images of each tile algebra piece and edge piece used in this book. You can copy the pieces and paste them into a text file (such as Word) or in to any paint program. In Word, double click on any piece to change it size and use the Drawing menu to rotate a piece as needed.

Graph and Grid Paper

- ✓ 0.25in.Grid.pdf
- ✓ 0.25in.Graph.pdf (grid paper with darker axes)
- ✓ 0.5in.Grid.pdf
- ✓ Two Column Algebra Piece-Symbolic Work Paper (for use starting with *Activity Set 2.4*)

CHAPTER ONE

INTEGERS AND INTEGER OPERATIONS

Activity Set 1.1

MODELING INTEGERS WITH BLACK AND RED TILES

PURPOSE

To learn how to model positive and negative integers with black and red tiles and how to determine the net value and opposite net value of collections of black and red tiles.

MATERIALS

Black and red* tiles: Black and red tiles are black on one side and red on the other side. 

✓ No calculators

INTRODUCTION

Integers

The set of Integers (\mathbf{Z}) is the set of positive and negative counting numbers and zero and is denoted by the letter \mathbf{Z} . $\mathbf{Z} = \{0, ^+1, ^+2, ^+3 \dots\}$

Black and Red Tiles

One black tile has value $^+1$, two black tiles have value $^+2$, one red tile has value $^-1$, two red tiles have value $^-2$, etc.

One black tile = $^+1$ Two black tiles = $^+2$ One red tile = $^-1$ Two red tiles = $^-2$



Black and red tiles are Opposite Colors and one black tile and one red tile cancel each other out.

One black tile and one red tile = $^+1 + ^-1 = 0$



A Collection of black and red tiles is any set of black and red tiles.

The Net Value of a collection of black and red tiles is the value of the remaining black or red tiles once all matching pairs of black and red tiles are removed. For example the net value of 1 black tile and 1 red tile is 0, the net value of 2 black tiles and 1 red tile is $^+1$ and the net value of 2 black tiles and 5 red tiles is $^-3$.



Net value = $^+1$



Net value = $^-3$

An Opposite Collection of black and red tiles is a collection in which all of the tiles have been flipped to the opposite color. For example, the opposite collection of 2 black tiles and 1 red tile is 2 red tiles and 1 black tile.

The Opposite Net Value of a collection of black and red tiles is the net value of the opposite collection of the black and red tiles.

* Red tiles are pictured in gray

1. Take a dozen or so black and red tiles and toss them on your table. Repeat as you fill out each row in the table; remove all matching black and red pairs before filling out the last two columns. As you toss, discuss your results. What observations can you make about black and red tiles? List all of your observations.

Collection	Total # Tiles	# Red Tiles	# Black Tiles	Net Value	
				#R or #B	Integer
Example	13	7	6	1 R	1
Toss 1					
Toss 2					
Toss 3					
Toss 4					
Toss 5					
Toss 6					

(*) Observations

2. Take a dozen or so black and red tiles and toss them on your table. Fill out the first row, and then flip over the tiles to form the opposite collection and fill out the second row. Repeat as you fill out each pair of rows in the table; remove all matching black and red pairs before filling out the last two columns. As you toss and flip, discuss your results. What observations can you make about collections, opposite collections, net value and opposite net value of collections of black and red tiles? List all of your observations.

Collection / Opposite Collection	Total # Tiles	# Red Tiles	# Black Tiles	Net Value	
				#R or #B	Integer
Toss 1					
Flip Toss 1					
Toss 2					
Flip Toss 2					
Toss 3					
Flip Toss 3					

Observations

3. (Partner work) Take turns filling out the outlined cells in each row of the table with numbers of your choice and have your partner determine how to fill out the rest of the row. Completely fill out the table on both partner's pages. What observations can you make about collections of black and red tiles in this setting? List all of your observations.

Collection	Total # Tiles	# Black	# Red	Net Value
Collection 1				
Collection 2				
Collection 3				
Collection 4				
Collection 5				
Collection 6				

Observations

Homework Questions 1.1

MODELING INTEGERS WITH BLACK AND RED TILES

Sketching Black and Red Tiles

For this homework set; you may wish to denote black tiles by B and red tiles by R rather than sketching and coloring square tiles. Although electronic images are available, at this stage just writing out B and R is much faster and much more efficient.

For each of the following collections of black and red tiles:

- i) Find the unknown number(s) of tiles. Sketch the resulting collection and mark it to show the given net value or explain why the collection cannot exist no matter what the unknown number of tiles is.
 - ii) If more than one such collection exists, give two different examples of collections that work and explain why there is more than one collection that meets the given conditions.
1. Collection I of black and red tiles contains exactly 6 red tiles, an unknown number of black tiles and has net value -2 .
 2. Collection II of black and red tiles contains exactly 5 black tiles, an unknown number of red tiles and has net value -2 .
 3. Collection III of black and red tiles contains exactly 6 red tiles, an unknown even number of black tiles and has net value $+3$.
 4. Collection IV of black and red tiles contains exactly 5 red tiles, an unknown even number of black tiles and has net value $+2$.
 5. Collection V of black and red tiles contains exactly 5 red tiles, an unknown even number of black tiles and has net value $+0$.
 6. Collection VI of black and red tiles contains an unknown even number of red tiles, an unknown even number of black tiles and has net value 0 .
 7. Collection VII of black and red tiles contains an unknown even number of red tiles, an unknown even number of black tiles and has net value $+3$.
 8. Collection VIII of black and red tiles contains an unknown odd number of red tiles, an unknown odd number of black tiles and has net value $+3$.
 9. Collection IX of black and red tiles contains an unknown odd number of red tiles, an unknown odd number of black tiles and has net value $+4$.
 10. Collection X of black and red tiles contains an unknown even number of red tiles, an unknown even number of black tiles and has net value $+6$.

Activity Set 1.2

ADDING INTEGERS WITH BLACK AND RED TILES

PURPOSE

To learn how to add integers using black and red tiles. To investigate the rule “When adding a negative number to a positive number, you can just subtract.”

MATERIALS

Black and red tiles

✓ No calculators



INTRODUCTION

Addition Terms

$2 + 3 = 5$: In this addition sentence, 2 and 3 are both Addends and 5 is the Sum.

SKETCHING TIPS

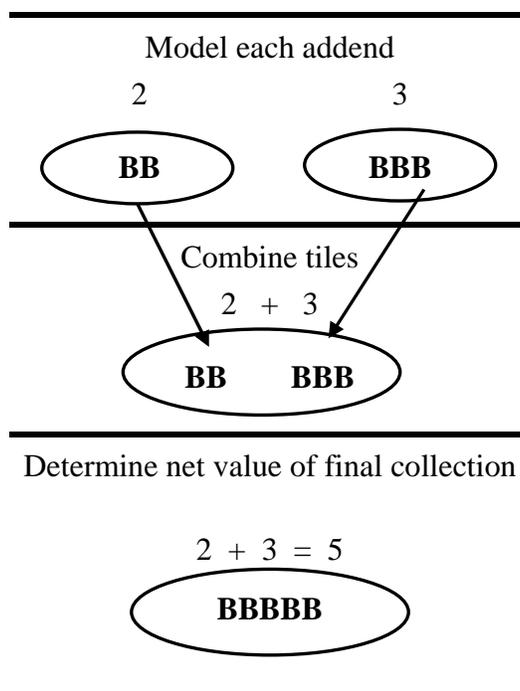
Sketching Tiles

While sketching integer addition in this activity set, you may wish to denote black tiles by B and red tiles by R rather than sketching and coloring square tiles.

Sketching Addition

When using sketches to show integer operations, label the steps and corresponding collections clearly so that another reader can follow your steps. Briefly explain your steps.

Example: $2 + 3 = 5$



1. Model $+4$ and model -6 with your black and red tiles. Explain how you would use your collections to show the sum in the addition question: $+4 + -6 = ?$ Sketch and label your work.
2. What observations can you make about finding the sum $+4 + -6 = ?$ with black and red tiles? Discuss and list your observations. Are there different collections of black and red tiles that can be used to model this sum? Explain
3. For the following addition questions, model each addend and then model the sum of the two addends. Sketch and label your work. Discuss any observations and note them by your sketches.

Observations

a. $+4 + +6 = ?$

b. (*) $-4 + -6 = ?$

Observations

c. $+6 + ^{-}4 = ?$

d. $^{-}6 + +4 = ?$

4. Using the black and red tile model and part c in the previous problem as a guide; explain why the rule “When adding a negative number to a positive number, you can just subtract” works.

Homework Questions 1.2

ADDING INTEGERS WITH BLACK AND RED TILES

Sketching Black and Red Tiles

For this homework set; you may wish to denote black tiles by B and red tiles by R rather than sketching and coloring square tiles.

1. For the following addition questions, use black and red tiles to model each addend and then the sum of the two addends. Sketch and label your work. Be sure to carry out the whole operation; don't short cut by changing signs. In each case, give the completed addition sentence.
 - a. $+8 + +3 = ?$
 - b. $+8 + -3 = ?$
 - c. $-8 + -3 = ?$

2. Describe the steps you would explain to an elementary school student about how to imagine using black and red tiles to help compute each of the following. You may assume the student knows how to add positive numbers. Be sure to explain the whole idea, not just how to short cut by changing signs.
 - a. $+155 + -125 = ?$
 - b. $-155 + -145 = ?$

3. Using the black and red tile model, show why the addition question $+8 + -3 = ?$ can be converted to the subtraction question $8 - 3 = ?$ to obtain the same result. Sketch, label and explain your work.

Activity Set 1.3

SUBTRACTING INTEGERS WITH BLACK AND RED TILES

PURPOSE

To learn how to subtract integers using black and red tiles. To investigate the rule “When subtracting a negative number you can just add.”

MATERIALS

Black and red tiles

✓ No calculators



INTRODUCTION

Subtraction Terms

$5 - 3 = 2$: In this subtraction sentence, 5 is the Minuend, 3 is the Subtrahend and 2 is the Difference.

Zero Pairs

A Zero Pair is a pair with net value 0—one black and one red tile.



Zero Pair

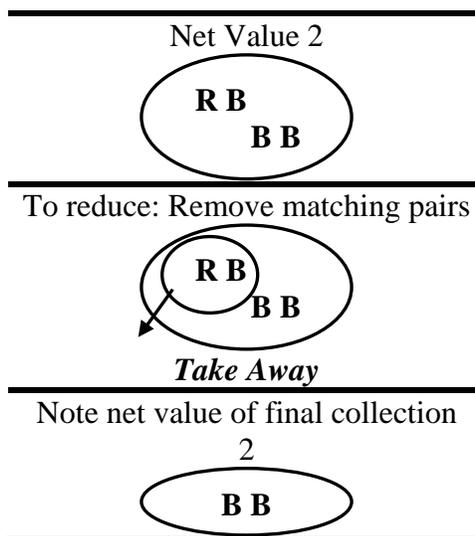
SKETCHING TIPS

Sketching Tiles

For sketching integer subtraction in this activity set, you may wish to denote black tiles by B and red tiles by R rather than sketching and coloring square tiles.

Sketching Take Away

When sketching taking away black and red tiles, one nice technique is to circle the tiles you are removing and label the circle as “Take Away” as shown in this example reducing a collection with net value 2.



1. Model $+6$ and model $+4$ with your black and red tiles. Explain how you would use your collections to show the difference in the subtraction question: $+6 - +4 = ?$ Sketch and label your work.
2. What observations can you make about finding the difference $+6 - +4 = ?$ with black and red tiles? Discuss and list your observations. Are there different collections of black and red tiles that can be used to model this subtraction question? Explain.
3. Model $+6$ and model $+4$ with your black and red tiles. Explain how you would use your collections to show the difference in the subtraction question: $+4 - +6 = ?$ Hint: The collection for $+4$ does not have to be only 4 black tiles. In order to take away 6 black tiles from your collection for $+4$, you must have at least 6 black tiles in the collection for $+4$. Form a collection with 6 black tiles and net value $+4$ and use this collection to model the difference for $+4 - +6 = ?$ Sketch and label your work.

4. (*) What observations can you make about starting with a collection that will allow you to find the difference $+4 - +6 = ?$ with black and red tiles? Discuss and list your observations. Are there different collections of black and red tiles that can be used to model this subtraction question? Explain
5. For the following subtraction questions, model the minuend and the subtrahend and then model the difference. Sketch and label your work. Discuss any observations and note them by your sketches. Note: Don't change subtracting an opposite to addition, actually carry out the subtraction.

Observations

a. $-4 - +6 = ?$

b. $-6 - -4 = ?$

c. $-4 - -6 = ?$

d. $+4 - -6 = ?$

6. Using the black and red tile model, explain why the rule “When subtracting a negative number you can just add” works. You may wish to use $+1 - -2 = ?$ to think about the question, but give a general answer that works for every situation.

Homework Questions 1.3

SUBTRACTING INTEGERS WITH BLACK AND RED TILES

Sketching Black and Red Tiles

For this homework set; you may wish to denote black tiles by B and red tiles by R rather than sketching and coloring square tiles.

1. For the following subtraction questions, use black and red tiles to model the minuend, the subtrahend and then the difference. Sketch and label your work. Be sure to carry out the whole operation; don't short cut by changing signs. In each case, give the completed subtraction sentence.
 - a. $+3 - +8 = ?$
 - b. $+3 - -8 = ?$
 - c. $-3 - +8 = ?$
 - d. $-3 - -8 = ?$

2. Describe the steps you would explain to an elementary school student about how to imagine the black and red tiles to help compute each of the following. You may assume the student knows how to subtract positive numbers. Be sure to explain the whole idea, not just how to short cut by changing signs.
 - a. $+150 - -175 = ?$
 - b. $-155 - -170 = ?$
 - c. $-170 - -135 = ?$

3. Using the black and red tile model, show why the subtraction question $+3 - -8 = ?$ can be converted to the addition question $3 + 8 = ?$ to obtain the same result. Sketch, label and explain your work.

Activity Set 1.4

ARRAYS WITH BLACK AND RED TILES

PURPOSE

To learn how to model rectangular arrays with black and red tiles and determine the net value of the arrays. To explore the result of flipping over columns and rows on the net value of a rectangular array. To learn how to use edge pieces to keep track of flipped columns and rows in a rectangular array of black and red tiles. To learn about minimal arrays, minimal collections and the empty array.

MATERIALS

Black and red tiles with black and red edge pieces

✓ No calculators



INTRODUCTION

Rectangular Arrays

A Rectangular Array is a rectangular arrangement of numbers or objects.

Rectangular Array Examples

This is a rectangular array of numbers.
This array has *3 rows* and *4 columns*; it is a
 3×4 Array
Row then column

1	4	5	2
5	5	4	7
2	0	10	1

This is a rectangular array of black tiles
This array has *2 rows* and *5 columns*; it is a
 2×5 Array



Rectangular Array Terms

Edge Pieces and Edge Sets are defined in context, see activity 2



Minimal Arrays, Minimal Collections and Minimal Edge Sets are defined in context, see activity 8.

The Empty Array and the Non-Empty Array are defined in context, see activity 8.

EQUIVALENT ARRAYS

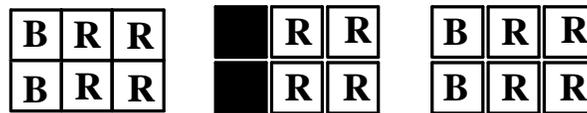
Orientation on the page does not distinguish arrays, a 1×3 array and a 3×1 array with the same edge sets are equivalent arrays and only one should be given as a solution to a question.



SKETCHING TIPS

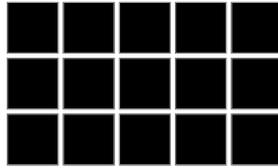
Sketching Arrays

When sketching an array of black and red tiles, show the rectangular shape of the array and the square shape of the tiles. It may be easiest to sketch blank squares, lightly shade the black tiles and label the interiors of the red tiles R or to skip shading and just label both the black and the red squares with a B or an R, respectively.



Tiles touching or Tiles not touching
Sketching options

1. Use your black and red tiles and form the following 3×5 array of black tiles.



Array 1

- a. (*) What is the net value of Array 1?
- b. (*) Using your model of Array 1, pick one column and flip over all of the tiles in that column; what is the ending net value of the array? Return Array 1 to all black tiles and flip over all of the tiles in another column. Does the ending net value depend on which column you flip over? Sketch two such arrays and explain.
- c. Return Array 1 to all black tiles; pick two columns and flip over all of the tiles in both columns; what is the ending net value of the array? Does the ending net value depend on which two columns you flip? Explain.
- d. Return Array 1 to all black tiles and flip over all of the tiles in Column 1 (C1) and then flip over all of the tiles in Row 1 (R1). Note the tile that is in both C1 and R1 will be flipped twice. What is the ending net value of the array? Does the ending net value depend on which column and row you flip? Experiment with several combinations (C2 and R3, C3 and R1, etc.) and explain what happens if you flip one column and then flip one row. Sketch at least one such array.

- e. Does the order of flipping matter? If you start with an all black Array 1, flip one row and then flip one column, what happens to the ending net value? Is the ending net value different than if you flip first the column and then the row?
- f. Return Array 1 to all black tiles and flip over all of the tiles in Column 1 (C1), in Column 2 (C2) and then in Row 1 (R1). What is the ending net value of the array? Does the ending net value depend on which two columns and one row you flip? Experiment with several combinations of two columns and one row and explain what happens. Sketch and label at least one such ending array.
- g. Summarize the ending net values you obtain by flipping various combinations of rows and columns in Array 1. Use black and red tiles to model each changing array as you fill out the table.

ARRAY 1: 3×5 array of black tiles

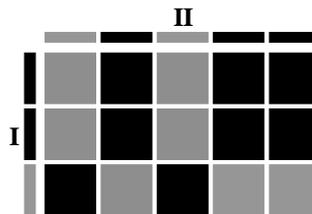
# of Columns To flip	# of Rows to flip	Ending Net Value
1	1	
1	2	
1	3	
2	1	
2	2	
2	3	
3	1	
3	2	
3	3	
4	1	
4	2	
4	3	
5	1	
5	2	
5	3	

Edge Sets

Note that it is difficult to look at an array of black and red tiles and determine which columns and rows have been flipped over. To keep track of the flipping information, we will use edge pieces and edge sets. Edge pieces indicate whether or not a row or column of an array has been flipped or turned over.

- ✓ An edge piece is a thin piece of black or red tile that is used to mark the edge dimensions of a black or red tiles (an edge piece has no height).
- ✓ A red edge piece at the end of a row or column indicates the row or column has been flipped.
- ✓ A black edge piece at the end of a row or column indicates the row or column has not been flipped.
- ✓ Edge pieces are designed to keep track of flipping and the dimensions of a rectangular array; the edge pieces themselves are not counted when determining the net value of an array.

Starting with an all black Array 1, these edge pieces indicate that Column 1, Column 3 and Row 3 have each been flipped over. The edge sets have been labeled I and II for reference.



Edge sets have net values just like tile collections as illustrated in this table:

Item	#Black	#Red	Net Value
Edge I	2B	1R	+1
Edge II	3B	2R	+1
Array	8B	7R	+1

2. (*) Use black and red tiles, with edge pieces, to explore the connection between edge pieces and arrays. Use black and red tiles to model each array as you fill out the table.

Edge I		Edge II		Array		Net Values		
R	B	R	B	R	B	Edge I	Edge II	Array
3	0	0	2					
3	0	1	1					
3	0	2	0					
2	1	0	2					
2	1	1	1					
2	1	2	0					
1	2	0	2					
1	2	1	1					
1	2	2	0					
0	3	0	2					
0	3	1	1					
0	3	2	0					

3. In terms of edges: When is a tile in an array black? When is a tile in an array red?

4. For each row in the table, determine an array, with edge pieces, that has net value 0. Find five different collections of edge sets that work. Model with black and red tiles as needed.

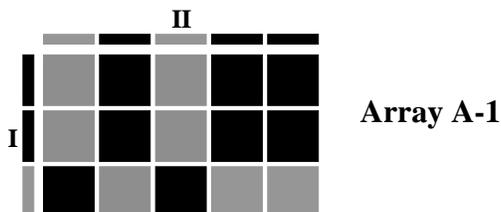
Edge I		Edge II		Array		Net Values		
R	B	R	B	R	B	Edge I	Edge II	Array
								0
								0
								0
								0
								0

5. (*) What do you notice about arrays of net value 0 and their corresponding edge sets? List your observations.

Inefficient Arrays

Many of the arrays and edge sets we have been working with contain net value zero pairs and do not seem to be the most efficient collections of black and red tiles or black and red edges.

6. Model the pictured *Array A-1* with black and red tiles and edge pieces.



- a. (*) Remove two rows (and their edge pieces) without changing the net value of the array or the net value of either edge. Sketch the new array, Array A-2, and the new edges for Array A-2.

- b. Now remove two columns (and their edge pieces) from Array A-2 without changing the net value of Array A-2 or the net value of either edge. Sketch this third array, Array A-3, include its edge pieces.
- c. Can you remove any additional pairs of rows or columns without changing the net value of Array A-3 or the net value of either edge? Sketch any additional arrays, include their edge pieces.
- d. Describe the pairs of rows and pairs of columns that you removed from the original array.

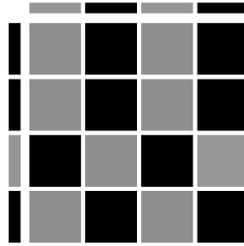
Minimal Arrays and Minimal Collections

An array of black and red tiles in which all net value zero row pairs and all net value zero column pairs are removed is a Minimal Array.

In the same way, a collection of black and red tiles in which all net value zero pairs have been removed is a Minimal Collection. A Minimal Edge Set is a minimal collection of edge pieces.

7. Describe the characteristics of a minimal array:
8. Describe the characteristics of a minimal edge set:

9. Consider the following non-minimal array with net value 0.



Remove two columns or two rows (and their edge pieces) at a time without changing the net value of the array or the net value of either edge. Sketch each new array, include the edge pieces. Note which columns or rows you have removed.

The Empty Array

The array you end up with in the previous activity is a special type of minimal array, it is the Empty Array.

A Non-Empty Array is any array in which there are some black and/or red tiles.

10. Explain the following statement: The only minimal array with net value 0 is the Empty Array.

11. For each row in the table, determine an array, with edge pieces, that has net value 6. Find four different collections of edge sets that work. Model with black and red tiles as needed.

Edge I		Edge II		Array		Net Values		
R	B	R	B	R	B	Edge I	Edge II	Array
								6
								6
								6
								6

12. (*) What do you notice about arrays of net value 6 and their corresponding edge sets? List your observations; include notes on whether your arrays are or are not minimal arrays.

Homework Questions 1.4

ARRAYS WITH BLACK AND RED TILES

1. Sketch three different and non-equivalent non-empty arrays (Arrays A—C) with net value 0. Include edge pieces and use this information to fill out a table like this:

	Edge I	Edge II	Array	Net Values			
	R	B	R	B	Edge I	Edge II	Array
Array A							0
Array B							0
Array C							0

2. Sketch four different non-equivalent minimal arrays (Arrays A—D) with net value $+4$. Include edge pieces and use this information to fill out a table like this:

	Edge I	Edge II	Array	Net Values			
	R	B	R	B	Edge I	Edge II	Array
Array A							$+4$
Array B							$+4$
Array C							$+4$
Array D							$+4$

3. For each of the following:
- i) Sketch a minimal array, with edge pieces, that satisfies the given conditions OR if no minimal array that meets the given conditions exists, explain why this is the case. Label the edge sets and arrays with their net value.
 - a. One edge has an odd number of black edge pieces and the net value of the array is $+2$.
 - b. One edge has an odd number of edge pieces and the net value of the array is -2 .
 - c. Both edges have an odd number of edge pieces and the net value of the array is $+8$.
 - d. One edge is all red and one edge is all black; the net value of the array is $+12$.
 - e. One edge is all red and one edge is all black; the net value of the array is -10 .
 - ii) If more than one such minimal array exists, give two different examples of minimal arrays that work

Activity Set 1.5

MULTIPLYING INTEGERS WITH BLACK AND RED TILES

PURPOSE

To learn how to multiply integers using black and red tile arrays. To investigate the various rules of multiplying integers “Two positives is a positive, two negatives is a positive, etc.” To explore the role of 0 in multiplication sentences.

MATERIALS

Black and red tiles with black and red edge pieces

✓ No calculators



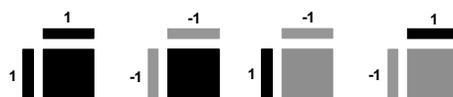
INTRODUCTION

Multiplication Terms

$2 \times 3 = 6$: In this multiplication sentence, 2 and 3 are both Factors and 6 is the Product.

Using Black and Red Edge Pieces to Measure Side Dimensions (Lengths)

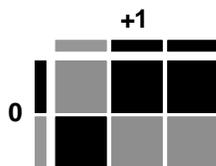
We saw in Activity Set 1.4 that we can use black and red edge pieces to indicate whether or not a row or column in an array of black and red tiles has been flipped. It also makes sense to think of a black and red edge pieces as measuring the side dimensions of arrays. For the rest of these materials, we will think of a black edge piece as measuring a side dimension of $+1$ unit and a red edge piece as measuring a side dimension of -1 unit. We will tend to call these “length 1” and “length -1 ,” although, of course, “length” is usually positive.



SKETCHING TIPS

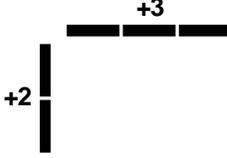
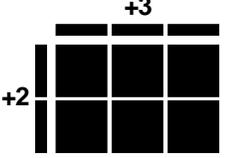
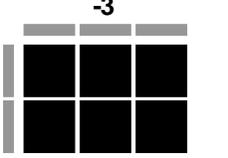
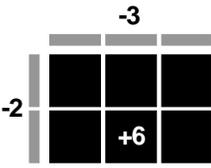
Labeling Edges

When sketching an array with edges, label the net value of each edge set as shown in this sketch.



Sketching Multiplication

When sketching multiplication, show and briefly explain each step as shown in the following example. Label edges throughout.

Find the solution to: $-2 \times -3 = ?$			
 <p style="text-align: center;">+3</p> <p style="text-align: left;">+2</p> <p style="text-align: center;">Lay out all black edges; note black edge values</p>	 <p style="text-align: center;">+3</p> <p style="text-align: left;">+2</p> <p style="text-align: center;">Fill in all black tiles</p>	 <p style="text-align: center;">+3</p> <p style="text-align: left;">-2</p> <p style="text-align: center;">Flip row and corresponding edge pieces for -2, note edge net value</p>	 <p style="text-align: center;">-3</p> <p style="text-align: center;">Flip column and edge pieces for -3, note edge net value</p>
<p>Use the array to determine the final product</p> <p>$-2 \times -3 = +6$</p>  <p style="text-align: center;">-3</p> <p style="text-align: left;">-2</p> <p style="text-align: center;">+6</p>			

c. $+3 \times^{-}4 = ?$

d. $(*)^{-}3 \times^{-}2 = ?$

3. Use your black and red tiles and form a non-empty Array 2 with Edge Set I = $+3$ and Edge Set II = 0 . Sketch and label your work

a. What is the net value of Array 2?

b. If you think of Array 2 and its edges as the multiplication of two factors; what are the two factors? What is the product of the two factors?

c. What multiplication sentence do the edge sets and the array show? If you have not already done so; label the net values of the edge sets and the array on your sketch.

d. Is this a minimal array? If you reduce Array 2 to a minimal array; what array will that be?

c. The product of one positive factor and one negative factor is _____

d. The product of one positive or negative factor and the factor 0 is _____

e. The product of the factor 0 and the factor 0 is _____

Homework Questions 1.5

MULTIPLYING INTEGERS WITH BLACK AND RED TILES

1. For the following multiplication questions, use black and red tiles and edge pieces to model a sequence of minimal arrays showing the multiplication steps. Sketch and label your work; label the net values of each edge set for each array and also the net value of the product on the last array. Briefly explain each step. Identify factors and products. In each case, give the completed multiplication sentence the array and edge sets shows.
 - a. $+3 \times +4 = ?$
 - b. $+3 \times -4 = ?$
 - c. $-3 \times -4 = ?$
2. Describe how you would explain to an elementary school student how the signs of factors relate to the sign of the product when multiplying integers. Be sure to explain the whole idea, not just how to short cut by changing signs. Use black and red tile arrays with edge pieces in your explanation.
3. Describe how you would explain to an elementary school student why multiplying an integer by a factor of 0 gives a product of 0. Use black and red tile arrays with edge pieces in your explanation.

Activity Set 1.6

DIVIDING INTEGERS WITH BLACK AND RED TILES

PURPOSE

To learn how to divide integers using black and red tile arrays. To investigate the various rules of dividing integers “Two positives is a positive, two negatives is a positive, etc.” To explore the role of 0 in division sentences.

MATERIALS

Black and red tiles with black and red edge pieces

✓ No calculators



INTRODUCTION

Division Terms

$12 \div 3 = 4$: In this division sentence, 12 is the Dividend, 3 is the Divisor and 4 is the Quotient

SKETCHING TIPS

Sketching Division

The steps of division are explored in context in this activity. As with multiplication; for your sketches; show and briefly explain each step. Label edge net values on each sketch.

2. Use your black and red tiles to model each of the following division questions in a series of steps. Sketch and label your work, including, for each step, labeling the net values of the edge sets and the array. Briefly explain each step. In each case, give the completed division sentence the array and edge sets shows.

a. $+10 \div +5 = ?$

b. $-8 \div +4 = ?$

c. $+12 \div -6 = ?$

d. $9 \div -3 = ?$

3. Use your black and red tiles to model the given array (dividend) and Edge Set I (divisor) to find the corresponding quotient (dividend \div divisor = quotient).
- Sketch and label your work, including, for each step, labeling the net values of the edge sets and the array.
 - Briefly explain each step.
 - In each case, give the completed division sentence the array and edge sets shows or explain why the set up does not result in a valid division sentence.
 - In each case, give the completed multiplication sentence the final diagram could show.
- a. (*) (Non-Empty) Array: Net Value 0 with 6 tiles, Edge Set I: Net Value: $+3$

b. (Non-Empty) Array: Net Value 0 with at least 4 tiles, Edge Set I: Net Value: -2

c. Array: Net Value $+6$, (Non-Empty) Edge Set I: Net Value: 0

d. Array: Net Value -4 , (Non-Empty) Edge Set I: Net Value: 0

e. (Non-Empty) Array: Net Value 0, Edge Set I: Net Value: 0. Hint: Double check that any solution you arrive at is the *only* possible solution.

Homework Questions 1.6

DIVIDING INTEGERS WITH BLACK AND RED TILES

1. For the following division questions, use black and red tiles and edge pieces to model a sequence of minimal arrays showing the division steps. Sketch and label your work; label the net values of each edge set for each array and also the net value of the quotient in the last step. Briefly explain each step. Identify dividend, divisor and quotient. In each case, give the completed division sentence the array and edge sets shows as well as the corresponding multiplication sentence the array and edge sets could show.
 - a. $+12 \div +4 = ?$
 - b. $+12 \div -3 = ?$
 - c. $-9 \div +3 = ?$
 - d. $-8 \div -2 = ?$
2. Describe how you would explain to an elementary school student how the signs of dividends and the divisors relate to the sign of the quotient when dividing integers. Be sure to explain the whole idea, not just how to short cut by changing signs. Use black and red tile arrays with edge pieces in your explanation.
3. Describe how you would explain to an elementary school student why when the dividend is 0 and the divisor is a nonzero integer, the quotient must be 0. Use black and red tile arrays with edge pieces in your explanation.
4. Describe how you would explain to an elementary school student why when the dividend is a nonzero integer and the divisor is 0, there is no possible quotient. Use black and red tile arrays with edge pieces in your explanation.
5. Describe how you would explain to an elementary school student why when the dividend and the divisor are both 0 the quotient is undefined. Use black and red tile arrays with edge pieces in your explanation.

CHAPTER 1 VOCABULARY AND REVIEW TOPICS

VOCABULARY

Activity Set 1.1

1. Integers (\mathbf{Z})
2. Black and Red Tiles
3. Collection
4. Net Value
5. Opposite Collection
6. Opposite Net Value

Activity Set 1.2

8. Addend
9. Sum

Activity Set 1.3

11. Minuend
12. Subtrahend
13. Difference
14. Zero Pair

Activity Set 1.4

16. Rectangular Array
17. Edge Pieces
18. Edge Sets
19. Minimal Array
20. Minimal Collection
21. Minimal Edge Set
22. Empty Array
23. Non-Empty Array

Activity Set 1.5

25. Factors
26. Product

Activity Set 1.6

28. Dividend
29. Divisor
30. Quotient

SKILLS AND CONCEPTS

Activity Set 1.1

- A. Working with collections and opposite collections of black and red tiles; determining missing numbers of black tiles and missing number of red tiles.
- B. Working with net value and opposite net value.

Activity Set 1.2

- C. Adding integers with black and red tiles.
- D. Addition rules: Converting adding an opposite to subtraction.

Activity Set 1.3

- E. Subtracting integers with black and red tiles.
- F. Subtraction rules: Converting subtracting an opposite to addition.

Activity Set 1.4

- G. Forming arrays of black and red tiles given net values of edges or the array.
- H. Relating the net values of the edge sets to the net value of the array.
- I. Minimal and non-minimal arrays; reducing non-minimal arrays to minimal arrays, connections between minimal and non-minimal arrays and their edge sets.

Activity Set 1.5

- J. Using black and red edge pieces to measure length
- K. Multiplying integers with black and red tiles.
- L. Integer multiplication sign rules
- M. Zero as a factor and a product.

Activity Set 1.6

- N. Dividing integers with black and red tiles.
- O. Integer division sign rules
- P. Zero as a dividend, a divisor and a quotient.
- Q. Connecting multiplication and division using the black and red tile array model.

CHAPTER 1 PRACTICE EXAM

1. Fill in the blanks or explain why no such collection exists..

Collection	Total # of tiles	# Black Tiles	# Red Tiles	Net Value
	12	7		
Opposite Collection	Total # of tiles	# Black Tiles	# Red Tiles	(Opposite) Net Value

2. Fill in the blanks or explain why no such collection exists..

Collection	Total # of tiles	# Black Tiles	# Red Tiles	Net Value
		6		
Opposite Collection	Total # of tiles	# Black Tiles	# Red Tiles	(Opposite) Net Value
		3		+2

3. Fill in the blanks or explain why no such collection exists..

Collection	Total # of tiles	# Black Tiles	# Red Tiles	Net Value
	19			-5
Opposite Collection	Total # of tiles	# Black Tiles	# Red Tiles	(Opposite) Net Value

4. Fill in the blanks or explain why no such collection exists..

Collection	Total # of tiles	# Black Tiles	# Red Tiles	Net Value
	7			-9
Opposite Collection	Total # of tiles	# Black Tiles	# Red Tiles	(Opposite) Net Value

- 5.
- If you know the net value of a collection is odd (positive or negative) when the total number of tiles is even; what can you say about the collection?
 - If you know the net value is even (positive or negative) when the total number of tiles is odd (positive or negative), what can you say about the collection?

For the following addition questions, use black and red tiles to model each addend and the sum of the two addends. Sketch and label your work. Be sure to carry out the whole operation; don't short cut by changing signs.

6. $+3 + +2 = ?$

7. $+3 + -2 = ?$

8. $-3 + -2 = ?$

9. How would you explain to an elementary school student that $+1000 + -400 = ?$ is the same idea as $1000 - 400$ without just saying "that is the rule"?

10. Explain how to imagine using black and red tiles to add two large negative numbers such as -1400 and -2345 .

For the following subtraction questions, use black and red tiles to model the minuend, the subtrahend and then the difference. Sketch and label your work. Be sure to carry out the whole operation; don't short cut by changing signs.

11. $+1 - +4 = ?$

12. $+1 - -4 = ?$

13. $-1 - +4 = ?$

14. $-1 - -4 = ?$

15. Explain why the subtraction question $+2 - -6 = ?$ can be converted to the addition question $2 + 6 = ?$ but the subtraction question $+2 - +6 = ?$ can not be converted to the addition question $2 + 6 = ?$

16. For each part, sketch a non-empty array with the given net value. Sketch and label corresponding edge sets. If more than one such (non-equivalent) array exists, explain. Note whether the array you give is or is not minimal.

- Net Value = -4
- Net Value = $+5$
- Net Value = -16
- Net Value = 0

17. If Edge Set I for an array has net value -3 ; what net values can the array have? Explain.

18. If Edge Set I for an array has an equal number of black and red tiles, what net values can the array have? Explain.

19. If Edge Set I for a minimal array is all red; can the array have a positive net value? Explain.

20. If Edge Set I for a minimal array is all black and Edge Set II has half black and half red tiles; can the array have a positive net value? Explain.

For the following multiplication questions, use black and red tiles and edge pieces to model a sequence of minimal arrays showing the multiplication steps. Sketch and label your work; briefly explain each step. In each case, give the completed multiplication sentence.

21. $+4 \times +6 = ?$

22. $+4 \times -6 = ?$

23. $-4 \times -6 = ?$

24. $+4 \times 0 = ?$

25. Describe the role of 0 in a multiplication sentence as possible factor or a possible product. In each case, give examples.

For the following division questions, use black and red tiles and edge pieces to model a sequence of arrays showing the division steps. Sketch and label your work; briefly explain each step. In each case, give the completed division sentence as well as the corresponding multiplication sentence the array and edge sets could show.

26. $+12 \div +6 = ?$

27. $+12 \div -6 = ?$

28. $-12 \div +6 = ?$

29. $-12 \div 0 = ?$

30. Describe the role of 0 in a division sentence as possible dividend, possible divisor and possible quotient. In each case, give examples.

CHAPTER TWO

LINEAR EXPRESSIONS, EQUATIONS AND GRAPHS

Activity Set 2.1

INTRODUCTION TO TOOTHPICK FIGURE SEQUENCES

PURPOSE

To learn how to analyze sequences of toothpick figures numerically, in words and symbolically to see and define an underlying algebraic pattern and to use that algebraic pattern to answer questions about the sequence of toothpick figures.

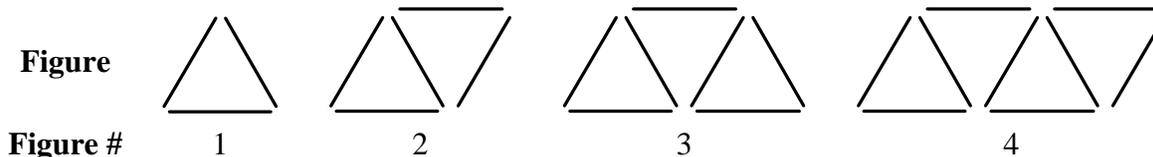
MATERIALS

Toothpicks; flat if possible

INTRODUCTION

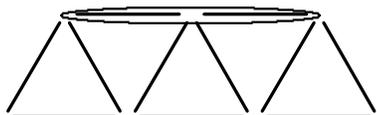
Figure Numbers

Figure Numbers, indexed here by the counting numbers; $n = 1, 2, 3 \dots$ indicate the figure positions in a sequence of figures.

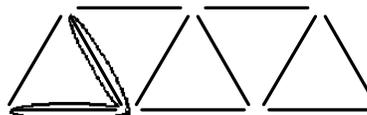


Counting using Loops

When counting the number of shapes in a figure, we can use Looping to mark off portions of the figure. A Loop is simply a shape drawn around a portion of a figure as illustrated here.



The top two toothpicks are looped.

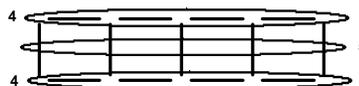


The bottom and right toothpick of the first triangle are looped together

When counting the number of toothpicks in a figure using loops; mark your counts on the figure and write out the complete numerical expression as shown in this example. Don't just simplify, for example; "13" is not as helpful for seeing this pattern as the addition expression $4 + 5 + 4$.



Figure 4



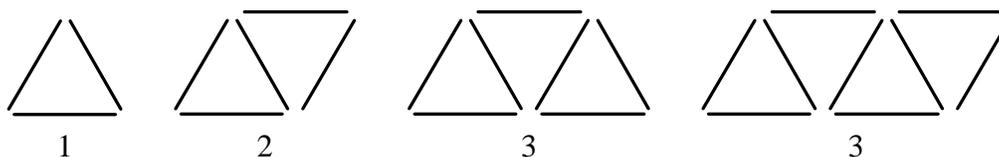
Looped figure with: $4 + 5 + 4$ toothpicks

Describing Looped Diagrams in Words

When describing looping in words, try to use simple phrases. These words and phrases should be clear and allow another person to see your counting technique. In the previous diagram you might describe the looping as "2 \times the figure for the top and bottom rows and 1 more than the figure for the vertical picks." Common types of phrases you might use while looping include:

- 1 more (or less) than the figure
- 1 more (or less) than twice the figure
- Half of the figure
- Half of 1 more (or less) than the figure
- Half of 1 more (or less) than twice the figure
- Three times a figure, etc.

1. Model the following sequence of toothpick figures, *Triangles*, with toothpicks. Carefully follow the given steps to discover a powerful technique for analyzing visual algebraic patterns such as the sequence of toothpick figures pictured here.

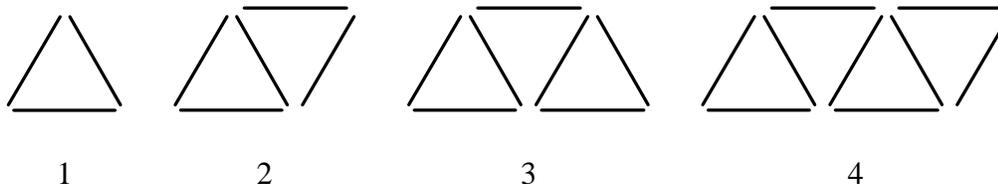


The first step is to look at the toothpick figures **Numerically**

- Use looping and count the total number of toothpicks in each figure without simply counting each individual toothpick.
- Use different looping patterns for set ONE, for set TWO and for set THREE. Your looping technique should be consistent for each toothpick figure within a set.
- Mark your individual counts by your loops and give each toothpick count as an addition expression (as illustrated in the **Introduction**).

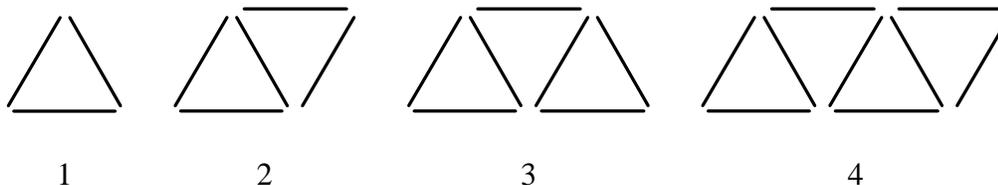
Repeat for sets TWO and THREE; use a different looping technique for each set.

(*) ONE



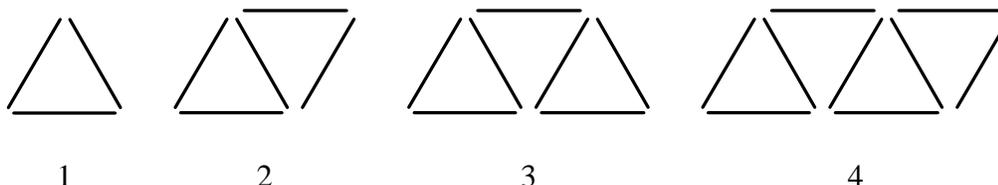
**Pick
count**

TWO



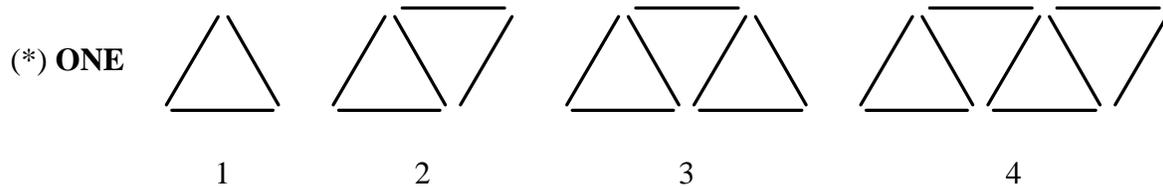
**Pick
count**

THREE

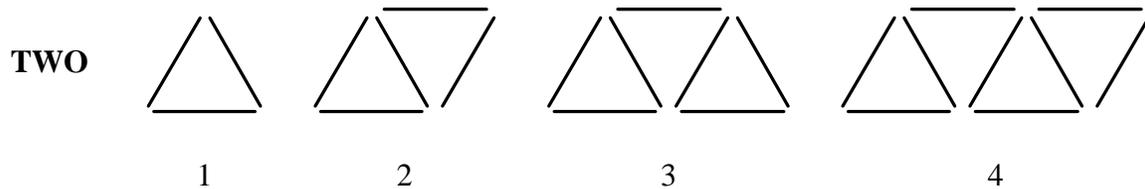


**Pick
count**

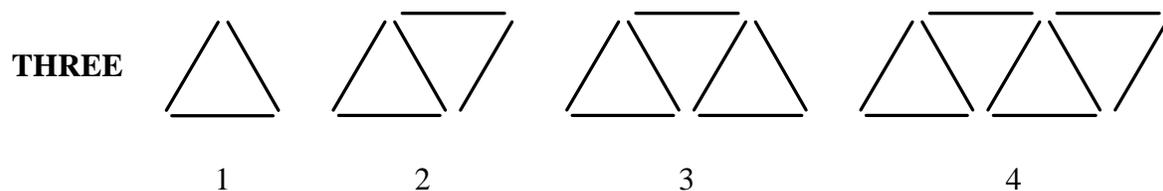
2. The second step is to describe your previous methods in **Words**. These words should describe the general technique you used for each toothpick figure in the set. To start, quickly re-sketch on the following diagrams, the looping/numbering from your work in activity 1. For each set of figures; describe your looping technique in **Words**. The extra space below **Words** will be used for the third step in the process of analyzing sequences of figures (described in activity 6).



Words:



Words:



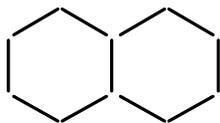
Words:

12. For the following sequence of toothpick figures, *Hexagons*, use a different looping technique and complete each of the following three steps for each set. Try to use looping techniques that extend easily to facilitate answering questions about additional toothpick figures.
- Step One: Loop** each figure and **Numerically** determine the total number of toothpicks in each figure. Mark your number counts on the figures. It may help to model the figures.
 - Step Two:** Convert your looping ideas into **Words**.
 - Step Three:** Convert your looping and word ideas into **Symbols**.

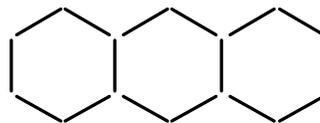
ONE



1



2



3

Pick count

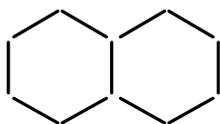
Words

Symbols

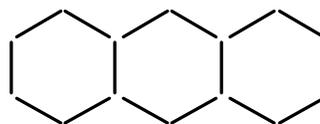
TWO



1



2



3

Pick count

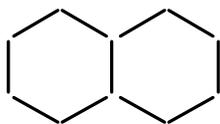
Words

Symbols

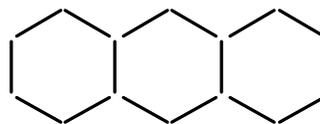
THREE



1



2



3

Pick count

Words

Symbols

13. Are all of your equations in the previous activity really the same? Simplify each of your $T =$ equations as completely as possible.

Equation One

Equation Two

Equation Three

14. How many toothpicks will there be in the 100th *Hexagons* figure?

15. For which *Hexagons* figure are there 1001 toothpicks?

Homework Questions 2.1

INTRODUCTION TO TOOTHPICK FIGURE SEQUENCES

1. Model the following sequence of toothpick figures, *Pentagons*, with toothpicks and:
 - a. Complete each of the three steps for three different looping techniques (Set One, Set Two and Set Three). Try to use looping techniques that extend easily to additional figures.

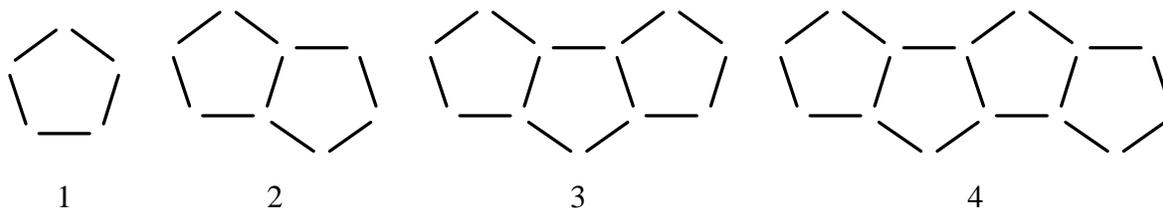
Step One: Loop each figure and **Numerically** determine the total number of toothpicks in each figure. Mark your number counts on the figures.

Step Two: Convert your looping ideas into **Words**.

Step Three: Convert your looping and word ideas into **Symbols**.

- ✓ See the provided Set One, Set Two and Set Three sketch pages for toothpick figures that you can draw on to show your work

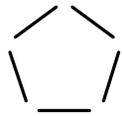
PENTAGONS



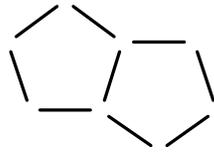
- b. Are all of your equations really the same? Simplify each of your $T =$ equations as completely as possible.
- c. How many toothpicks will there be in the 75th *Pentagons* figure?
- d. For which *Pentagons* figure are there 1001 toothpicks?

Homework 2.1: Introduction to Toothpick Figure Sequences
PENTAGONS SKETCH PAGE

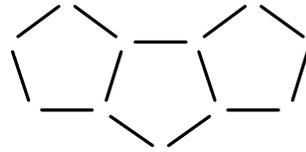
ONE



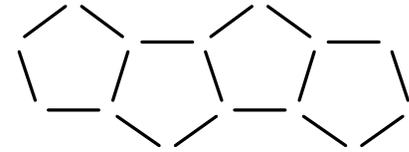
1



2



3



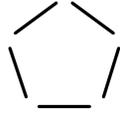
4

Pick count

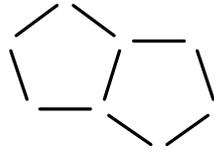
Words

Symbols

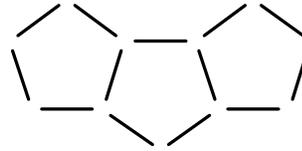
TWO



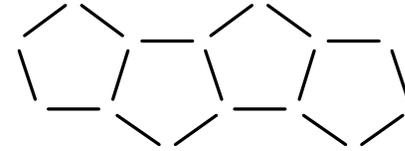
1



2



3



4

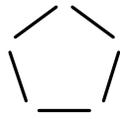
Pick count

Words

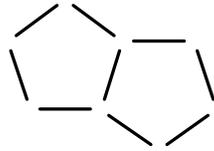
Symbols

Homework 2.1: Introduction to Toothpick Figure Sequences
PENTAGONS SKETCH PAGE

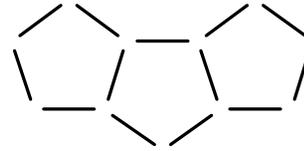
THREE



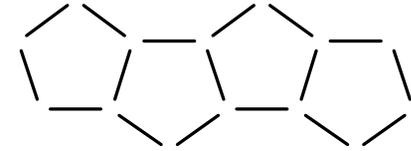
1



2



3



4

Pick count

Words

Symbols

Activity Set 2.2

ALTERNATING TOOTHPICK FIGURE SEQUENCES

PURPOSE

To learn how to analyze sequences of toothpick figures that alternate every other figure or every three figures.

MATERIALS

Toothpicks; flat if possible

Appendix A: *Alternating Sequence Tables*

INTRODUCTION

Alternating Even and Odd Number Sequences

Consider the following number sequence indexed by the counting numbers $n = 1, 2, 3, \dots$

Index n	1	2	3	4	5	6	7	8	9	10
Number Sequence S	1	4	3	8	5	12	7	16	9	20

If you look carefully, you can see that there are two sets of numbers in the number sequence S :

- 1) The numbers indexed by odd n s and 2) The numbers indexed by even n s:

Odd n	1	3	5	7	9
Number Sequence S	1	3	5	7	9
Even n	2	4	6	8	10
Number Sequence S	4	8	12	16	20

It is easy to see the symbolic expression for the odd n s is simply the index number n , and you may also see the symbolic expression for the even n s is twice the index number or $2n$

You can express the entire symbolic equation for the number sequence, S , by writing S as a split equation as illustrated here:

$$S = n \quad \text{when } n \text{ is odd}$$

$$S = 2n \quad \text{when } n \text{ is even}$$

Alternating “Every Three” Number Sequences

Consider the following number sequence:

Index n	1	2	3	4	5	6	7	8	9	10	11	12
Number Sequence T	1	0	1	4	3	2	7	6	3	10	9	4

If you look carefully, you can see there are three sets of numbers in the number sequence T : the numbers indexed by:

- 1) $\{1, 4, 7, \dots\}$ n is 1 more than a multiple of 3 ($n = 3k + 1$)
- 2) $\{2, 5, 8, \dots\}$ n is 2 more than a multiple of 3 ($n = 3k + 2$)
- 3) $\{3, 6, 9, \dots\}$ n is a multiple of 3 ($n = 3k$)
where $k = 1, 2, 3, \dots$ is a placeholder index

For this example, by splitting up the “view” of the three components of the number sequence, it is much easier to develop the symbolic expression for each of the three parts of the number sequence.

$n = 3k + 1$	1		4		7		10		
Number Sequence T	1		4		7		10		
Symbolic Expression	n		n		n		n		
$n = 3k + 2$		2		5		8		11	
Number Sequence T		0		3		6		9	
Symbolic Expression		$n - 2$		$n - 2$		$n - 2$		$n - 2$	
$n = 3k$			3		6		9		12
Number Sequence T			1		2		3		4
Symbolic Expression			$\frac{n}{3}$		$\frac{n}{3}$		$\frac{n}{3}$		$\frac{n}{3}$

All together:

$$T = n \quad n = 3k + 1$$

$$T = n - 2 \quad n = 3k + 2$$

$$T = \frac{n}{3} \quad n = 3k$$

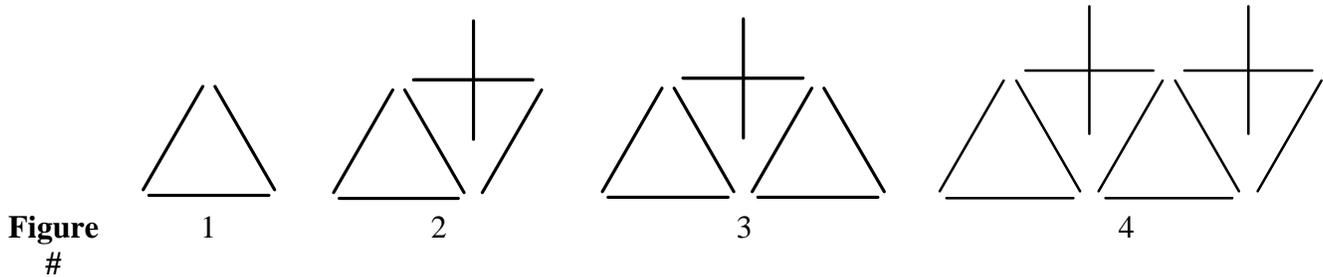
$$k = 1, 2, 3, \dots$$

Visual Guide to Alternating Number Sequences

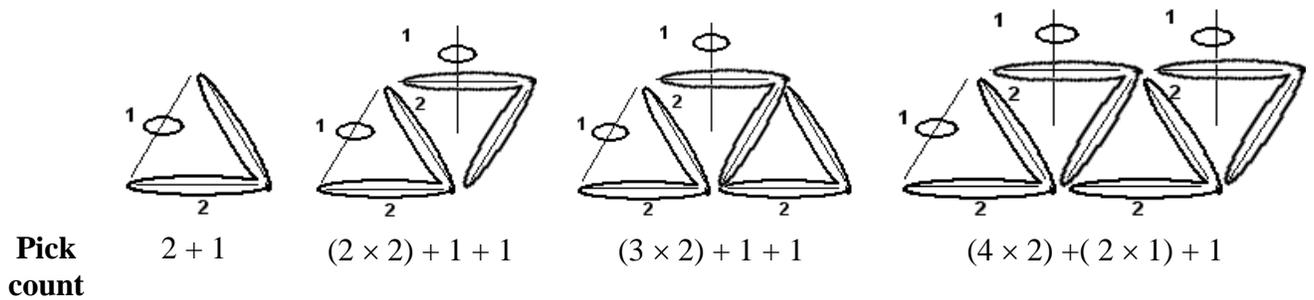
The alternating number sequences you will analyze will be modeled by toothpick figures. When you loop and count the toothpicks in the various figures, the “even / odd” and “every three” nature of the patterns will usually be *visually* obvious.

Analyzing Figures in Pieces

The following sequence of toothpick figures clearly alternates between the even figures and the odd figures. There are many ways to loop the picks to count them; some looping techniques may be harder to describe in words and then generalize to a symbolic pattern. It often helps to view a pattern as individual components and then analyze the figures one component at a time.

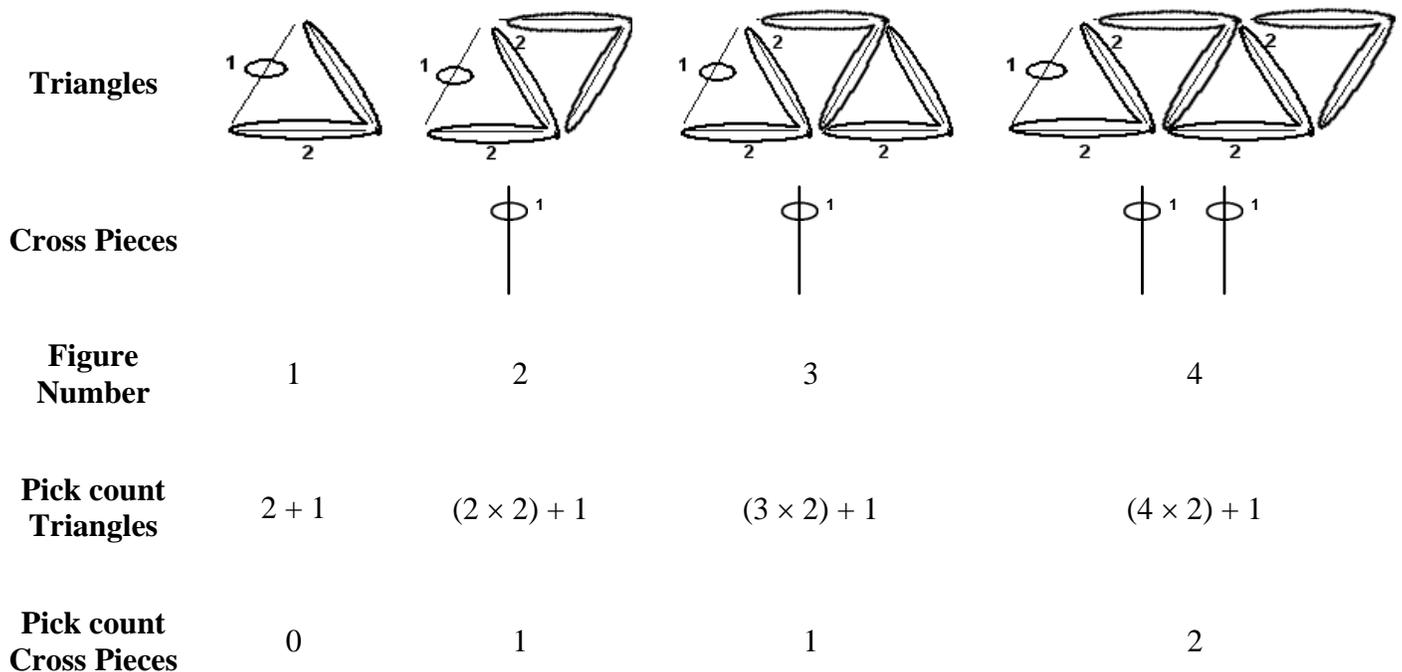


Here is a combined looping and loop count of the figures:



Now suppose you think of the figures as two set of pieces:

- One—the triangles
- Two—the alternating cross pieces



You may find the pattern now easier to analyze in two pieces

Figure Number	1	2	3	4
Pick count Triangles	$2 + 1$	$(2 \times 2) + 1$	$(3 \times 2) + 1$	$(4 \times 2) + 1$
Words	2 for each figure number + 1 for the first			
Symbols	$T = 2n + 1$ for both even and odd figures			

Continuing on, we now analyze the vertical cross piece count:

Figure Number	1	2	3	4
Pick count Cross Piece	0	1	1	2
Words	Half of 1 less than the figure number	Half of the figure number	Half of 1 less than the figure number	Half of the figure number
Symbols	$T = \frac{n-1}{2}$	$T = \frac{n}{2}$		

Therefore, all together, we have:

When n is even:

Words: 2 for each figure plus 1 for the end, plus half of the figure number for the cross pieces.

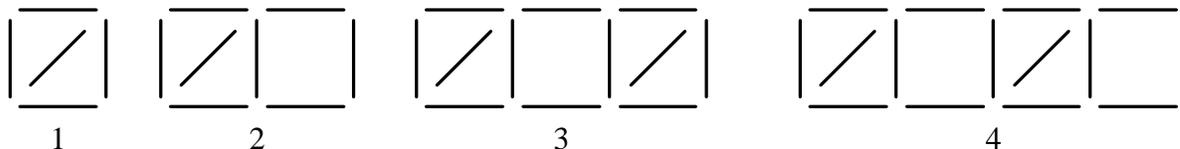
Symbols: $T = 2n + 1 + \frac{n}{2}$

When n is odd

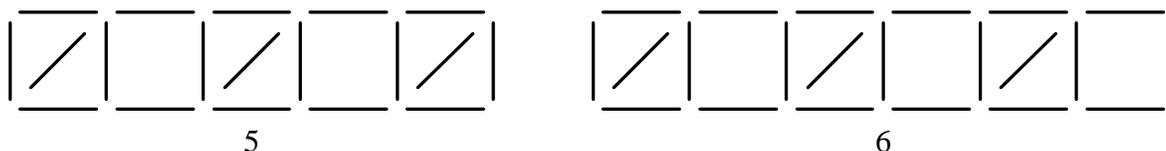
Words: 2 for each figure plus 1 for the end, plus half of one more than the figure number for the cross pieces.

Symbols: $T = 2n + 1 + \frac{n-1}{2}$

1. (*) For the following sequence of toothpick figures, *Squares with Diagonals*, model the figures with toothpicks and notice that the odd figures and the even figures are slightly different.
 - a. **Step One: Loop** and number each figure and **Numerically** determine the total number of toothpicks in each figure. Mark your number counts on the figures.
 - b. **Step Two: Convert** your looping ideas into **Words**—use one set of words for the odd figures and one set of words for the even figures.
 - c. **Step Three: Convert** your looping and word ideas into **Symbols***—use one symbolic equation for the odd figures and one symbolic equation for the even figures. Simplify your symbolic equations and check them for $n = 1, 2, 3, 4, 5$ and 6 .



Pick count _____



Pick count _____

Odd Words

Even Words

Odd Symbols

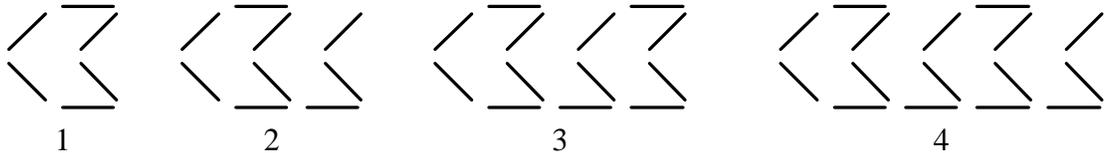
Even Symbols

2. (*) How many toothpicks will there be in the 99th *Squares with Diagonals* figure? How many toothpicks will there be in the 100th *Squares with Diagonals* figure?

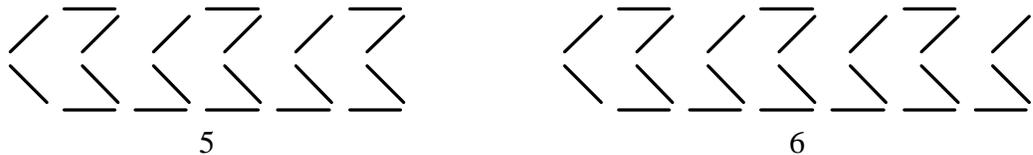
3. (*) For which *Squares with Diagonals* figure are there 54 toothpicks? For which *Squares with Diagonals* figure are there 120 toothpicks?"

* You may find Appendix A: *Alternating Sequence Tables* helpful at this stage.

4. For the following sequence of toothpick figures, *Arrows*, model the figures with toothpicks and notice that the odd figures and the even figures are slightly different.
 - a. **Step One: Loop** and number each figure and **Numerically** determine the total number of toothpicks in each figure. Mark your number counts on the figures.
 - b. **Step Two:** Convert your looping ideas into **Words**—use one set of words for the odd figures and one set of words for the even figures.
 - c. **Step Three:** Convert your looping and word ideas into **Symbols**—use one symbolic equation for the odd figures and one symbolic equation for the even figures. Simplify your symbolic equations and check them for $n = 1, 2, 3, 4, 5$ and 6 .



Pick count _____ _____ _____ _____



Pick count _____ _____

Odd Words

Even Words

Odd Symbols

Even Symbols

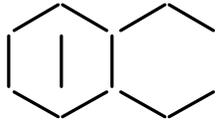
5. How many toothpicks will there be in the 99th *Arrows* figure? How many toothpicks will there be in the 100th *Arrows* figure?

6. For which *Arrows* figure are there 41 toothpicks? For which *Arrows* figure are there 107 toothpicks?

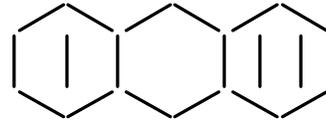
7. (*) For the following sequence of toothpick figures, *Hexagon Belt*, model the figures with toothpicks and notice that when $n = 3k + 1$, $n = 3k + 2$ and $n = 3k$, the figures are different (remember, $k = 1, 2, 3, \dots$ is just a placeholder index).
- a. **Step One: Loop** and number each figure and **Numerically** determine the total number of toothpicks in each figure. Mark your number counts on the figures.



1



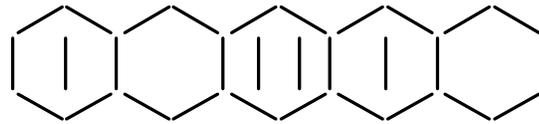
2



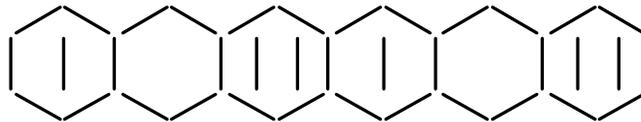
3



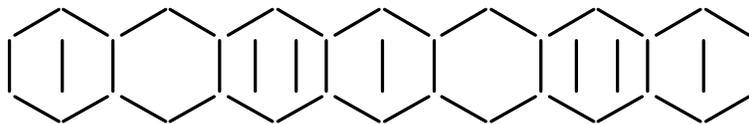
4



5



6



7

- b. **Step Two:** Convert your looping ideas into **Words**—use one set of words for the $n = 3k + 1$ figures, one set of words for the $n = 3k + 2$ figures and one set of words for the $n = 3k$ figures.
- c. **Step Three:** Convert your looping and word ideas into **Symbols**—use one symbolic equation for $n = 3k + 1$, one for $n = 3k + 2$ and one for $n = 3k$. Simplify your symbolic equations and check them for $n = 1, 2, 3, 4, 5$ and 6 .

$n = 3k + 1$ Words
($n = 1, 4, 7, \dots$)

$n = 3k + 2$ Words
($n = 2, 5, 8, \dots$)

$n = 3k$ Words
($n = 3, 6, 9, \dots$)

$n = 3k + 1$ Symbols

$n = 3k + 2$ Symbols

$n = 3k$ Symbols

8. How many toothpicks will there be in the 30th *Hexagon Belt* figure? How many toothpicks will there be in the 31st *Hexagon Belt* figure? How many toothpicks will there be in the 32nd *Hexagon Belt* figure?
9. For which *Hexagon Belt* figure are there 223 toothpicks? For which *Hexagon Belt* figure are there 282 toothpicks? For which *Hexagon Belt* figure are there 433 toothpicks?

Homework Questions 2.2

ALTERNATING TOOTHPICK FIGURE SEQUENCES

1. Model the following sequence of toothpick figures, *Dashes and Stars*, with toothpicks and:

a.

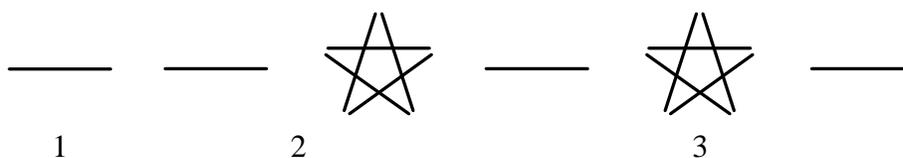
Step One: Loop and number each figure and **Numerically** determine the total number of toothpicks in each figure. Mark your number counts on the figures.

Step Two: Convert your looping ideas into **Words**.

Step Three: Convert your looping and word ideas into **Symbols**. Simplify your symbolic equations and check them for $n = 1, 2, 3, 4, 5$ and 6.

- ✓ See the provided sketch pages for figures 1 – 6 that you can draw on to show your work

DASHES AND STARS



- b. How many toothpicks will there be in the 99th *Dashes and Stars* figure?
- c. How many toothpicks will there be in the 100th *Dashes and Stars* figure?
- d. How many toothpicks will there be in the 101st *Dashes and Stars* figure?
- e. For which *Dashes and Stars* figure are there 84 toothpicks?
- f. For which *Dashes and Stars* figure are there 97 toothpicks?
- g. For which *Dashes and Stars* figure are there 1011 toothpicks?
2. Model the following sequence of toothpick figures, *Rectangles with Pluses*, with toothpicks and:

a.

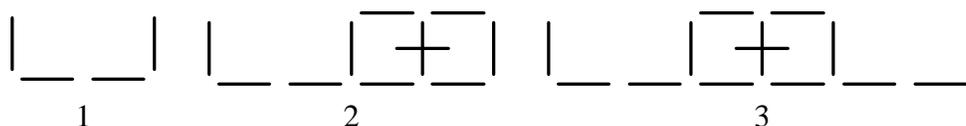
Step One: Loop and number each figure and **Numerically** determine the total number of toothpicks in each figure. Mark your number counts on the figures.

Step Two: Convert your looping ideas into **Words**.

Step Three: Convert your looping and word ideas into **Symbols**. Simplify your symbolic equations and check them for $n = 1, 2, 3, 4, 5$ and 6.

- ✓ See the provided sketch pages for figures 1 – 6 that you can draw on to show your work

RECTANGLES WITH PLUSES



- b. How many toothpicks will there be in the 99th *Rectangles with Pluses* figure?
 - c. How many toothpicks will there be in the 100th *Rectangles with Pluses* figure?
 - d. How many toothpicks will there be in the 101st *Rectangles with Pluses* figure?
 - e. For which *Rectangles with Pluses* figure are there 124 toothpicks?
 - f. For which *Rectangles with Pluses* figure are there 171 toothpicks?
 - g. For which *Rectangles with Pluses* figure are there 1009 toothpicks?
3. Model the following sequence of toothpick figures, *Squares with Slides*, with toothpicks and:
- a.

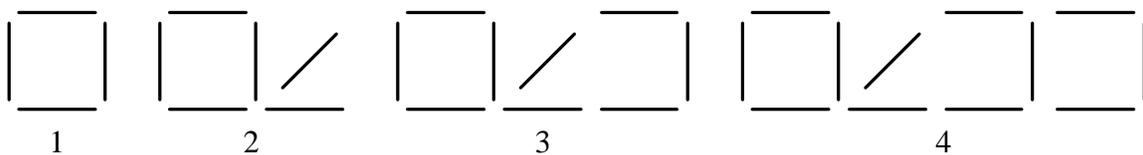
Step One: Loop and number each figure and **Numerically** determine the total number of toothpicks in each figure. Mark your number counts on the figures.

Step Two: Convert your looping ideas into **Words**.

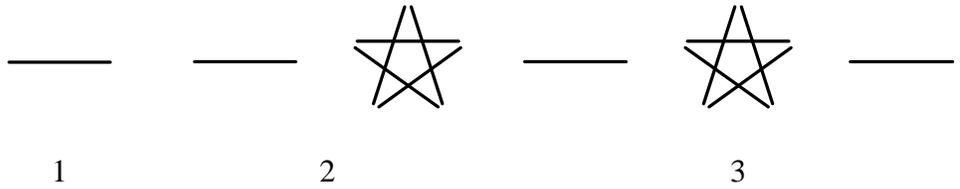
Step Three: Convert your looping and word ideas into **Symbols**. Simplify your symbolic equations and check them for $n = 1, 2, 3, 4, 5$ and 6.

✓ See the provided sketch pages for figures 1 – 6 that you can draw on to show your work

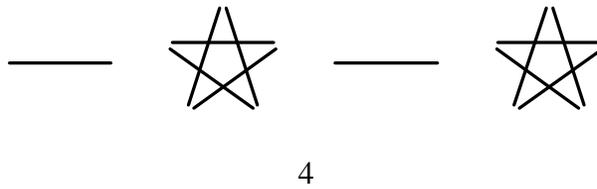
SQUARES WITH SLIDES



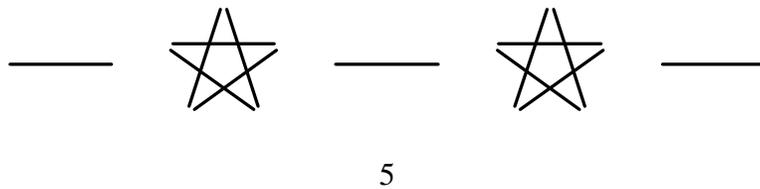
- b. How many toothpicks will there be in the 99th *Squares with Slides* figure?
- c. How many toothpicks will there be in the 100th *Squares with Slides* figure?
- d. How many toothpicks will there be in the 101st *Squares with Slides* figure?
- e. For which *Squares with Slides* figure are there 76 toothpicks?
- f. For which *Squares with Slides* figure are there 94 toothpicks?
- g. For which *Squares with Slides* figure are there 297 toothpicks?



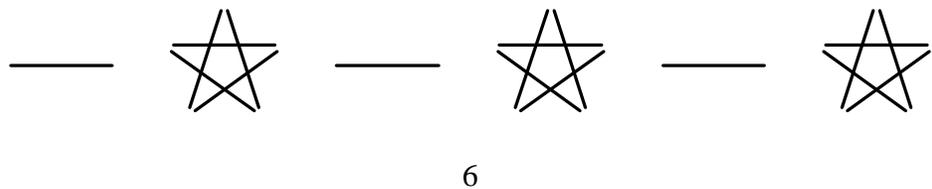
Pick count



Pick count



Pick count



Pick count

Words

Symbols

Homework 2.2: Alternating Toothpick Figure Sequences
RECTANGLES WITH PLUSES SKETCH PAGE



1

2

3

Pick count



4

Pick count



5

Pick count

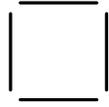


6

Pick count

Words

Symbols



1

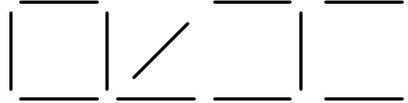
Pick count



2



3



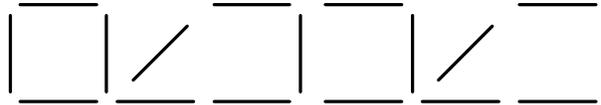
4

Pick count



5

Pick count



6

Pick count

Words

Symbols

Activity Set 2.3

INTRODUCTION TO TILE FIGURE SEQUENCES

PURPOSE

To learn how to analyze sequences of tile figures numerically, in words and symbolically to see, define and explore an underlying algebraic pattern. To learn how to model algebraic patterns with black tiles and black n -strips and to represent the algebraic pattern graphically.

MATERIALS

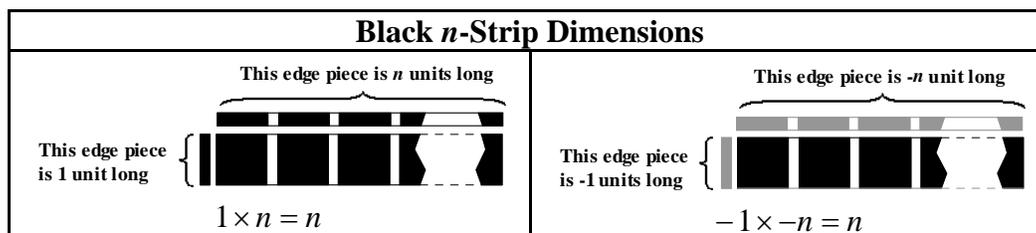
Black tiles and black n -strips



INTRODUCTION

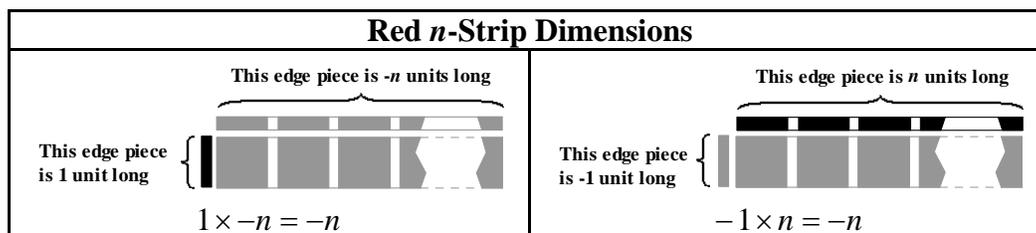
Black and Red n -Strips*

A Black n -Strip, represented by the variable n , represents a positive (output) number; $n = 1, 2, 3 \dots$. The output value of a black n -strip is the same as the input number ($n = 1, 2, 3 \dots$). The height of the black n -strip is fixed and is the same length as the edges of the black and red tiles (1 or -1), but the length of the black n -strip is an arbitrary length. Since it is black, the dimensions of the black n -strip can be $1 \times n$ or $-1 \times -n$.



Red n Strips

A Red n -Strip, represented by the variable $-n$, represents a negative (output) number; $n = -1, -2, -3 \dots$. The output value of a red n -strip is the opposite of the input number ($n = 1, 2, 3 \dots$). The height of the red n -strip is fixed and is the same length as the edges of the black and red tiles (1 or -1), but the length of the red n -strip is an arbitrary length. Since it is red, the dimensions of the red n -strip can be $1 \times -n$ or $-1 \times n$.



* Note: Only black n -strips will be used in this activity set.

Using Black and Red n -Strip Edge Pieces to Measure Length

In Chapter 1, we used black and red edge pieces to measure lengths of 1 unit and -1 unit. In the same way, a thin piece of a black n -strip (black n -strip edge piece) measures a length of n units and a thin piece of a red n -strip (red n -strip edge piece) measures a length of $-n$ units. These edge pieces are shown, in context, in the previous descriptions of black n -strips and red n -strips.

Modeling with Black n -Strips

A sequence of tile figures whose n th figure is modeled by a black n -strip might look like one of the following horizontal or vertical sets of tile figures:

Horizontal Tile Figures					...		or	
	Figure #	1	2	3		4	n	
Size	1×1	1×2	1×3	1×4		$1 \times n$		

Vertical Tile Figures					...		or	
	Figure #	1	2	3		4	n	
Size	1×1	2×1	3×1	4×1		$n \times 1$		

Using Black n -Strips to Model the n th Figure of a Tile Sequence

The black n -strip can be all or part of the n th figure for a tile sequence and can be combined with other algebra pieces and black and red tiles.

SKETCHING TIPS

Sketching Black n -Strips

To sketch a black n -strip, you may wish to simply sketch an outline of the strip. In future activity sets you will also need to sketch red n -strips. To distinguish the black n -strip outline, label a black n -strip with a B for black.



Sketching Black and Red n -Strip Edge Pieces

To sketch a black or red n -strip edge piece, simply sketch an outline of the strip, label it Bn or $R-n$ or shade the strip and label it n or $-n$.



Sketching Tiles for Tile Sequences

To sketch black or red tiles for tile sequences, the shape of the tile matters. It may be easiest to sketch blank squares, lightly shade the black tiles and label the interiors of the red tiles R or to just label both the black and the red squares with a B or an R respectively.

**Sketching vs. Art Work**

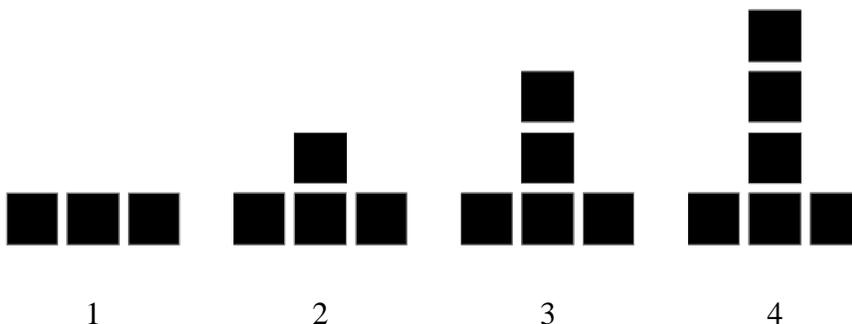
The overall goal of this activity set is to think about and work with patterns and algebraic relationships. For all of the activities in this activity set, in future activity sets and in homework, develop a quick (but clear) sketch technique to use when recording your algebra piece models. There is no intellectual value in creating work-of-art algebra piece drawings and extra time spent drawing perfect pictures is time away from mathematical thinking.

1. (*) Model the following sequence, *Tees*, of black tile figures with black tiles and:
- Step One: Loop** and number each figure and **Numerically** determine the total number of black tiles in each figure. Mark your number counts on the figures.
 - Step Two:** Convert your looping ideas into **Words**.
 - Step Three:** Convert your looping and word ideas into **Symbols**. Use n for the figure number and T for the total number of black tiles. Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4 .

Do each of these three steps for Set One and for Set Two; use a different looping technique for each set.

TEES SET ONE

TILE FIGURES



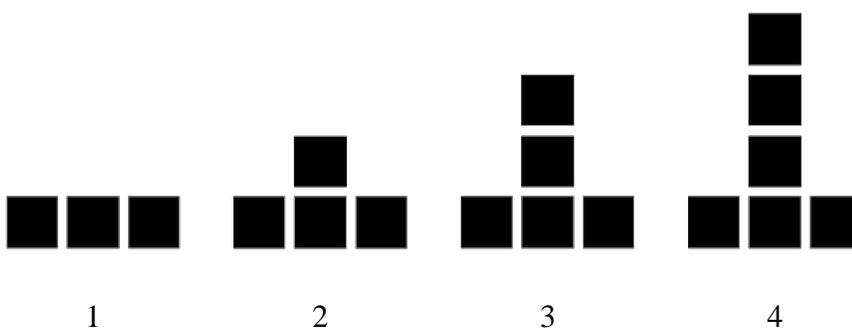
Tile count

Words

Symbols

TEES SET TWO

TILE FIGURES



Tile count

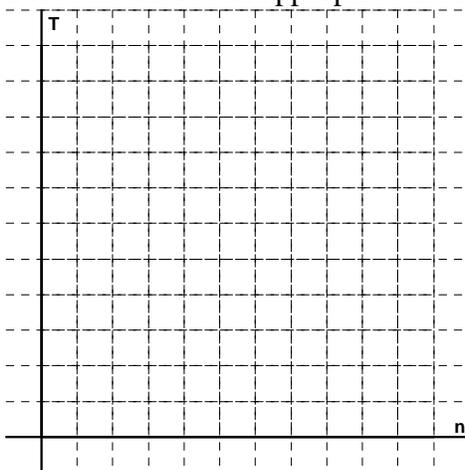
Words

Symbols

2. (*) Describe the 100th *Tee* figure. What does it look like? How many black tiles are in it?
3. (*) Which *Tee* figure will have 2002 black tiles? Describe the figure.
4. (*) What does the n th *Tee* figure look like? Use your black n -strips and tiles, as needed, to model the figure; be sure to have the pieces oriented to look like the other tile figures in the *Tee* sequence (that is; like an upside down T). You do not need to sketch edge pieces. Sketch and describe the n th *Tee* figure here. Label the pieces clearly.
5. (*) For the tile sequence *Tees*, complete the second row of output values in the following t-table for the total number of black tiles, T , in each indicated figure.

n	1	2	3	4	5	6	n
T	3	4					

6. (*) On the following grid; plot the ordered pairs from the t-table associated with $n = 1, 2 \dots 6$ for the tile sequence *Tees*. Label the axes with appropriate numbers.

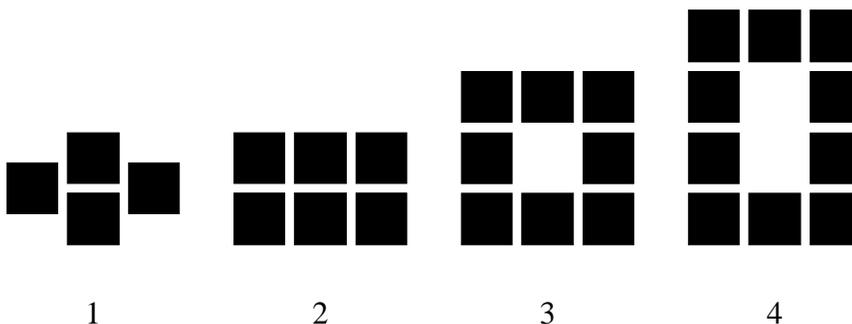


- a. Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What T (output) value do you estimate for $n = 10$ by just looking at the pattern of the graph? Check this value by using your previous symbolic work.

- b. List at least three observations about the *Tees* tile sequence and the graph associated with this tile sequence.

7. Model the following sequence, *Windows*, of black tile figures with black tiles and:
- Step One: Loop** and number each figure and **Numerically** determine the total number of black tiles in each figure. Mark your number counts on the figures.
 - Step Two:** Convert your looping ideas into **Words**.
 - Step Three:** Convert your looping and word ideas into **Symbols**. Use n for the figure number and T for the total number of black tiles. Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4 .

TILE FIGURES



Tile count

Words

Symbols

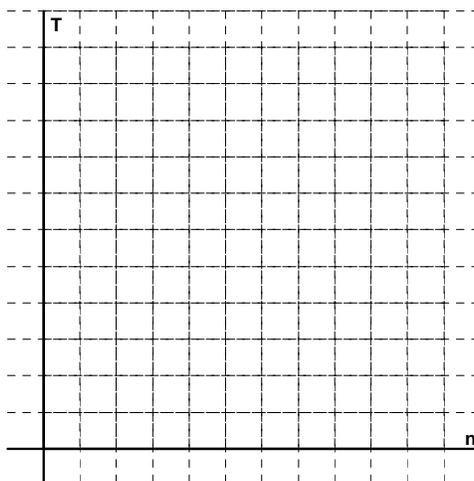
8. Describe the 100th *Window* figure. What does it look like? How many black tiles are in it?
9. Which *Window* figure will have 2002 black tiles? Describe the figure.

10. What does the n th *Window* figure look like? Use your black n -strips and tiles, as needed, to model this figure; be sure to have the pieces oriented to look like the other tile figures in the *Window* sequence. You do not need to sketch edge pieces. Sketch, label and describe the figure.

11. For the tile sequence *Windows*, fill out the second row of output values in the following t-table for the total number of black tiles, T , in each indicated figure.

n	1	2	3	4	5	6	n
T							

12. On the following grid; plot the ordered pairs from the t-table associated with $n = 1, 2 \dots 6$ for the tile sequence *Windows*. Label the axes with appropriate numbers.



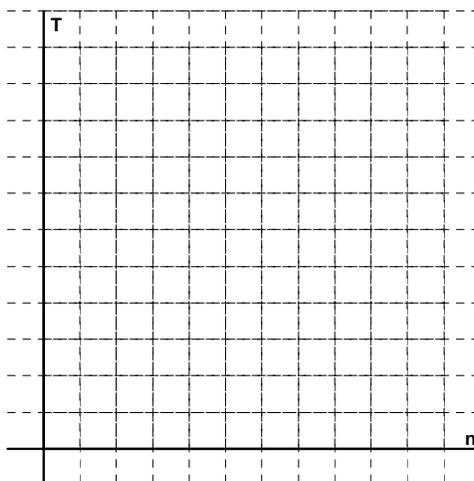
- Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What T (output) value do you estimate for $n = 10$ by just looking at the pattern of the graph? Check this value by using your previous symbolic work.
- List at least three observations about the *Windows* tile sequence and the graph associated with this tile sequence.

16. What does the n th *Fork* figure look like? Use your black n -strips and tiles, as needed, to model this figure; be sure to have the pieces oriented to look like the other tile figures in the *Fork* sequence. You do not need to sketch edge pieces. Sketch, label and describe the figure.

17. For the tile sequence *Forks*, fill out the second row of output values in the following t-table for the total number of black tiles, T , in each indicated figure.

n	1	2	3	4	5	6	n
T							

18. On the following grid; plot the ordered pairs from the t-table associated with $n = 1, 2 \dots 6$ for the tile sequence *Forks*. Label the axes with appropriate numbers.



- Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What T (output) value do you estimate for $n = 10$ by just looking at the pattern of the graph? Check this value by using your previous symbolic work.
- List at least three observations about the *Forks* tile sequence and the graph associated with this tile sequence.

19. For the tile sequences, *Tees*, *Windows* and *Forks*, the n th figures involved only black tiles and black n -strips. Describe the features of the tile figure sequences that generalize to these types of n th figures and relate these features to their graphs.

Homework Questions 2.3

INTRODUCTION TO TILE FIGURE SEQUENCES

1. Model the following sequence, E_s , of tile figures with black tiles and:

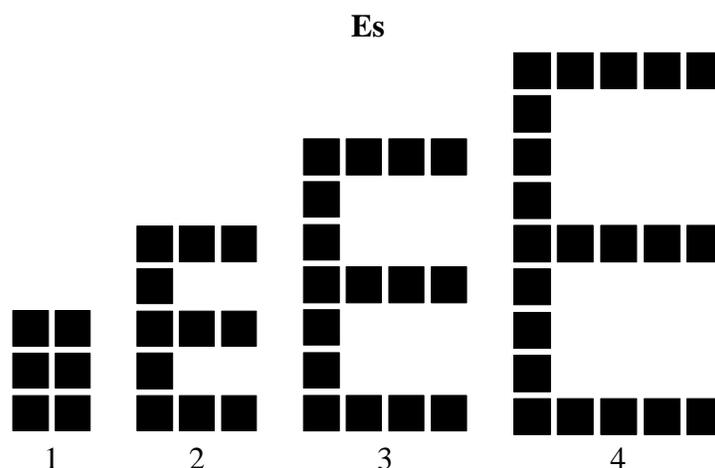
a.

Step One: Loop and number each figure and **Numerically** determine the total number of black tiles in each figure. Mark your number counts on the figures.

Step Two: Convert your looping ideas into **Words**.

Step Three: Convert your looping and word ideas into **Symbols**. Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4.

✓ See the provided sketch pages for figures that you can draw on to show your work



- b. Describe the 100th E figure. What does it look like? How many black tiles are in it?
- c. Which E figure will have 2001 black tiles? Describe the figure.
- d. What does the n th E figure look like? Use your black n -strips and tiles, as needed, to model this figure; be sure to have the pieces oriented to look like the other tile figures in the E sequence. You do not need to sketch edge pieces. Sketch, label and describe the figure.
- e. Create a t-table for the total number of black tiles, T , in the E tile sequence, for figure number inputs $n = 1, 2 \dots 6$.
- f. Plot the ordered pairs from the t-table on graph paper. Label the axes with appropriate numbers.
- g. Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What T (output) value do you estimate for $n = 10$ by just looking at the pattern of the graph? Check this value by using your previous symbolic work.

2. Model the following sequence, *Squares with Diagonals*, of tile figures with black tiles and:
- a.

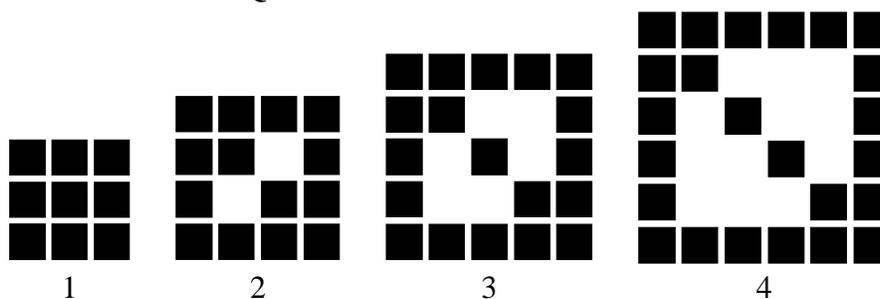
Step One: Loop and number each figure and **Numerically** determine the total number of black tiles in each figure. Mark your number counts on the figures.

Step Two: Convert your looping ideas into **Words**.

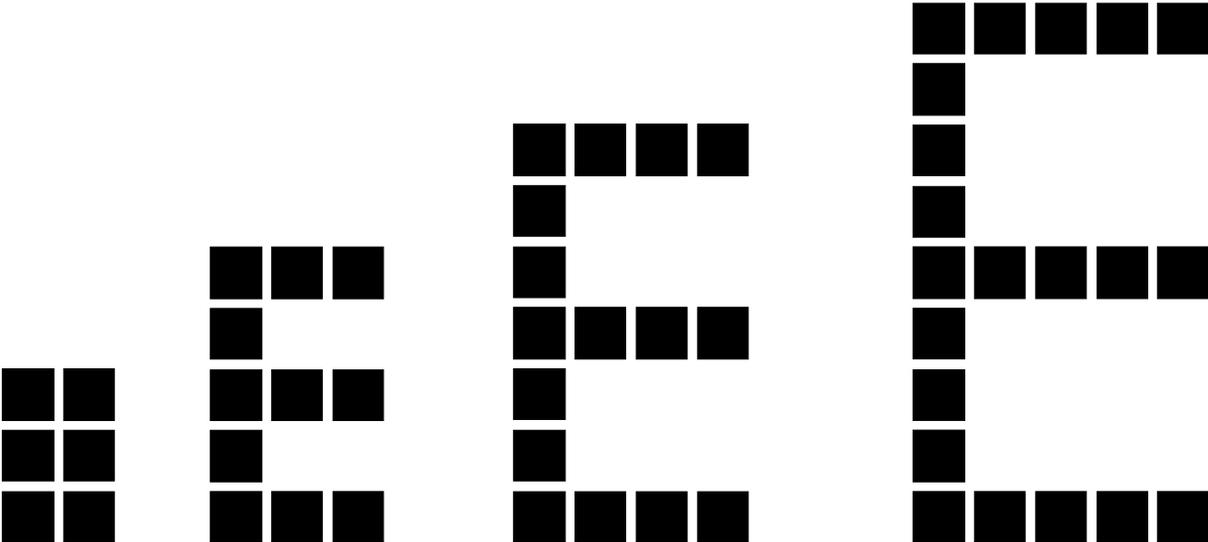
Step Three: Convert your looping and word ideas into **Symbols**. Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4.

- ✓ See the provided sketch pages for figures that you can draw on to show your work

SQUARES WITH DIAGONALS



- b. Describe the 100th *Square with Diagonal* figure. What does it look like? How many black tiles are in it?
- c. Which *Square with Diagonal* figure will have 2504 black tiles? Describe the figure.
- d. What does the n th *Square with Diagonal* figure look like? Use your black n -strips and tiles, as needed, to model this figure; be sure to have the pieces oriented to look like the other tile figures in the *Squares with Diagonals* sequence. You do not need to sketch edge pieces. Sketch, label and describe the figure.
- e. Create a t-table for the total number of black tiles, T , in the *Squares with Diagonals* tile sequence, for figure number inputs $n = 1, 2 \dots 6$.
- f. Plot the ordered pairs from the t-table on graph paper. Label the axes with appropriate numbers.
- g. Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What T (output) value do you estimate for $n = 10$ by just looking at the pattern of the graph? Check this value by using your previous symbolic work.



1

2

3

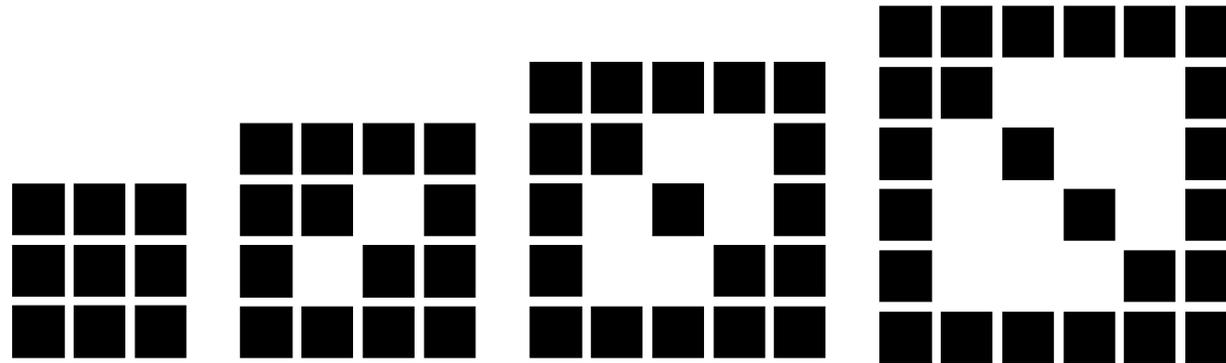
4

Tile count

Words

Symbols

Homework 2.3: Introduction to Tile Figure Sequences
SQUARES WITH DIAGONALS SKETCH PAGE



1

2

3

4

Tile count

Words

Symbols

Activity Set 2.4

TILE FIGURES AND ALGEBRAIC EQUATIONS

PURPOSE

To learn how to use algebra piece models to answer questions about sequences of tile figures and to solve basic algebra problems. To learn how to use algebraic equations to build tile figure sequences. To learn how to connect the work with algebra piece models to their corresponding symbolic steps.

MATERIALS

Black and red tiles and black n -strips



INTRODUCTION

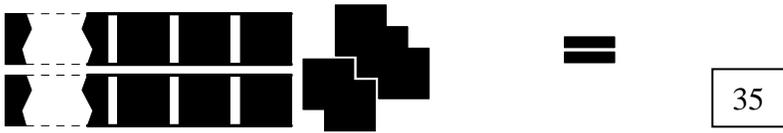
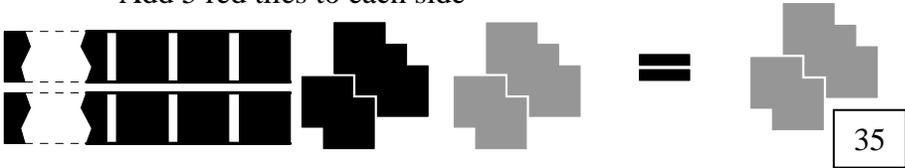
Modeling an Equal Symbol

Two black edge pieces make an excellent = symbol while working with algebra pieces and equations. 

Using Algebra Pieces to Solve Problems

When using algebra pieces to solve problems, set the pieces up on your table and use them, as you have been using black and red tiles, to work out the question you are considering. In many cases, there will be a large number involved that is impractical to model with black or red tiles. You may wish to keep track of these large numbers by jotting them on scraps of paper.

Example: For the equation $T = 2n + 5$, use your algebra piece representation of the n th $T = 2n + 5$ figure to determine which figure has 35 black tiles. Use a table to sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

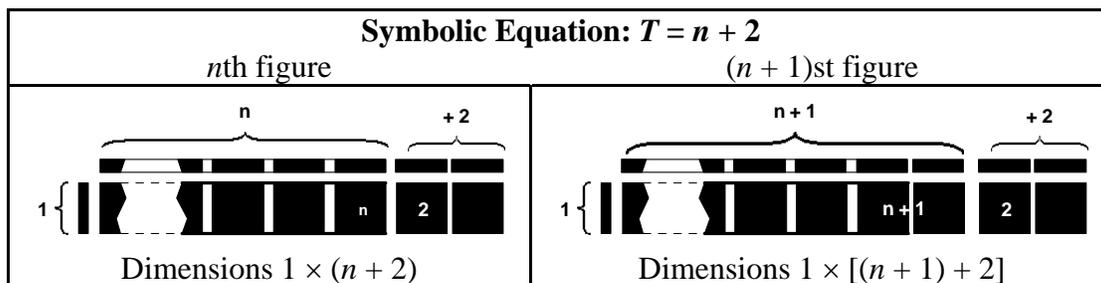
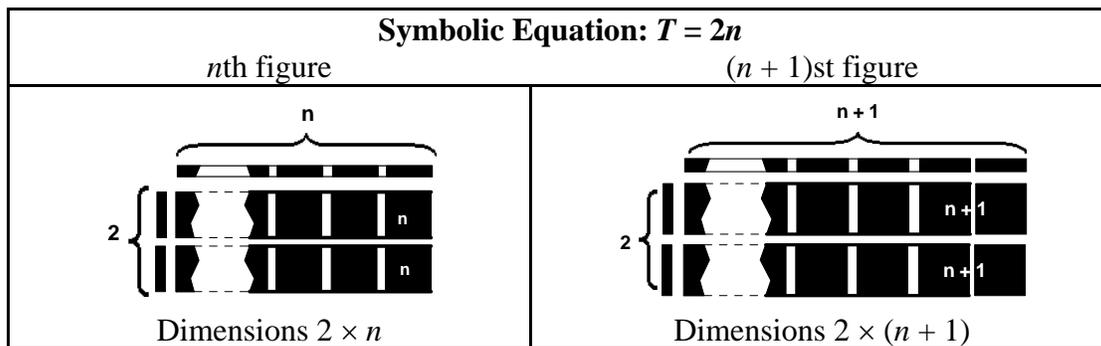
ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK
<p style="text-align: center;">Set up pieces</p> 	<p style="text-align: center;">Set up equation</p> $2n + 5 = 35$
<p style="text-align: center;">Add 5 red tiles to each side</p> 	$2n + 5 = 35$ $+5 \quad +5$

ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK
<p style="text-align: center;">Simplify</p> 	$2n = 30$
<p>Divide each side into 2 (equally sized) groups</p>  	$\frac{2n}{2} = \frac{30}{2}$
<p style="text-align: center;">Simplify</p> 	$n = 15$
<p>CHECK $2(15) + 5 = 35\checkmark$</p>	

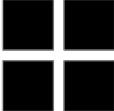
Modeling (n + 1)st Figures

The (n + 1)st figure in a tile sequence is the next figure after the nth figure in a tile sequence. As you work with algebra piece models you will often be asked to model both the nth figure and the (n + 1)st figure (two arbitrary, consecutive figures) for a given tile sequence. Let’s look at a two examples to see how (n + 1)st figures are constructed.

To build a (n + 1)st figure from an nth figure, change all side dimensions **n** in the nth figure to a side dimensions **n + 1** for the (n + 1)st figure. Note fixed lengths such as 1 or 2 do not change. Edge pieces are shown here to emphasize the dimension changes from the nth figure to the (n + 1)st figure.



1. Model the following sequence, *Rectangles*, with black tiles and:
 - a. **Step One: Loop** and number each figure and **Numerically** determine the total number of black tiles in each figure. Mark your number counts on the figures.
 - b. **Step Two:** Convert your looping ideas into **Words**.
 - c. **Step Three:** Convert your looping and word ideas into **Symbols**. Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4.

TILE FIGURES				
Figure #	1	2	3	4
Tile count	_____	_____	_____	_____
Words			Symbols	

2. Describe the 100th *Rectangle* figure. What does it look like? How many black tiles are in it?

3. Which *Rectangle* figure will have 2002 black tiles? Describe the figure.

4. What does the n th *Rectangle* figure look like? Use your black n -strips and black tiles, as needed, to model the figure, be sure to have the pieces oriented to look like the other tile figures in the *Rectangles* sequence. You do not need to sketch edge pieces. Sketch and describe the n th *Rectangle* figure here. Label the pieces clearly.

5. (*) If the collection of tiles in a certain *Rectangle* figure is tripled and 10 more black tiles are added, there will be a total of 160 black tiles. Use three copies of your algebra piece representation of the n th *Rectangle* to help determine which *Rectangle* figure this is. Use the table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

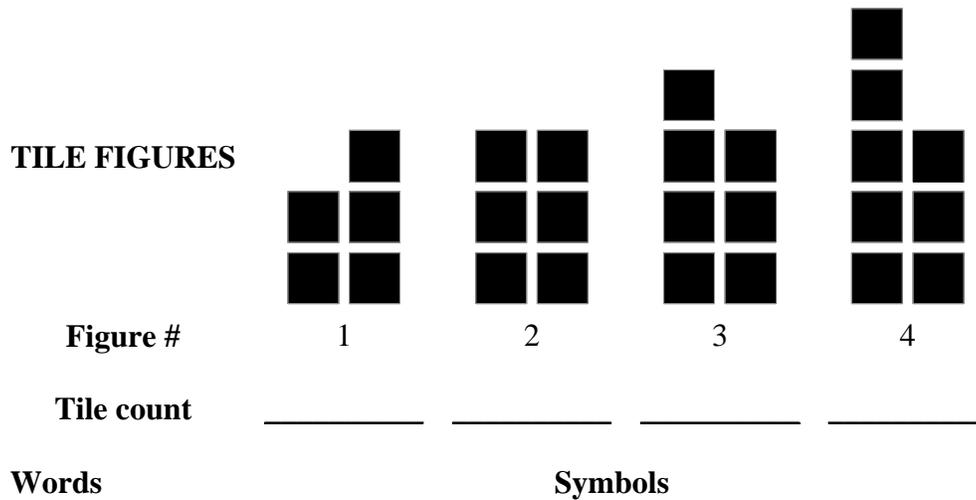
ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK

6. (*) What does the $(n + 1)$ st *Rectangle* figure look like? Use your black n -strips and black tiles, as needed, to model the figure, be sure to have the pieces oriented to look like the other tile figures in the *Rectangles* sequence. You do not need to show the edge pieces for the $(n + 1)$ st figure, but you may find them useful for constructing the $(n + 1)$ st figure. Sketch and describe the $(n + 1)$ st *Rectangle* figure here. Label the pieces clearly.

7. Two consecutive *Rectangle* figures (the n th and $(n + 1)$ st figures) have a total of 802 black tiles. Which two figures are they? Use your algebra piece representation of the n th *Rectangle* and the $(n + 1)$ st *Rectangle* to determine the answer. Use the table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK

8. Model the following sequence, *Chairs*, with black tiles and:
- Step One: Loop** and number each figure and **Numerically** determine the total number of black tiles in each figure. Mark your number counts on the figures.
 - Step Two:** Convert your looping ideas into **Words**.
 - Step Three:** Convert your looping and word ideas into **Symbols**. Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4 .



9. Describe the 100th *Chair* figure. What does it look like? How many black tiles are in it?

10. Which *Chair* figure will have 2002 black tiles? Describe the figure.

11. What do the n th and the $(n + 1)$ st *Chair* figures look like? Use your black n -strips and black tiles, as needed, to model these figures; be sure to have the pieces oriented to look like the other tile figures in the *Chairs* sequence. You do not need to show edge pieces in your final figures. Sketch and describe the figures here. Label the pieces clearly. The $(n + 1)$ st figure will be used for activity 13.
12. If 17 black tiles are added to a certain *Chair* figure, there will be a total of 40 black tiles. Use your algebra piece representation of the n th *Chair* to help determine which *Chair* figure this is. Use the table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK

13. Two consecutive *Chair* figures have a total of 1009 black tiles. Which two figures are they? Use your algebra pieces to determine the answer. Use the table and sketch your algebra piece work in the left column and write the corresponding symbolic steps in the right column.

ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK

14. Consider the equation $T = 3n + 4$. Use your black tiles to build figures 1, 2, 3 and 4 for a tile sequence corresponding to the equation $T = 3n + 4$. Try to make your figures “look like” $3n + 4$ and use a consistent orientation for each figure. Sketch the figures here and give the number of black tiles in each figure.

TILE FIGURES

Figure #	1	2	3	4
# black tiles	_____	_____	_____	_____

15. Use algebra pieces to model the n th figure for $T = 3n + 4$ that matches your tile figures. Sketch the figure. Label the algebra pieces clearly.
16. Describe the 100th figure for $T = 3n + 4$. What does it look like? How many black tiles are in it?
17. Which $T = 3n + 4$ figure will have 2005 black tiles? Describe the figure.

18. Two consecutive $T = 3n + 4$ figures have a total of 6011 black tiles. Which two figures are they? Model the question with your algebra pieces to determine the answer. Use the table and sketch your algebra piece work in the left column and write the corresponding symbolic steps in the right column. Check your final solution.

ALGEBRA PIECE WORK with notes

**CORRESPONDING
SYMBOLIC WORK**

Homework Questions 2.4

TILE FIGURES AND ALGEBRAIC EQUATIONS

1. Model the following sequence, I_n , with black tiles and:

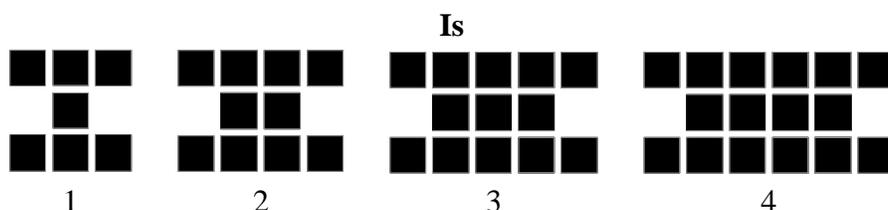
a.

Step One: Loop and number each figure and **Numerically** determine the total number of black tiles in each figure. Mark your number counts on the figures.

Step Two: Convert your looping ideas into **Words**.

Step Three: Convert your looping and word ideas into **Symbols**. Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4 .

✓ See the provided sketch pages for figures that you can draw on to show your work



- b. Describe the 100th I figure. What does it look like? How many black tiles are in it?
- c. Which I figure will have 202 black tiles? Describe the figure.
- d. What does the n th I figure look like? Use your black n -strips and black tiles, as needed, to model the figure, be sure to have the pieces oriented to look like the other tile figures in the I_n sequence. You do not need to show edge pieces. Sketch and describe the n th I figure. Label the pieces clearly.
- e. If the collection of tiles in a certain I figure is doubled and 4 more black tiles are added, there will be a total of 192 black tiles. Use your algebra pieces to help determine which I figure this is. Use a two column table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.
- f. Two consecutive I figures have a total of 131 black tiles. Which two figures are they? Use your algebra pieces to help determine which I figures these are. Use a two column table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

2. Model the following sequence, Z_s , with black tiles and:

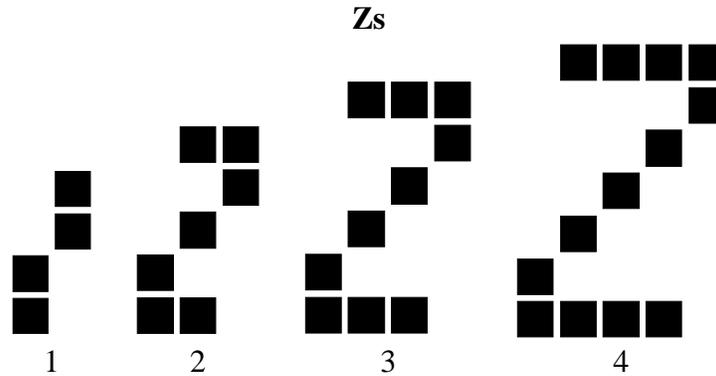
a.

Step One: Loop and number each figure and **Numerically** determine the total number of black tiles in each figure. Mark your number counts on the figures.

Step Two: Convert your looping ideas into **Words**.

Step Three: Convert your looping and word ideas into **Symbols** Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4.

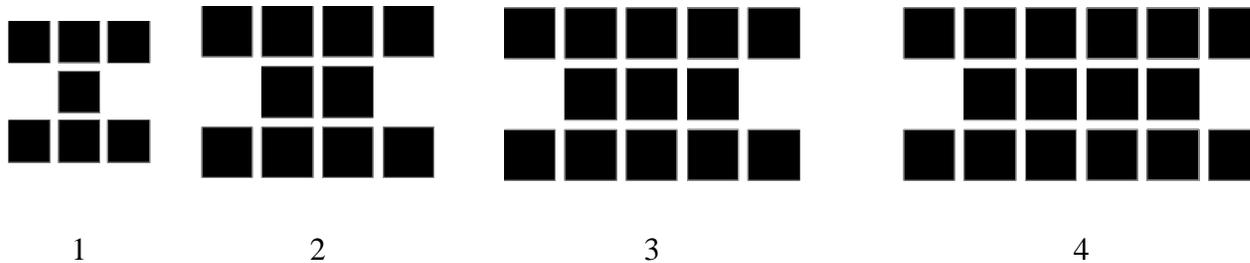
✓ See the provided sketch pages for figures that you can draw on to show your work



- b. Describe the 100th Z figure. What does it look like? How many black tiles are in it?
- c. Which Z figure will have 202 black tiles? Describe the figure.
- d. What does the n th Z figure look like? Use your black n -strips and black tiles, as needed, to model the figure, be sure to have the pieces oriented to look like the other tile figures in the Z_s sequence. You do not need to show edge pieces. Sketch and describe the n th Z figure. Label the pieces clearly.
- e. If 4 black tiles are added to four copies of a certain Z figure, there will be a total of 152 black tiles. Use your algebra pieces to help determine which Z figure this is. Use a two column table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.
- f. Two consecutive Z figures have a total of 269 black tiles. Which two figures are they? Use your algebra pieces to help determine which Z figures these are. Use a two column table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

3. Consider the equation $T = 4n + 5$.
- Use your black tiles to build figures 1, 2, 3 and 4 for a tile sequence that corresponds to the equation $T = 4n + 5$. Try to make your figures “look like” $4n + 5$ and use a consistent orientation for each figure. Sketch the figures and give the number of black tiles in each figure.
 - Use algebra pieces to model the n th figure for $T = 4n + 5$ that matches your tile figures. Sketch the figure. Label the algebra pieces clearly.
 - Describe the 100th figure for $T = 4n + 5$. What does it look like? How many black tiles are in it?
 - Which $T = 4n + 5$ figure will have 2005 black tiles? Describe the figure.
 - Two consecutive $T = 4n + 5$ figures have a total of 262 black tiles. Which two figures are they? Use your algebra pieces to help determine which figures these are. Use a two column table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.
 - Create a t-table for the total number of black tiles, T , for figure number inputs $n = 1, 2 \dots 6$.
 - Plot the ordered pairs from the t-table on graph paper. Label the axes with appropriate numbers.
 - Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What T (output) value do you estimate for $n = 10$ by just looking at the pattern of the graph? Check this value by using the symbolic formula $T = 4n + 5$.
4. For any tile sequence whose n th figure can be modeled with just black n -strips and black tiles:
- What is the shape of the graph if you plot coordinate pairs for $n = 1, 2, 3, \dots$? Explain why you think this is the case.
 - What is the difference between the n th figure and the $(n + 1)$ st figure? Explain. How does this relate to the graph of the coordinate pairs?

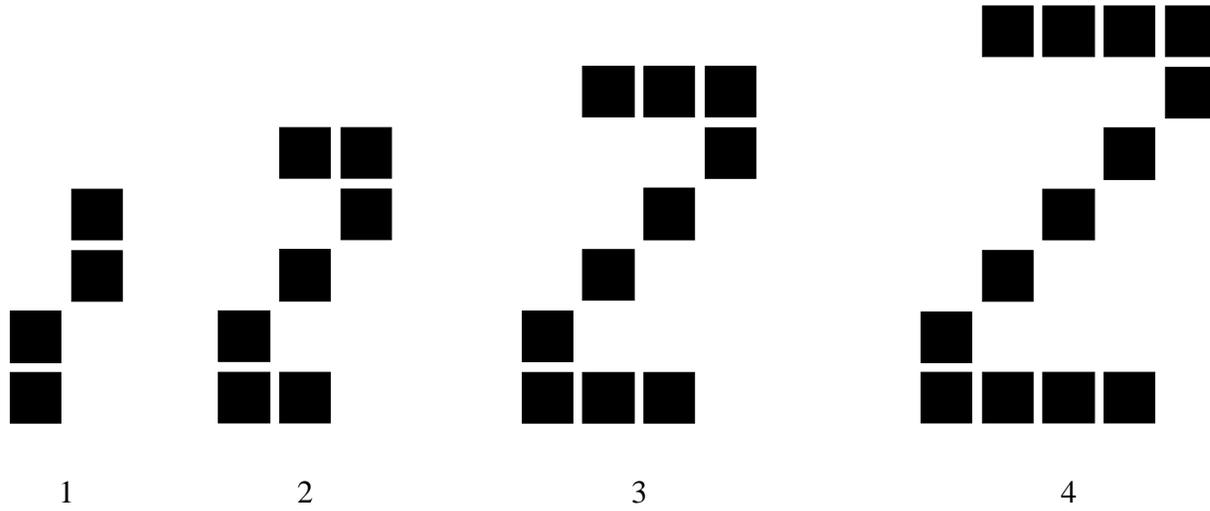
Homework 2.4: Black Tile Figure Sequences
Is SKETCH PAGE



Tile count

Words

Symbols



Tile count

Words

Symbols

Horizontal Tile Figures					...	
Figure #	1	2	3	4		N
Size	1×-1 or -1×1	1×-2 or -1×2	1×-3 or -1×3	1×-4 or -1×4		$1 \times -n$ or $-1 \times n$

Vertical Tile Figures					...	
Figure #	1	2	3	4		N
Size	-1×1 or 1×-1	-2×1 or 2×-1	-3×1 or 3×-1	-4×1 or 4×-1		$-n \times 1$ or $n \times -1$

Using Red n -Strips to Model the n th Figure of a Tile Sequence

The red n -strip can be all or part of the n th figure for a tile sequence and can be combined with other algebra pieces and black and red tiles.

Modeling with Black and Red n -Strips

A sequence of tile figures such as the following combines the use of black and red tiles; hence, in this example, this n th figure combines the use of black n -strips and red n -strips. In this sequence of tile figures we are looking at the pattern and we are not concerned about reducing the figures to minimal collections of black or red tiles.

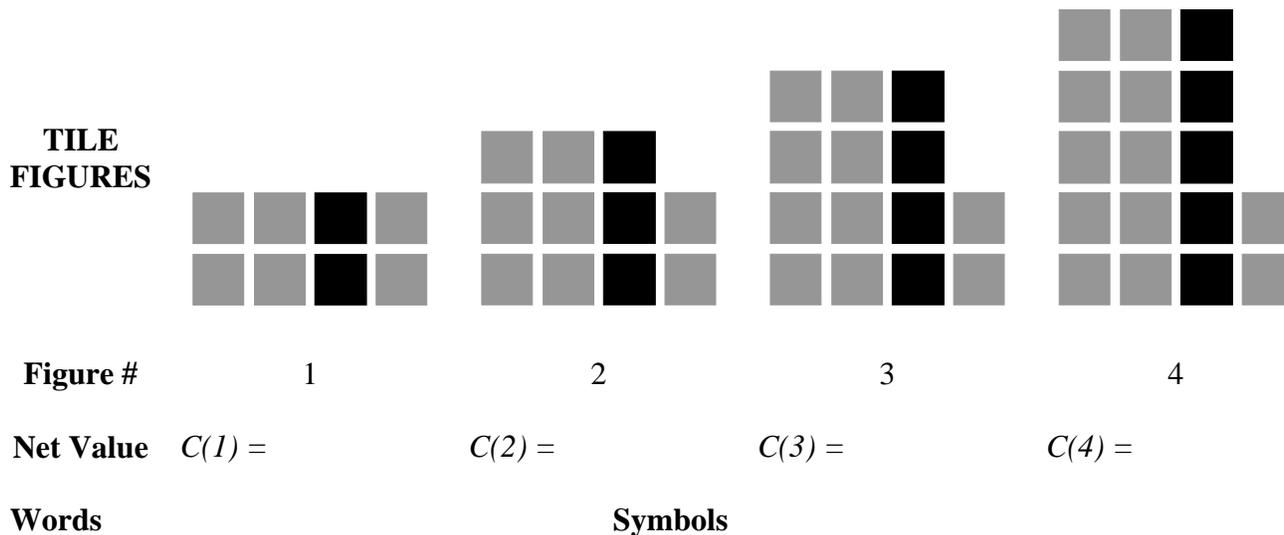
Tile Figures				...	
Figure #	1	2	3		n
Net Value	$-2 + 1$	$-3 + 2$	$-4 + 3$		$-(n + 1) + n = -1$

SKETCHING TIPS**Sketching Red n -Strips**

To sketch a red n -strip, you may wish to simply sketch an outline of the strip. To distinguish the red n -strip outline, label a red n -strip with a R for red.



1. (*) Model the following sequence, *Black-Red Chairs* with black and red tiles and:
 - a. **Step One: Loop** and number each figure and **Numerically** determine the net value of each figure. Use $C(n)$ for the *Black-Red Chairs* net value function name and use function notation to express these totals (but don't simplify). Mark your number counts on the figures.
 - b. **Step Two:** Convert your looping ideas into **Words**.
 - c. **Step Three:** Convert your looping and word ideas into **Symbols**. Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4.



2. (*) Describe the 100th *Black-Red Chair*. What does it look like? What is $C(100)$?

3. (*) What does the n th *Black-Red Chair* look like? Use your black and red n -strips and black and red tiles, as needed, to model the figure; you do not need to show edge pieces. Sketch and describe the n th *Black-Red Chair* here. Label the pieces clearly.

4. (*) Which *Black-Red Chair* will have a total of 2405 black and red tiles? Describe the figure including the number of black tiles and the number of red tiles. For this n , what is $C(n)$?

5. (*) For which *Black-Red Chair* will $C(n) = -2405$? Describe the figure including the number of black tiles and the number of red tiles.
6. If the collection of tiles in a certain *Black-Red Chair* is doubled and 22 black tiles are added, the net value of the new collection of black and red tiles will be 4. Which *Black-Red Chair* is it? Start with a complete version of the n th figure (don't cancel out zero pairs yet). Use the table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

ALGEBRA PIECE WORK with notes**CORRESPONDING
SYMBOLIC WORK**

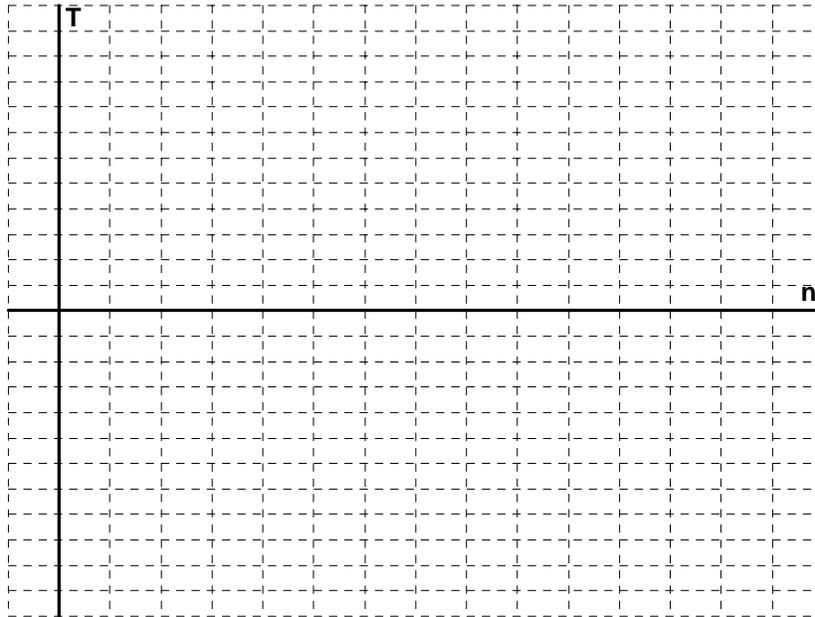
7. What does the $(n + 1)$ st *Black-Red Chair* look like? Use your black and red n -strips and black and red tiles, as needed, to model the figure; you do not need to show edge pieces. Sketch and describe the $(n + 1)$ st *Black-Red Chair* here. Label the pieces clearly.
8. Two consecutive *Black-Red Chairs* have a combined net value of -51. Which two figures are they? For the first step; start with complete versions of the n th and $(n + 1)$ st figures (don't cancel out zero pairs yet). Use the table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK

9. For the *Black-Red Chairs* tile sequence, fill out the second row of output values in the following t-table for the net value, $C(n) = T$ for each indicated figure.

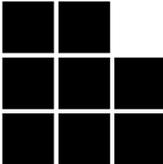
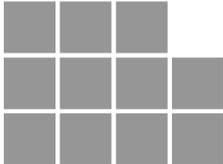
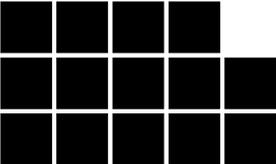
n	1	2	3	4	5	6	n
$C(n)$							

10. On the following grid; plot the ordered pairs from the t-table associated with $n = 1, 2 \dots 6$ for the *Black-Red Chairs* tile sequence. Label the axes with appropriate numbers.



- a. Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What do you estimate $C(10)$ to be by just looking at the pattern of the graph? Check this value by using your previous symbolic work.
- b. List at least three observations about the *Black-Red Chairs* tile sequence and the graph associated with this tile sequence.

11. Model the following sequence, *Alternating Chairs*, $A(n)$, with black and red tiles and:
- Step One: Loop** and number each figure and **Numerically** determine the net value of each figure (but don't simplify). Mark your number counts on the figures.
 - Step Two:** Convert your looping ideas into **Words**—Hint: This is a split function with different expressions for the odd terms and the even terms.
 - Step Three:** Convert your looping and word ideas into **Symbols**. Simplify your symbolic equations and check them for $n = 1, 2, 3$ and 4 .

TILE FIGURES				
Figure #	1	2	3	4
Net Value	$A(1) =$	$A(2) =$	$A(3) =$	$A(4) =$
Words				
			Symbols	

12. Describe the 100th *Alternating Chair* figure. What does it look like? What is $A(100)$?

13. What does the n th *Alternating Chair* figure look like for an even n and for an odd n ? Use your black and red n -strips and tiles, as needed, to model the figures; you do not need to show edge pieces. Sketch and describe the even n th and the odd n th *Alternating Chair* figures here. Label the pieces clearly.

14. Which *Alternating Chair* figure will have a total of 2399 black or red tiles? Which color will they be? For this n , what is $A(n)$?

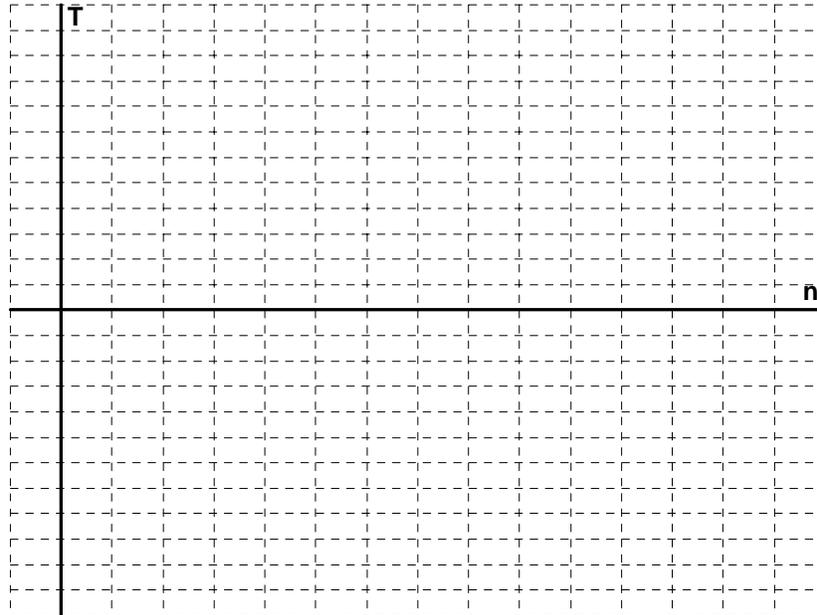
15. If the collection of tiles in a certain *Alternating Chair* figure is tripled and 35 black tiles are added, the new collection of tiles in the figure will reduce to a minimal collection of 2 black tiles. Which *Alternating Chair* figure is it? Use algebra and/or algebra pieces to determine the solution. Clearly show your work and explain your thinking.
16. For the *Alternating Chairs* tile sequence, the net value of the difference between two consecutive figures is 73. Which two figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show and explain your work.

17. The net value of the combination of the collections of tiles in two consecutive figures in the *Alternating Chairs* tile sequence is -3 . Which two figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show and explain your work. Hint—there may be more than one solution.

18. For the *Alternating Chairs* tile sequence, fill out the second row of output values in the following t-table for the net value, $A(n) = T$, in each indicated figure

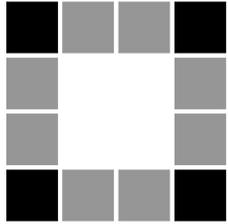
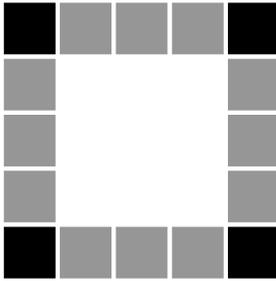
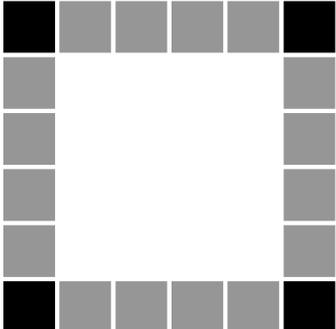
n	1	2	3	4	5	6	n odd	n even
$T = A(n)$								

19. On the following grid; plot the ordered pairs from the t-table associated with $n = 1, 2 \dots 6$ for the *Alternating Chairs* tile sequence. Label the axes with appropriate numbers.



- Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$ and $n = 11$. What do you estimate $A(10)$ and $A(11)$ to be by just looking at the pattern of the graph? Check these values by using your previous symbolic work.
- List at least three observations about the *Alternating Chairs* tile sequence and the graph associated with this tile sequence.

20. Model the following sequence, *Black-Red Squares*, $S(n)$, with black and tiles and:
- Step One: Loop** and number each figure and **Numerically** determine the net value of each figure (but don't simplify). Mark your number counts on the figures.
 - Step Two:** Convert your looping ideas into **Words**.
 - Step Three:** Convert your looping and word ideas into **Symbols**. Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4.

TILE FIGURES				
	1	2	3	4
Net Value	$S(1) =$	$S(2) =$	$S(3) =$	$S(4) =$
Words			Symbols	

21. Describe the 100th *Black-Red Square*. What does it look like? What is $S(100)$?
22. What does the n th *Black-Red Square* look like? Use your black and red n -strips and tiles, as needed, to model the figure; you do not need to use edge pieces. Sketch and describe the n th *Black-Red Square* here. Label the pieces clearly.
23. Which *Black-Red Square* will have a total of 2404 black and red tiles? Describe the figure including the number of black tiles and the number of red tiles. For this n , what is $S(n)$?

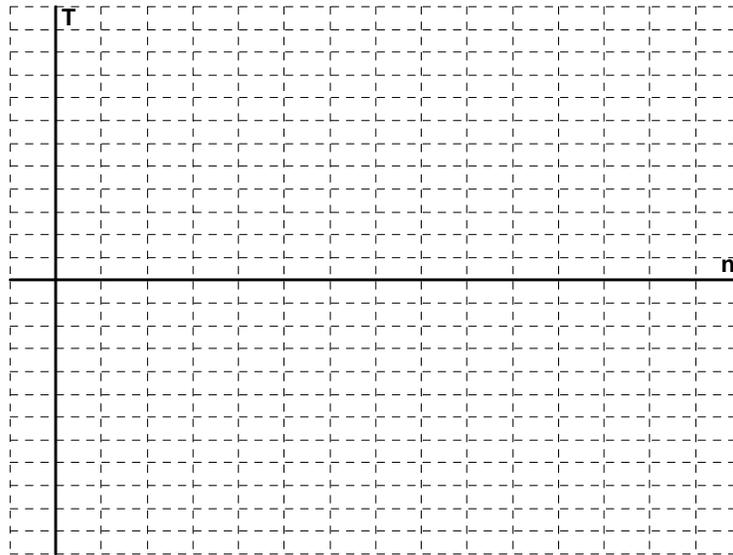
24. For which figure is $S(n) = -2404$? Describe the figure including the number of black tiles and the number of red tiles.
25. If the collection of tiles in a certain *Black-Red Square* figure is doubled and 60 black tiles are added, the new collection of tiles will reduce to a minimal collection of 60 tiles. There are two such *Black-Red Square* figures, one where the resulting 60 tiles are all red and one where the 60 tiles are all black. Which two *Black-Red Square* figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show your work and explain your thinking. Hint—there may be more than one solution.

26. For the *Black-Red Squares* tile sequence, the combined net value of two figures that are two figure numbers apart is -200. Which two figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show and explain your work.
27. The combination of two consecutive tile figures in the *Black-Red Squares* tile sequence will result in a non-minimal collection of 108 black and red tiles. Which two figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show and explain your work.

28. For the *Black-Red Squares* tile sequence, fill out the second row of output values in the following t-table for the net value, $S(n) = T$, in each indicated figure

n	1	2	3	4	5	6	n
$T = S(n)$							

29. On the following grid; plot the ordered pairs from the t-table associated with $n = 1, 2 \dots 6$ for the *Black-Red Squares* tile sequence. Label the axes with appropriate numbers.



- Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What do you estimate $S(10)$ to be by just looking at the pattern of the graph? Check these values by using your previous symbolic work.
- List at least three observations about the *Black-Red Squares* tile sequence and the above graph.

Homework Questions 2.5

LINEAR EXPRESSIONS AND EQUATIONS

1. Model the following sequence, *Black-Red Ls*, $L(n)$, with black and red tiles and:

a.

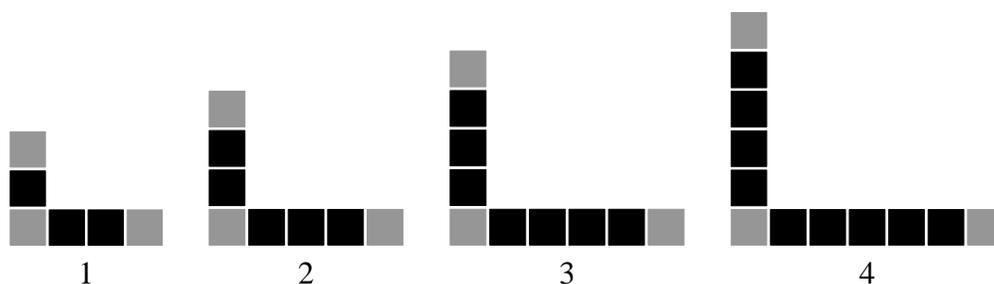
Step One: Loop and number each figure and **Numerically** determine the net value of each figure (but don't simplify). Mark your number counts on the figures.

Step Two: Convert your looping ideas into **Words**.

Step Three: Convert your looping and word ideas into **Symbols** Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4.

✓ See the provided sketch pages for figures that you can draw on to show your work

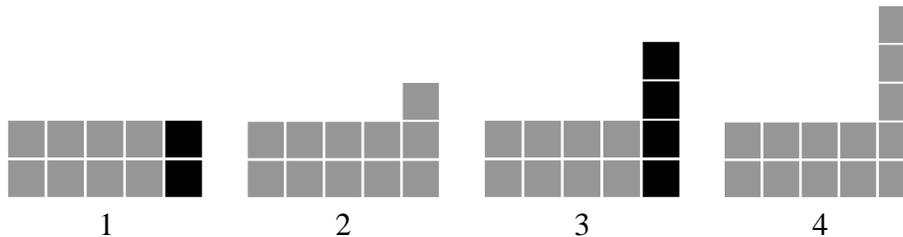
BLACK RED Ls



- b. Describe the 100th *Black-Red L*. What does it look like? What is $L(100)$?
- c. What does the n th *Black-Red L* look like? Use your black and red n -strips and tiles, as needed, to model the figure; you do not need to show edge pieces. Sketch and describe the n th *Black-Red L*. Label the pieces clearly.
- d. Which *Black-Red L* will have a total of 2400 black and red tiles? Describe the figure including the number of black tiles and the number of red tiles. For this n , what is $L(n)$?
- e. If the collection of tiles in a certain *Black-Red L* figure is tripled and 30 black tiles are removed, the new collection of tiles will reduce to a minimal collection of 30 tiles. There are two such *Black-Red L* figures, one where the resulting 30 tiles are all red and one where the 30 tiles are all black. Which two *Black-Red L* figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show your work and explain your thinking. Hint—there may be more than one solution.
- f. For the *Black-Red Ls* tile sequence, the combined net value of two figures that are two figure numbers apart is 64. Which two figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show and explain your work.
- g. For the *Black-Red Ls* tile sequence, create a t-table for the net value, $L(n)$, for figure number inputs $n = 1, 2 \dots 6$.
- h. Plot the ordered pairs from the t-table on graph paper. Label the axes with appropriate numbers.

- i. Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What do you estimate $L(10)$ to be by just looking at the pattern of the graph? Check this value by using your previous symbolic work.
2. Model the following sequence, *Black-Red Walls*, $W(n)$, with black and red tiles and:
- a.
- Step One: Loop** and number each figure and **Numerically** determine the net value of each figure (but don't simplify). Mark your number counts on the figures.
- Step Two:** Convert your looping ideas into **Words**.
- Step Three:** Convert your looping and word ideas into **Symbols** Simplify your symbolic equation and check it for $n = 1, 2, 3$ and 4.
- ✓ See the provided sketch pages for figures that you can draw on to show your work

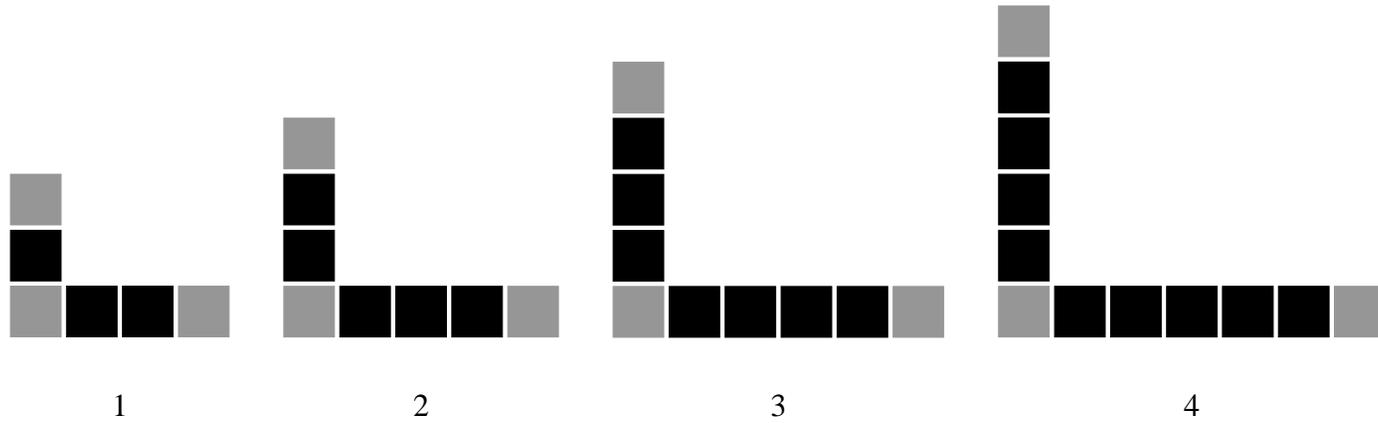
BLACK RED WALLS



- b. Describe the 100th and the 101st *Black-Red Wall* figures. What do they look like? What are $W(100)$ and $W(101)$?
- c. What does the n th *Black-Red Wall* figure look like for an even n and for an odd n ? Use your black and red n -strips and tiles, as needed, to model the figures; you do not need to show edge pieces. Sketch and describe the even n th and the odd n th *Black-Red Wall* figure. Label the pieces clearly.
- d. Which *Black-Red Wall* figure will have a total of 2400 black and red tiles? Describe the figure including the number of black tiles and the number of red tiles. For this n , what is $W(n)$?
- e. If 24 black tiles are removed from four copies of the collection of tiles in a certain *Black-Red Wall* figure, the net value of the new collection of tiles will be 0. Which *Black-Red Wall* figure is it? Use algebra and/or algebra pieces to determine the solution. Clearly show your work and explain your thinking.
- f. For the *Black-Red Walls* tile sequence, the net value of the difference between two consecutive figures is -33. Which two figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show and explain your work.
- g. For the *Black-Red Walls* tile sequence, create a t-table for the net value, $W(n)$, for figure number inputs $n = 1, 2 \dots 8$
- h. Plot the ordered pairs from the t-table on graph paper. Label the axes with appropriate numbers.

- i. Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 12$ and $n = 13$. What do you estimate $W(12)$ and $W(13)$ to be by just looking at the pattern of the graph? Check these values by using your previous symbolic work.
- j. List at least three observations about the *Black-Red Walls* tile sequence and the graph associated with this tile sequence.

Homework 2.5: Linear Expressions and Equations
 BLACK-RED Ls SKETCH PAGE



Net Value $L(1) =$

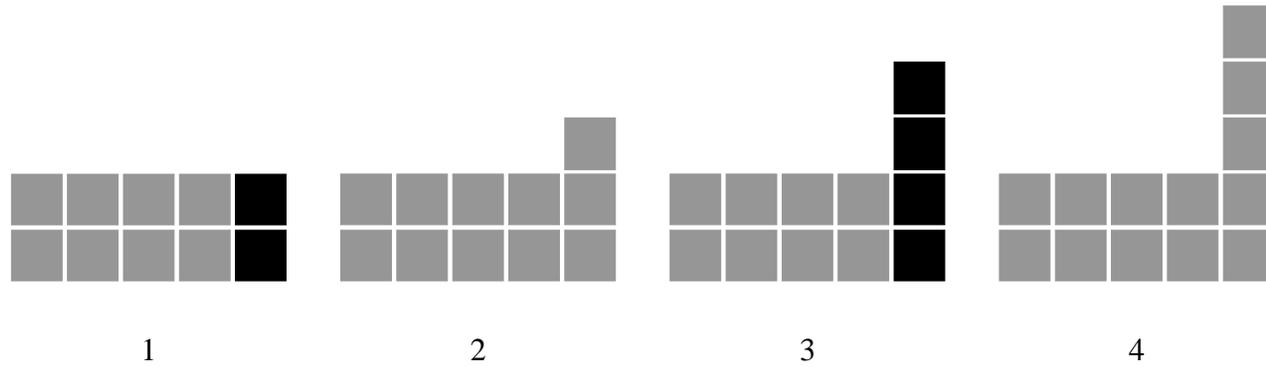
$L(2) =$

$L(3) =$

$L(4) =$

Words

Symbols



Net Value $W(1) =$

$W(2) =$

$W(3) =$

$W(4) =$

Words

Symbols

Activity Set 2.6

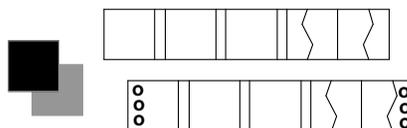
EXTENDED SEQUENCES AND LINEAR FUNCTIONS

PURPOSE

To learn how to extend sequences and graphs to an integer index set $n = 0, \pm 1, \pm 2 \dots$. To learn to use white and opposite white n -strips to model extended sequences. To investigate the features of linear functions such as intersections, intercepts and slope.

MATERIALS

Black and red tiles, white and opposite white n -strips



INTRODUCTION

Extended Graphs and Extended Sequences

As you may have noticed, it sometimes seems reasonable to extend graphical representations of tile figure sequences “backwards” where $n \leq 0$. For example, the graph for $T(n) = n + 2$ begins like this:

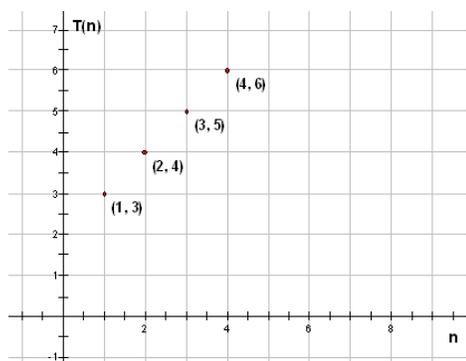


Figure 1

However, by adding additional coordinate pairs on the same line, we can visually extend this pattern “backwards” where $n \leq 0$. We can see these new coordinate pairs also follow the function relationship $(n, n + 2)$.

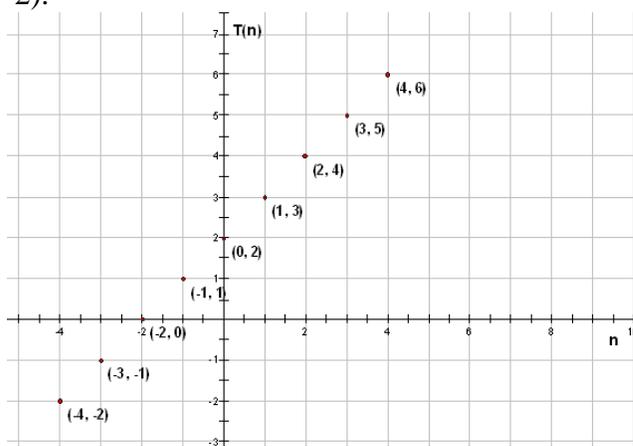
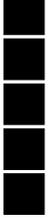


Figure 2

It can also make sense to visually extend tile figure sequence patterns, “backwards” for $n \leq 0$. We can create an Extended Sequence by letting our index n range over all integers. I.e., instead of just allowing $n = 1, 2, 3 \dots$ we can use integers as our input index set with $n = 0, \pm 1, \pm 2 \dots$

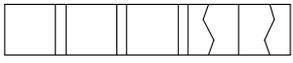
Black and Red Tile Figure Sequence with Integer Index Set (Figure 3)

Extended Tile Sequence								
Index	-4	-3	-2	-1	0	1	2	3
Coordinate Pair	(-4, -2)	(-3, -1)	(-2, 0)	(-1, 1)	(0, 2)	(1, 3)	(2, 4)	(3, 5)

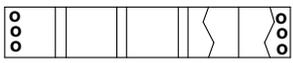
You can see the input, output pairs for the extended tile figure sequence in the above diagram matches the graph in Figure 2 perfectly. Suppose that you would like to model the n th term of this sequence with algebra pieces. Since the black n -strips are always black and always positive, a black n -strip and 2 black tiles will not work as a model for $T(n) = n + 2$; $n = 0, \pm 1, \pm 2 \dots$. For example, if $n = -3$, there is no way to make a black n -strip and 2 black tiles look like 1 red tile and have output value -1. To address this; we introduce White and Opposite n -Strips.

White and Opposite White n -Strips

A White n -Strip, represented by the variable n , represents an integer (output) number; $n = 0, \pm 1, \pm 2 \dots$. The (output) value of a white n -strip is the same as the index (input) number ($n = 0, \pm 1, \pm 2 \dots$).

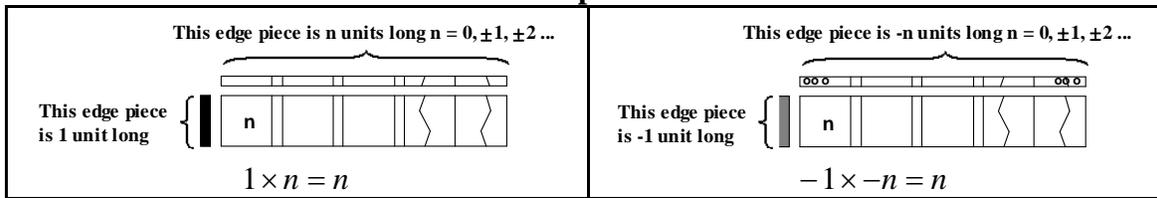
White n -Strip Values		
 n	Input	Output
	$n < 0$	n -strip = $n = -1, -2, -3 \dots$
	$n = 0$	n -strip = $n = 0$
	$n > 0$	n -strip = $n = 1, 2, 3 \dots$

An Opposite White n -Strip, represented by the variable $-n$, represents an integer (output) number; $-n = 0, \pm 1, \pm 2 \dots$. The (output) value of an opposite white n -strip is the opposite of the index (input) number ($n = 0, \pm 1, \pm 2 \dots$). The opposite side of the white n -strip is marked with Os for “opposite.”

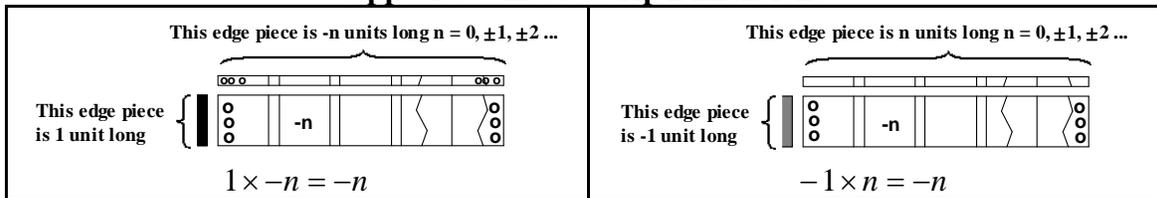
Opposite White n -Strip Values		
 $-n$	Input	Output
	$n < 0$	$-n$ -strip = $-n = 1, 2, 3 \dots$
	$n = 0$	$-n$ -strip = $-n = -0 = 0$
	$n > 0$	$-n$ -strip = $-n = -1, -2, -3 \dots$

The height of the white and opposite white n -strips is fixed and is the same length as the edges of the black and red tiles (1 or -1), but the length of the white and opposite white n -strips are an arbitrary length. Since the white n -strip represents the variable n , its dimensions are $1 \times n$ or $-1 \times -n$. Since the opposite white n -strip represents the variable $-n$, its dimensions are $1 \times -n$ or $-1 \times n$ as illustrated in the following diagrams.

White n -Strip Dimensions



Opposite White n -Strip Dimensions

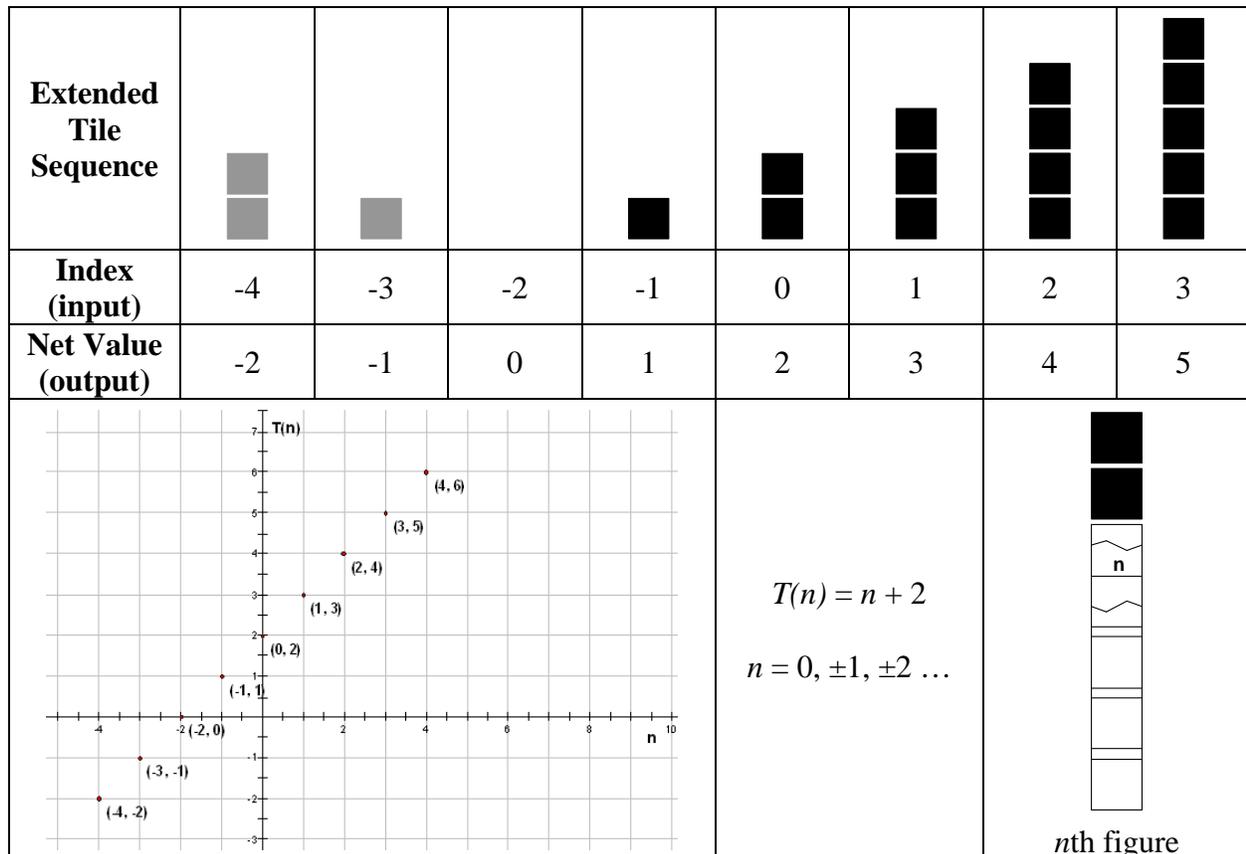


Using White and Opposite White n -Strip Edge Pieces to Measure Length

Just like with black and red n -strip edge pieces, a thin piece of a white n -strip (white n -strip edge piece) measures a length of n units and a thin piece of a opposite white n -strip (opposite white n -strip edge piece) measures a length of $-n$ units (where n is an integer). These edge pieces are shown, in context, in the previous descriptions of white n -strips and opposite white n -strips.

Connecting Extended Sequences, Graphs, Symbolic Functions and White n -Strip Models

We can now combine the graph in figure 2, the symbolic formula $T(n) = n + 2$ and the white n -strip to see the following connected representations of the extended black and red tile figure sequence in figure 3.



White and Opposite White n -Strips Replace Black n -Strips and Red n -Strips

For the remaining activity sets and other course materials, we will no longer use black n -strips or red n -strips. The white and opposite white n -strips are generalizations of the black and red n -strips and will serve in their place.

White and Opposite White n -Strip Zero Pairs

Just like with black and red n -strips, a white n -strip and a white opposite n -strip combined to have a net value of 0 ($n + -n = 0$).

Graphing Terms

Intersect and Points of Intersection are defined in context, see activity 9.

T Intercepts, n -intercepts and slope are defined in context, see activity 11.

Parallel Lines are defined in context, see activity 14.

SKETCHING TIPS

Sketching White and Opposite White n -Strips

To sketch a white or an opposite white n -strip, simply sketch an outline of the strip. To distinguish the white n -strips and opposite white n -strips, label the white n -strip with n and the opposite white n -strip with $-n$ and with several Os.

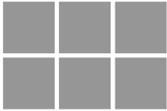


Sketching White and Opposite White n -Strip Edge Pieces

To sketch a white or an opposite white n -strip edge piece, simply sketch an outline of the strip and label it n or $-n$ (the fact that n is an integer should be clear in context). You may also wish to mark the opposite white n -strip edge pieces with several Os.

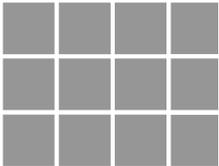
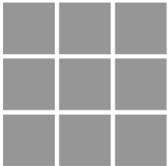
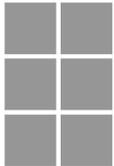
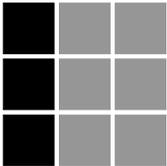
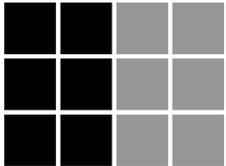


1. (*) Analyze the following extended sequence of black and red tile figures; $A(n)$, using the three-step numerical-words-symbols framework.

Tile Figures					
Input	-2	-1	0	1	2
Output	$A(-2) =$	$A(-1) =$	$A(0) =$	$A(1) =$	$A(2) =$
Words					
				Symbols	

2. (*) Using white or opposite white n -strips and black and red tiles; sketch a representation of the n th $A(n)$ tile figure.

3. Analyze the following extended sequence of black and red tile figures; $B(n)$, using the three-step numerical-words-symbols framework.

Tile Figures					
Input	-2	-1	0	1	2
Output	$B(-2) =$	$B(-1) =$	$B(0) =$	$B(1) =$	$B(2) =$
Words					
				Symbols	

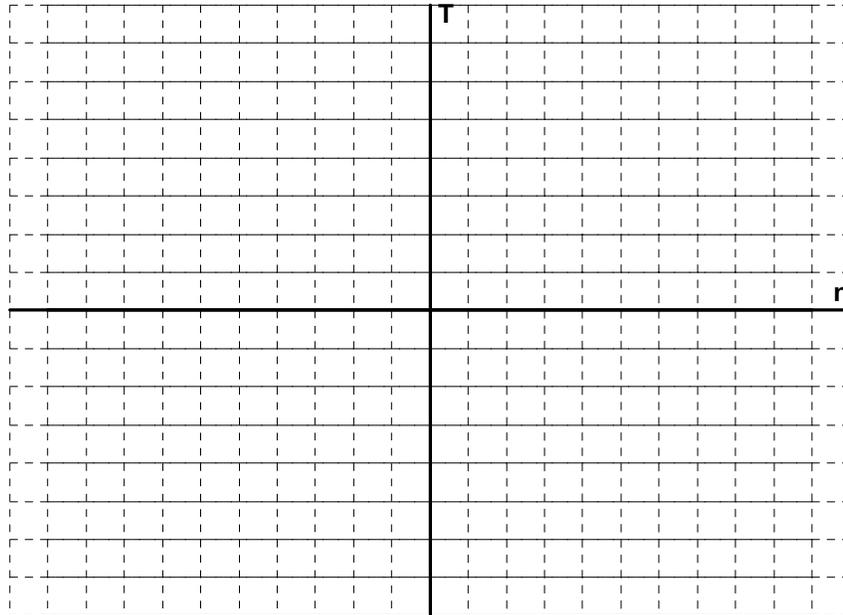
4. Using white or opposite white n -strips and black and red tiles; sketch a representation of the n th $B(n)$ tile figure.

5. (*) Use t-tables to record net values for the $A(n)$ and $B(n)$ extended sequences.

n	-3	-2	-1	0	1	2	3	n
$A(n)$								

n	-3	-2	-1	0	1	2	3	n
$B(n)$								

6. (*) On the following grid; plot the ordered pairs associated with $n = -3$ to 3 for both the $A(n)$ and $B(n)$ extended sequences. To distinguish the points, mark the $A(n)$ values with a dot or small circle and the $B(n)$ values with a small x. Label the axes with appropriate numbers.



7. Inspect the plotted ordered pairs and visually extend the patterns for $A(n)$ and $B(n)$.
- For which n do you visually estimate that $A(n) = B(n)$?
 - What output value do you visually estimate $A(n) = B(n)$ to be for the n you found in part a?

8. Solve the question; for which n is $A(n) = B(n)$? by using your n th figure white n -strip models for $A(n)$ and $B(n)$. Use the table and sketch your algebra piece work in the left column (include brief notes about what you are doing) and write the corresponding symbolic steps in the right column. Check your final solution.

ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK

9. What is the output value, $A(n) = B(n)$, for the n you found in the previous activity? The coordinate pair $(n, A(n))$ (which is the same coordinate pair as $(n, B(n))$) is where the graphs of $A(n)$ and $B(n)$ intersect and is sometimes called the point of intersection of $A(n)$ and $B(n)$. Mark this point on your graph of $A(n)$ and $B(n)$.
10. For the n where $A(n) = B(n)$, sketch the $A(n)$ and the $B(n)$ tile figures. Do the two tile figures for this n need to look the same or do they just need to have the same net value? Explain.

11. Use the graphs and t-tables for $A(n)$ and $B(n)$, algebra and/or algebra pieces to determine each of the following:
- In general, for any function $T(n)$, the coordinate pair $(0, T(0))$ is called the T intercept*. Why do you think this is called the T intercept?
 - What is $(0, A(0))$? Which portion of the n th figure of $A(n)$ is the T intercept output value $A(0)$?
 - What is the T intercept for $B(n)$? Which portion of the n th figure of $B(n)$ is the T intercept output value $B(0)$?
 - For any n th figure, $T(n)$, made of white or opposite white n -strips and black and red tiles, which portion of the n th figure of $T(n)$ is the T intercept output value $T(0)$?
 - In general, for any function $T(n)$, the coordinate pair $(n, 0)$ is called the n intercept for $T(n)$. Why do you think this is called the n intercept?
 - What is the n intercept for $A(n)$? How can you use the n th figure of $A(n)$ to determine this?
 - What is the n intercept for $B(n)$? How can you use the n th figure of $B(n)$ to determine this?
 - For any n th figure, $T(n)$, made of white or opposite white n -strips and black and red tiles, how can you use the n th figure of $T(n)$ to find the n intercept for $T(n)$?

* T is our general dependent variable and n as our general independent variable

- i. What is the difference in output values for consecutive $A(n)$ figures? How can you use the n th figure of $A(n)$ to determine this? How does this relate visually to the graph of $A(n)$?

 - j. What is the difference in output values for consecutive $B(n)$ figures? How can you use the n th figure of $B(n)$ to determine this? How does this relate visually to the graph of $B(n)$?

 - k. For any n th figure, $T(n)$, made of white or opposite white n -strips and black and red tiles, how can you use the n th figure of $T(n)$ to determine the difference in output values for consecutive $T(n)$ figures? In this case, this difference is called the slope of the $T(n)$ graph. How does this relate visually to the graph of $T(n)$?
12. We have seen that any function, $T(n)$, whose n th figure can be modeled with only white or opposite white n -strips and black and red tiles, has a graph that is in the shape of a line. In this case, the symbolic formula for $T(n)$ can be simplified to $T(n) = mn + b$ where m and b can be any integers.
- a. What is m ? How does this relate to the n th figure of $T(n)$?

 - b. Lines that go up from the left to the right are said to have positive slope. How does this relate to m ? How does this relate to the n th figure of $T(n)$?

 - c. Lines that go down from the left to the right are said to have negative slope. How does this relate to m ? How does this relate to the n th figure of $T(n)$?

d. Lines that are flat are said to have zero slope. How does this relate to m ? How does this relate to the n th figure of $T(n)$?

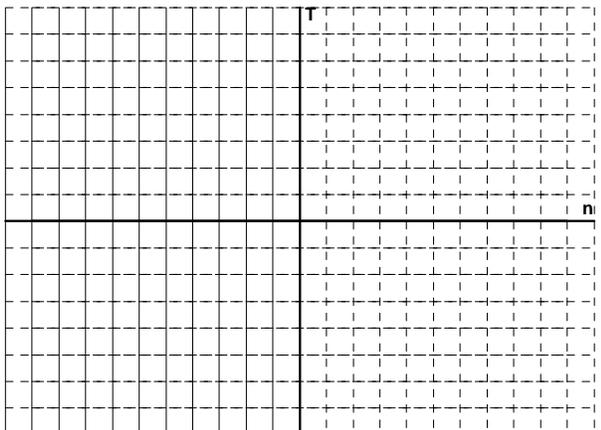
e. What is b ? How does this relate to the n th figure of $T(n)$?

13. Use algebra pieces to make up and model an n th figure for each of the following linear function conditions. Sketch the n th figures and label the pieces clearly. Give the symbolic function and sketch the graph of the line corresponding to your example.

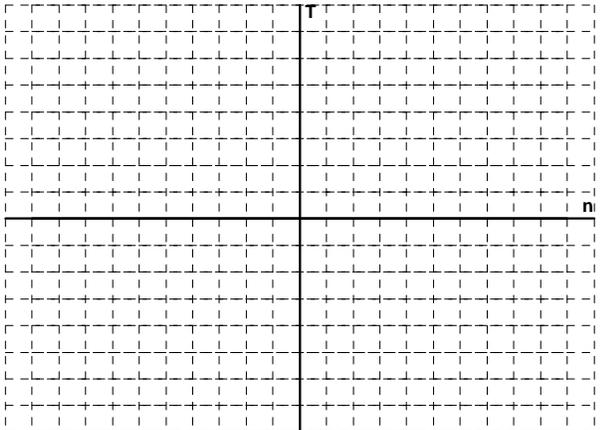
a.	nth figure	<u>Positive Slope / Negative T Intercept</u>	
	Function	Graph	

b.	nth figure	<u>Positive Slope / Negative n Intercept</u>	
	Function	Graph	

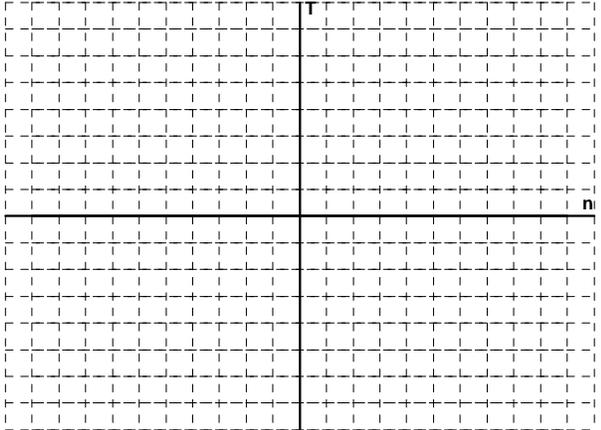
c. **Negative Slope / Positive T Intercept**

nth figure	Graph 
Function	

d. **Positive Slope / Positive n Intercept**

nth figure	Graph 
Function	

e. **Zero Slope / Positive T Intercept**

nth figure	Graph 
Function	

f.	<i>n</i>th figure	<u>Zero Slope / Negative <i>T</i> Intercept</u>	
			Graph
	Function		

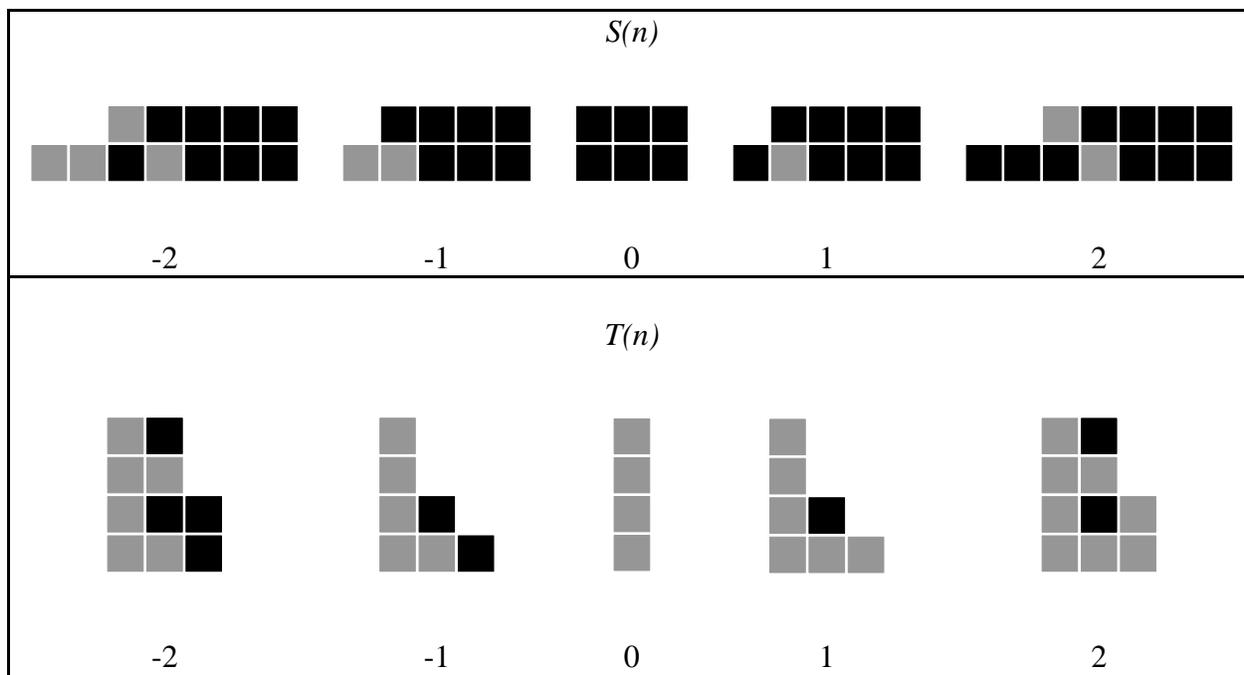
g.	<i>n</i>th figure	<u>Zero Slope / Zero <i>T</i> Intercept</u>	
			Graph
	Function		

14. Lines that have the same slope are called parallel lines. How does this relate to the m for each of the parallel lines? How does this relate to the n th figures for both of the parallel lines?

Homework Questions 2.6

EXTENDED SEQUENCES AND LINEAR FUNCTIONS

1. Refer to the given extended sequence $S(n)$ and $T(n)$ to answer the following questions.



- a. Analyze $S(n)$ and $T(n)$; what functions are they? Sketch the n th $S(n)$ and n th $T(n)$ figures.
 - b. Create t-tables for $n = -3$ to 3 for both $S(n)$ and $T(n)$.
 - c. Plot the ordered pairs from the t-tables; to distinguish the points, mark the $S(n)$ values with a dot or small circle and the $T(n)$ values with a small x. Label the axes with appropriate numbers.
 - d. What are the n -intercepts and T -intercepts for $S(n)$ and $T(n)$? Mark these points on your graphs and show your work for finding these four points.
 - e. Where do the graphs of $S(n)$ and $T(n)$ intersect? Use algebra and/or algebra pieces to determine the solution symbolically and mark this point on your graph of $S(n)$ and $T(n)$. Clearly show your work and explain your thinking.
 - f. List several observations about the graphs of $S(n)$ and $T(n)$.
- 2.
- a. Create an extended tile sequence $C(n)$ whose graph has slope -1 and T intercept 5 . Sketch the tile figures for $n = -2, -1, 0, 1, 2$ and n .
 - b. Create an extended tile sequence $D(n)$ whose graph has slope 1 and T intercept -3 . Sketch the tile figures for $n = -2, -1, 0, 1, 2$ and n .
 - c. Create t-tables for $n = -3$ to 3 for both $C(n)$ and $D(n)$.

- d. Plot the ordered pairs from the t-tables; to distinguish the points, mark the $C(n)$ values with a dot or small circle and the $D(n)$ values with a small x. Label the axes with appropriate numbers.
- e. What are the n -intercepts for $C(n)$ and $D(n)$? Mark these points on your graphs and show your work for finding these two points.
- f. Where do the graphs of $C(n)$ and $D(n)$ intersect? Use algebra and/or algebra pieces to determine the solution symbolically and mark this point on your graph of $C(n)$ and $D(n)$. Clearly show your work and explain your thinking.
- g. List several observations about the graphs of $C(n)$ and $D(n)$.

CHAPTER 2 VOCABULARY AND REVIEW TOPICS

VOCABULARY

Activity Set 2.1

1. Figure Numbers
2. Looping (loop)

Activity Set 2.3

3. Black n -Strip
4. Red n -Strip
5. Black and Red n -Strip Edge Pieces

Activity Set 2.5

6. Functions
7. Function Notation
8. Independent Variable
9. Dependent Variable

Activity Set 2.6

10. Extended Sequence
11. White and Opposite n -Strips
12. White and Opposite White n -Strip Edge Pieces
13. Point of Intersection
14. n & T -intercepts
15. Slope

SKILLS AND CONCEPTS

Activity Set 2.1

- A. Counting and describing components of figures using looping.
- B. Finding underlying algebraic patterns using a three step framework: Numerical-Words-Symbols.

Activity Set 2.2

- C. Working with and analyzing algebraic patterns for Alternating Even and Odd Sequences.
- D. Working with and analyzing algebraic patterns for Alternating “Every Three” Sequences.

Activity Set 2.3

- E. Using black and red n -strip edge pieces to measure length
- F. Using the three step framework to analyze, and using black n -strips and black tiles to model, sequences of black tile figures.
- G. Answering algebraic questions about black tile figures.
- H. Plotting ordered pairs corresponding to black tile figure sequences.

Activity Set 2.4

- I. Using black n -strips and black and red tiles to answer questions and to solve basic algebra problems.
- J. Connecting algebra piece model work to corresponding symbolic steps.

Activity Set 2.5

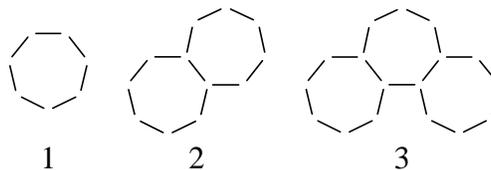
- K. Using the three step framework to analyze, and using black and red n -strips and black and red tiles to model, sequences of black and red tile figures.
- L. Using functions and function notation.
- M. Answering algebraic questions about consecutive figures.
- N. Answering algebraic questions about total number of tiles vs. number of black or red tiles vs. net values of figures.

Activity Set 2.6

- O. Extending tile figure sequences to an integer input set
- P. Connecting extended sequences to extended graphs
- Q. Using the three step framework to analyze, and using white and opposite white n -strips and black and red tiles to model, extended sequences of black and red tile figures.
- R. Determine using algebra piece models where the graphs of two extended tile figure sequences intersect.
- S. Investigate features of linear function; intercepts and slope.

CHAPTER 2 PRACTICE EXAM

HEXAGONS



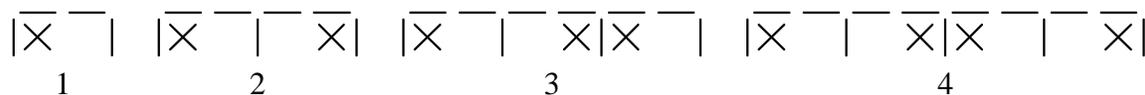
1. Show two visually different methods for using the three step Numerical-Words-Symbols framework to analyze the *Hexagons* toothpick figure sequence. Show that both methods result in equivalent symbolic equations.

FENCES



2. Use the three step Numerical-Words-Symbols framework to analyze the *Fences* toothpick figure sequence.
3. How many toothpicks will there be in the:
 - a. 75th *Fences* figure?
 - b. 76th *Fences* figure?
 - c. 77th *Fences* figure?
4. For which *Fences* figures are there:
 - a. 2014 toothpicks?
 - b. 2019 toothpicks?
 - c. 2036 toothpicks?

RECTANGLES WITH Xs

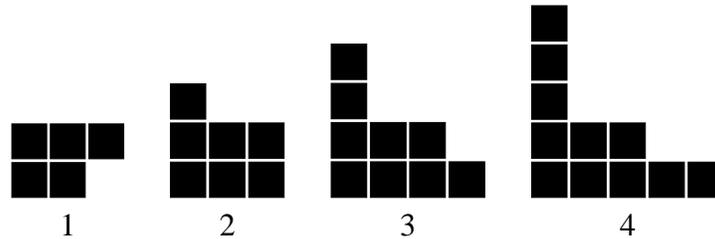


5. Use the three step Numerical-Words-Symbols framework to analyze the *Rectangles with Xs* toothpick figure sequence.

6. How many toothpicks will there be in the:
- 75th *Rectangles with Xs* figure?
 - 76th *Rectangles with Xs* figure?
 - 77th *Rectangles with Xs* figure?
7. For which *Rectangles with Xs* figures are there:
- 2007 toothpicks?
 - 2035 toothpicks?
 - 2046 toothpicks?



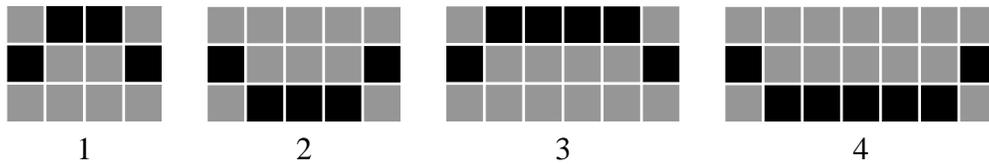
8. Use the three step Numerical-Words-Symbols framework to analyze the *Fences with Xs* toothpick figure sequence.
9. How many toothpicks will there be in the:
- 75th *Fences with Xs* figure?
 - 76th *Fences with Xs* figure?
 - 77th *Fences with Xs* figure?
10. For which *Fences with Xs* figures are there:
- 2005 toothpicks?
 - 2010 toothpicks?
 - 2016 toothpicks?

BLACK STEPS

11. Use the three step Numerical-Words-Symbols framework to analyze the *Black Steps* tile figure sequence.
12. Describe the 100th *Black Step* tile figure. What does it look like? How many black tiles are in it?
13. Which *Black Step* tile figure will have 2003 tiles? Describe the figure.
14. Sketch and describe the n th *Black Step* figure
15. For the *Black Step* tile sequence.
 - a. Create a t-table for the number of tiles, T , for figure number inputs $n = 1, 2 \dots 6$.
 - b. Plot the ordered pairs from the t-table on graph paper. Label the axes with appropriate numbers.
 - c. Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What T (output) value do you estimate for $n = 10$ by just looking at the pattern of the graph? Check this value by using your symbolic analysis of the tile figure sequence.
16. If 6 black tiles are added to three copies of a certain *Black Step* figure, there will be a total of 141 black tiles. Which *Black Step* tile figure is this? Show your algebra piece and symbolic work in a two column table.
17. Sketch the $(n + 1)$ st figure for the *Black Steps* tile sequence. Label the figure clearly.
18. Two consecutive *Black Steps* figures have a total of 180 black tiles. Which two *Black Step* figures are they? Show your algebra piece and symbolic work in a two column table.
19. Use your black tiles to build figures 1, 2, 3, 4 and n for the tile sequence that matches the equation the equation $T = 5n + 5$. Give the number of black tiles in each figure.
20.
 - a. Create a t-table for the number of black tiles, $T = 5n + 5$, for figure number inputs $n = 1, 2 \dots 6$.
 - b. Plot the ordered pairs from the t-table on graph paper. Label the axes with appropriate numbers.

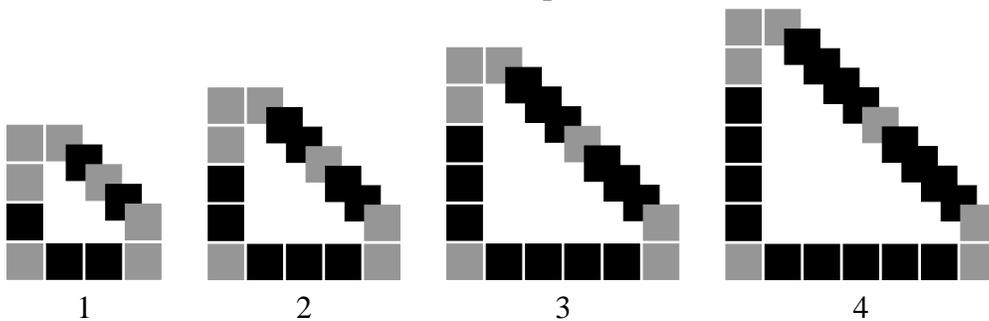
- c. Inspect the plotted ordered pairs and visually extend the pattern you see to $n = 10$. What T (output) value do you estimate for $n = 10$ by just looking at the pattern of the graph? Check this value by using the symbolic formula $T = 5n + 5$.

A(n) Tile Sequence

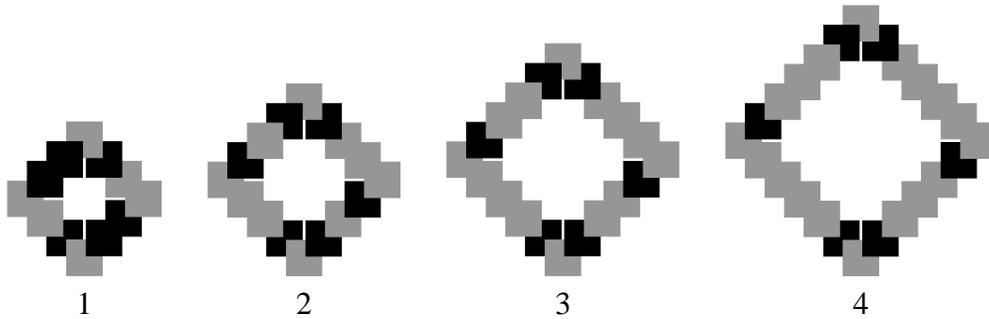


21. The combined net value of two $A(n)$ figures that are three figure numbers apart is -109 . Which two $A(n)$ figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show and explain your work.
22. If 100 red tiles are removed from five copies of the collection of tiles in a certain $A(n)$ figure, the net value of the new collection of tiles will be 0. Which $A(n)$ figure is it? Use algebra and/or algebra pieces to determine the solution. Clearly show your work and explain your thinking.

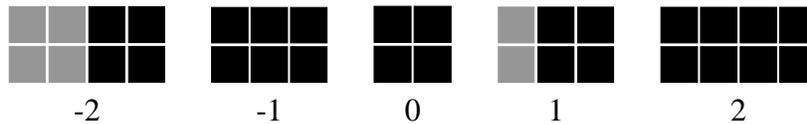
B(n) Tile Sequence



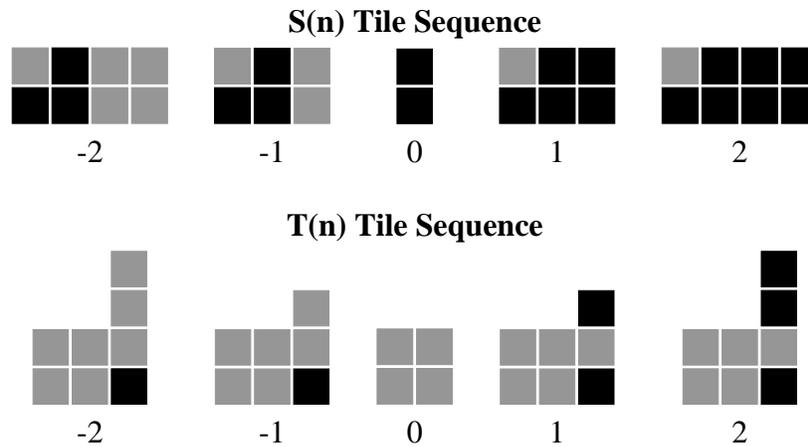
23. The combined net value of two $B(n)$ figures that are five figure numbers apart is 336. Which two $B(n)$ figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show and explain your work.
24. There are a total of 204 tiles in two consecutive $B(n)$ figures. Which two $B(n)$ figures are they? Use algebra and/or algebra pieces to determine the solution. Clearly show and explain your work and describe both figures, including the number of black and the number of red tiles in each figure.

C(n) Tile Sequence

25. If the collection of tiles in a certain $C(n)$ figure is tripled and 96 black tiles are added, the new collection of tiles will reduce to a minimal collection of 96 tiles (possibly red and possibly black). Which $C(n)$ figure is it? Use algebra and/or algebra pieces to determine the solution. Clearly show your work and explain your thinking. Is there more than one solution?

D(n) Tile Sequence

26. Analyze the extended $D(n)$ tile figure sequence.
- Sketch and describe the n th $D(n)$ figures. Label the pieces clearly.
 - Plot the ordered pairs for $D(n)$ corresponding to $n = 0, \pm 1, \pm 2, \pm 3$ and ± 4 on graph paper.
27. If the collection of tiles in a certain $D(n)$ figure is tripled and 15 black tiles are added, the new collection of tiles will reduce to a minimal collection of 15 tiles (possibly red and possibly black). Which $D(n)$ figure is it? Use algebra and/or algebra pieces to determine the solution. Clearly show your work and explain your thinking. Is there more than one solution?



28. Analyze the extended tile sequences $S(n)$ and $T(n)$. What functions are they? Create t-tables for $n = -3$ to 3 for both $S(n)$ and $T(n)$ and plot the ordered pairs from the t-tables; to distinguish the points, mark the $S(n)$ values with a dot or small circle and the $T(n)$ values with a small x. Label the axes with appropriate numbers.
29. What are the n -intercepts and T -intercepts for $S(n)$ and $T(n)$? Mark these points on your graphs and show your work for finding these four points.
30. Where do the graphs of $S(n)$ and $T(n)$ intersect? Use algebra and/or algebra pieces to determine the solution symbolically and mark this point on your graph of $S(n)$ and $T(n)$. Clearly show your work and explain your thinking.

CHAPTER THREE

REAL NUMBERS AND QUADRATIC FUNCTIONS

Activity Set 3.1

GRAPHING WITH REAL NUMBERS

PURPOSE

To learn how to extend the idea of working with discrete data points (individual tile figures) to working symbolically and graphically with real number function inputs and outputs. To learn how to find the domain and range of basic functions. To learn how to describe domains and ranges using number line and inequality notation. To learn about absolute value tile sequences and their corresponding symbolic and graphical forms.

MATERIALS

Black and red tiles



INTRODUCTION

Domain of a Function

The Domain of a Function is the set of allowable inputs for a function. We used a domain of the counting numbers ($n = 1, 2, 3 \dots$) to describe the input set (figure numbers) for Activity Sets 2.1 through 2.5. In Activity Set 2.6 we extended our domain to the set of Integers and considered all of the tile figures corresponding to input numbers $n = 0, \pm 1, \pm 2 \dots$

Range of a Function

The Range of a Function is the set of possible outputs for the function. We have seen many possible ranges. For example, for the function $f(n) = 2n + 1$, $n = 1, 2, 3 \dots$ our range is the set of numbers 3, 5, 7... (why?) and for the function $g(n) = -2n$, $n = 1, 2, 3 \dots$ our range is the set of numbers -2, -4, -6 ... (again, why?).

Real numbers

The set of Real Numbers (R) is the set of all of the numbers on the number line and is the union of all of the rational numbers (**Q** for quotient) and irrational numbers (**I**): $\mathbf{R} = \mathbf{Q} \cup \mathbf{I}$.

Rational Numbers are the numbers that can be written as a ratio of two integers with a nonzero denominator. Irrational Numbers are the numbers, such as π and $\sqrt{2}$, that can not be written as a ratio of two integers.

Graphing and Modeling “All” of the Points

In Chapter 2, we plotted distinct (discrete) coordinate pairs corresponding to functions defined with figure number inputs and net value outputs. In many cases, it probably felt natural to “connect the dots” even though there were no figures corresponding to input numbers such as 0.5 and -2.1. You may have also noticed that even though there were no figures corresponding to such numbers, symbolically, it did make sense to use numbers other than $n = 1, 2, 3 \dots$ or $n = 0, \pm 1, \pm 2 \dots$ as function inputs (as the function domain).

We will now extend our working domains and ranges to the real numbers and subsets of the real numbers so that we may, in fact, “connect the dots” and graph all of the points. In some cases, we will be able to sketch actual figures that correspond to rational number inputs such as $\frac{1}{2}$ and

$\frac{1}{3}$. Of course, we won't be able to physically model these figures unless we break up our algebra pieces, but we can sketch pictures of partial tiles. Overall, we will use extended tile figure sequences to find patterns and then use the symbolic function rules we find to determine additional coordinate pair points on our function graphs. We can then precisely “connect the dots” and complete our graphs.

Using x as the Independent Variable and y as the Dependent Variable

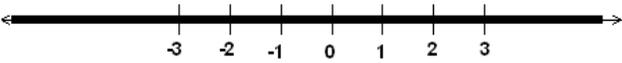
We will now use x for the independent input variable and y as the dependent output variable to help us keep track of our change to using real numbers. For functions we will use x and y notation such as $y = f(x)$ (instead of $T = f(n)$).

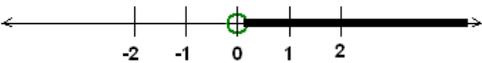
We will call our white strips “white and opposite white x -strips” instead of “white and opposite white n -strips” and we will label the strips with x and $-x$ instead of n and $-n$. We will start using white and opposite white x -strips in Activity Set 3.2.

Showing and Describing Real Numbers and Subsets of Real Numbers

In order to answer questions about domain and range, we will need to be able to describe subsets of the real numbers. The two ways we will use to do this are:*

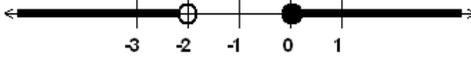
- 1) Using a Number Line (visual)
- 2) Using Inequality Notation (symbolic)

Example 1—All real numbers (R)	
Number Line	
Inequality Notation	$-\infty < x < \infty$ Read as “All x greater than negative infinity and less than positive infinity”

Example 2—All real numbers greater than 0	
Number Line	
Inequality Notation	Either $0 < y < \infty$, $0 < y$ or $y > 0$ Read as “All y greater than 0”

Example 3—All real numbers greater than or equal to -1 and less than 3	
Number Line	
Inequality Notation	$-1 \leq x < 3$ Read as “All x greater than or equal to -1 and less than 3”

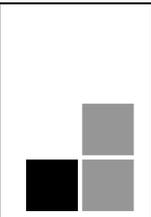
* We will not use Interval Notation such as $(-2, 7)$ [for the interval $-2 < x < 7$]. This notation does not indicate a variable and an interval such as $(-2, 7)$ is easily confused with a coordinate pair.

Example 4—All real numbers less than -2 or greater than or equal to 0	
Number Line	
Inequality Notation	$y < -2$ or $y \geq 0$ (both pieces are needed here) Read as “All y less than -2 or greater than or equal to 0”

Note:

- Examples 1 and 3 use the variable x which we will use to answer questions (occasionally) about the domain of functions
- Examples 2 and 4 use the variable y which we will use to answer questions (regularly) about the range of functions

1. Refer to “Showing and Describing Real Numbers and Subsets of Real Numbers” in the **Introduction** to answer these questions:
 - a. On a number line: What does an open circle around a number mean? What does the solid circle mean?
 - b. What words correspond to the symbols $<$ and $>$?
 - c. What words correspond to the symbols \leq and \geq ?
2. (*) Analyze the following extended sequence of tile figures, $y = f(x)$, with domain, \mathbf{R} and:
 - a. Sketch figures corresponding to $x = -1.5$ and $x = 2.5$.

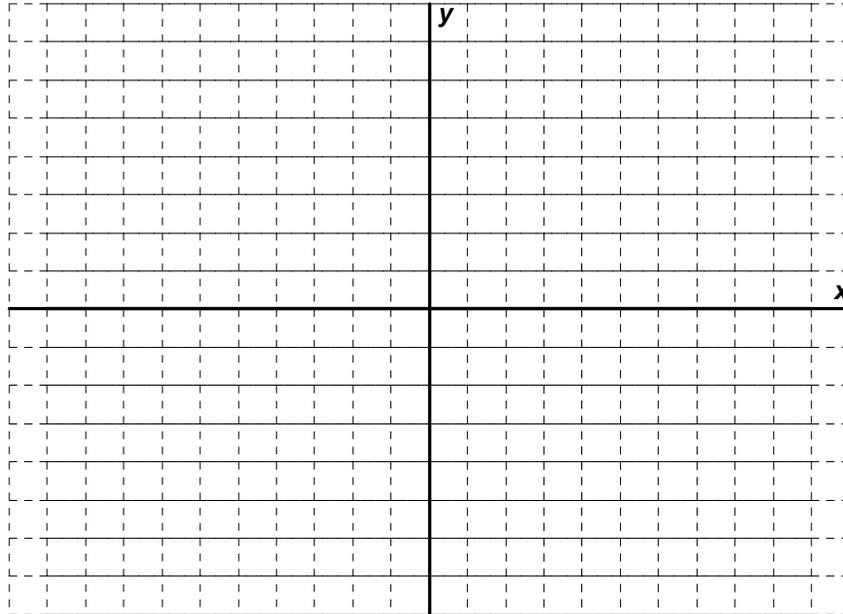
$y = f(x)$							
x	-2	-1.5	-1	0	1	2	2.5

- b. Fill out the indicated function values in the following t-table.

x	-2.5	-2	-1.5	-1	-.5	0	.5	1	1.5	2	2.5	x
$y = f(x)$												

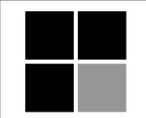
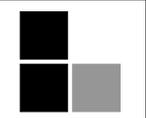
- c. By looking at the symbolic form of the function $y = f(x)$, what possible output values do you think you can obtain, i.e., what is the range of $y = f(x)$? Give this range on a number line and using inequality notation.

- d. Label the axes with appropriate numbers, plot enough coordinate pairs on the following grid so that you can connect the points and sketch the function $y = f(x)$. Because we are using the real numbers, \mathbf{R} , as our domain, the function will fill in between the plotted points and continues in both directions (even though the grid ends). You can show the idea of “continuing on” by sketching arrows on each end of your function sketch.



- e. How does your answer for part c. about range show visually on the graph in part d.? Explain the connection.
3. What is the domain of a function of the form $y = f(x) = mx + b$ where $m, b \in \mathbf{R}$? How does this show visually on the graph of $y = f(x) = mx + b$?
4. What is the range of a function of the form $y = f(x) = mx + b$ where $m \in \mathbf{R}$ and $b \in \mathbf{R}$? Are there any special cases? How does this show visually on the graph of $y = f(x) = mx + b$?

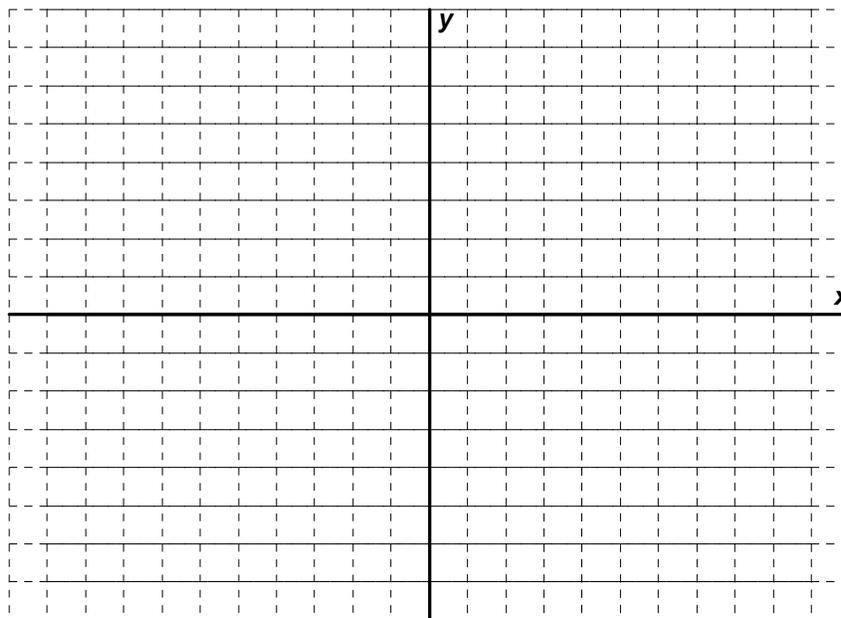
5. Analyze the following extended sequence of tile figures, $y = g(x)$, with domain, \mathbf{R} and:
- a. Sketch figures corresponding to $x = -1.5$ and $x = 1.5$.

$y = g(x)$							
x	-2	-1	-0.5	0	0.5	1	2

- b. Fill out the indicated function values in the following t-table.

x	-2	-1.5	-1	-0.5	0	.5	1	1.5	2	2.5
$y = g(x)$										

- c. When $x \leq 0$, what is the symbolic form of the function ?
- d. When $x \geq 0$, what is the symbolic form of the function ?
- e. By looking at the symbolic form of the function, what is the range of $y = g(x)$? Give this range on a number line and using inequality notation.
- f. Label the axes with appropriate numbers, plot enough coordinate pairs so that you can sketch the function and sketch $y = g(x)$ (don't forget the end arrows).



- g. How does your answer for part c. about range show visually on the graph in part d.? Explain the connection.

6. The Absolute Value of a number x is denoted $|x|$. $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Determine each of the following:

a. $|-2.4| = \underline{\hspace{2cm}}$

b. $|12.7| = \underline{\hspace{2cm}}$

c. $|0| = \underline{\hspace{2cm}}$

By thinking about the function, $y = |x|$, over two intervals, $y = |x|$ can be written as two linear functions.

d. When $x \geq 0$, $y = \underline{\hspace{3cm}}$

e. When $x \leq 0$, $y = \underline{\hspace{3cm}}$

7. Can you think of a way to symbolically describe the function $y = g(x)$ in activity 5 using absolute value notation?

8. Analyze the following extended sequence of tile figures, $y = h(x)$, with domain, \mathbf{R} and:

- a. Sketch figures corresponding to $x = -2.5$ and $x = .5$.

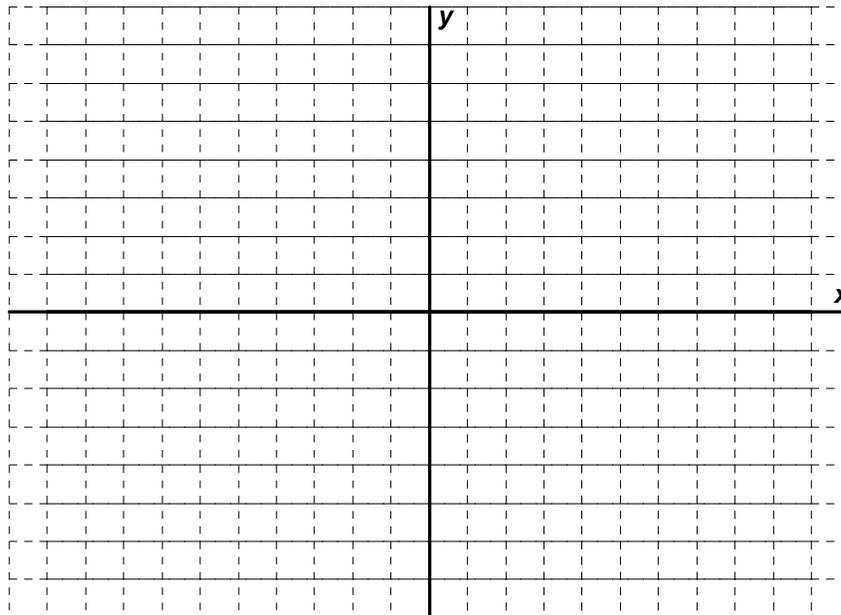
$y = h(x)$							
x	-2.5	-2	-1	0	.5	1	2

- b. Fill out the indicated function values in the following t-table. For the columns with x values marked $x \leq \underline{\hspace{1cm}}$ and $x \geq \underline{\hspace{1cm}}$, fill in the blanks and determine the corresponding linear formulas for $y = h(x)$. If you can think of a way to describe $y = h(x)$ with one symbolic rule, write that rule in the column with just x .

x	-2.5	-2	-1	0	.5	1	2	$x \leq \underline{\hspace{1cm}}$	$x \geq \underline{\hspace{1cm}}$	x
$y = h(x)$										

- c. By looking at the symbolic form of the function, what is the range of $y = h(x)$? Give this range on a number line and using inequality notation.

- d. Label the axes with appropriate numbers, plot enough coordinate pairs so that you can sketch the function and sketch $y = h(x)$ (don't forget the end arrows).



- e. How does your answer for part c. about range show visually on the graph in part d.? Explain the connection.

- f. If you have not already done so, can you describe the function $y = h(x)$ using absolute value notation as part of the function description?

9. Consider the following extended sequence of tile figures, $y = j(x)$, with domain, \mathbf{R} .

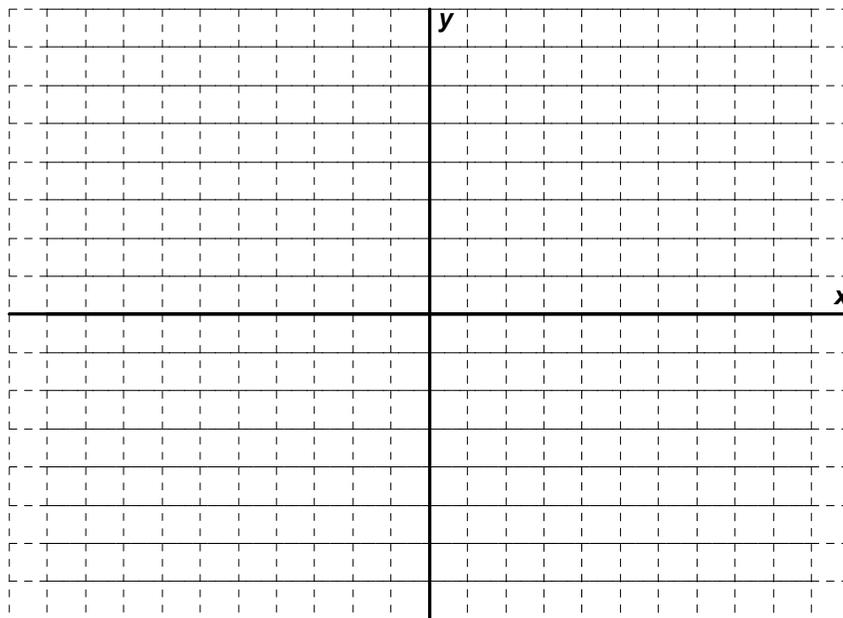
$y = j(x)$						
x	-3	-2	-1	0	1	2

- a. (*) Fill out the indicated function values in the following t-table. For the columns with x values marked $x \leq \underline{\quad}$ and $x \geq \underline{\quad}$, fill in the blanks and determine the corresponding linear formulas for $y = j(x)$. If you can think of a way to describe $y = j(x)$ with one symbolic rule, write that rule in the column with just x .

x	-3	-2	-1	0	1	2	3	$x \leq \underline{\quad}$	$x \geq \underline{\quad}$	x
$y = j(x)$										

- b. (*) By looking at the symbolic form of the function, what is the range of $y = j(x)$? Give this range on a number line and using inequality notation.

- c. (*) Label the axes with appropriate numbers, plot enough coordinate pairs so that you can sketch the function and sketch $y = j(x)$ (don't forget the end arrows).



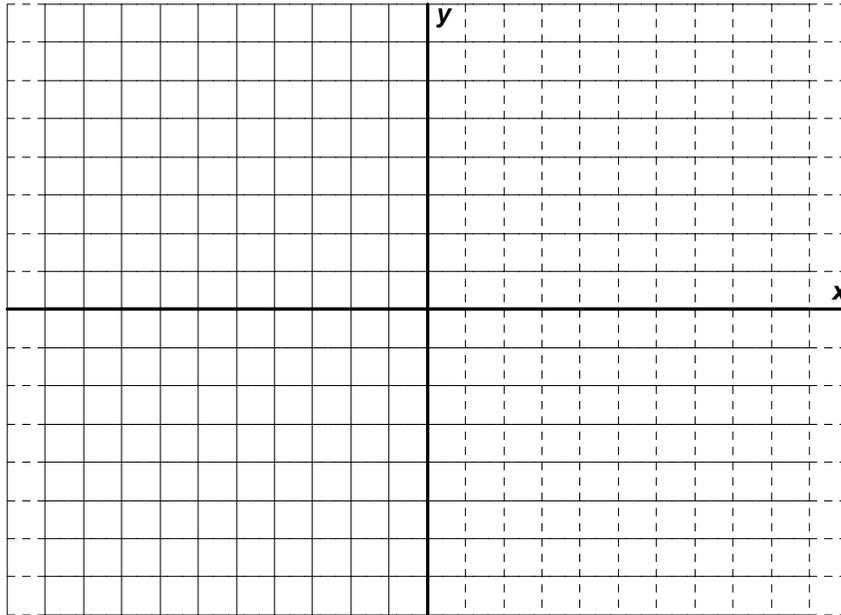
- d. How does your answer for part b. about range show visually on the graph in part c.? Explain the connection.
- e. If you have not already done so, can you describe the function $y = j(x)$ using absolute value notation as part of the function description?

10. Consider the function $y = |x| - 2$.

- a. Use your black and red tiles to build figures -2, -1, 0, 1 and 2 for an extended sequence of tile figures matching this function. Sketch your figures and sketch figures for $x = -3.5$ and 3.5.

$y = x - 2$							
x	-3.5	-2	-1	0	1	2	3.5

- b. Label the axes with appropriate numbers, plot enough coordinate pairs so that you can sketch the function and sketch $y = |x| - 2$.



- c. What is the range of $y = |x| - 2$? Give this range on a number line and using inequality notation.

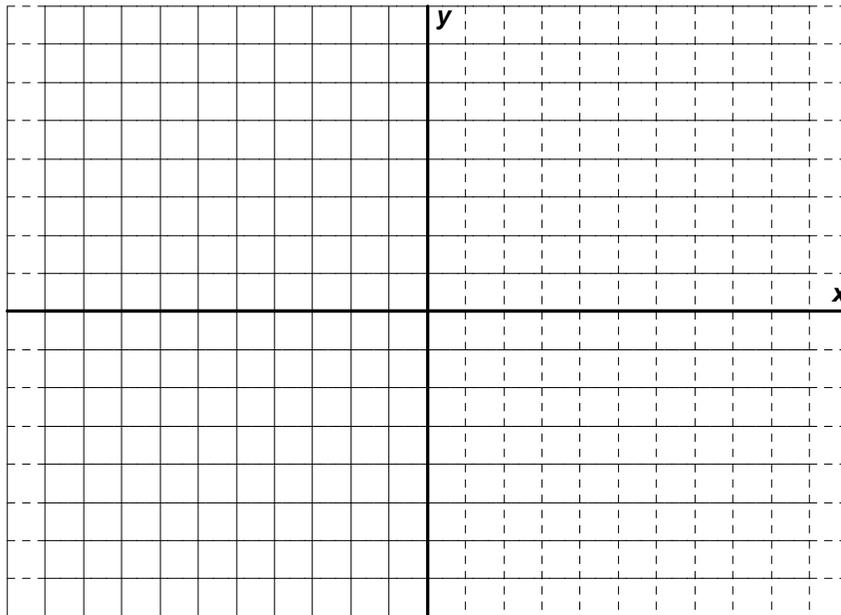
- d. List at least three observations about the function $y = |x| - 2$.

11. Consider the function $y = -|x-2|+3$.

- a. Use your black and red tiles to build figures -2, -1, 0, 1 and 2 for an extended sequence of tile figures matching this function. Sketch your figures and sketch figures for $x = -.5$ and $.5$.

$y = - x-2 +3$							
x	-1	0	1	2	3	4	5

- b. Label the axes with appropriate numbers, plot enough coordinate pairs so that you can sketch the function and sketch $y = -|x-2|+3$.



- c. How does the graph of $y = -|x-2|+3$ differ from the graph of $y = |x|$? Describe in terms of shifting or stretching.
- d. What is the range of $y = -|x-2|+3$? Give this range on a number line and using inequality notation.

- e. Where is the vertex of $y = -|x - 2| + 3$?
- f. What is the linear equation for the half of $y = -|x - 2| + 3$ when $x \leq$ the x value of the vertex of $y = -|x - 2| + 3$? Describe a quick and simple way you can use the absolute value form of the function to find this linear function.
- g. What is the linear equation for the half of $y = -|x - 2| + 3$ when $x \geq$ the x value of the vertex of $y = -|x - 2| + 3$? Describe a quick and simple way you can use the absolute value form of the function to find this linear function.

Homework Questions 3.1

GRAPHING WITH REAL NUMBERS

1. Consider the functions $f(x) = |x + 2|$ and $g(x) = |x - 3|$.
 - a. Sketch $f(x)$, $g(x)$ and $y = |x|$ on one set of axes, label each function graph clearly.
 - b. How do the graphs of $f(x)$ and $g(x)$ differ from the graph of $y = |x|$? Describe in terms of shifting or stretching.
 - c. What is the range of $f(x)$? Of $g(x)$?
 - d. What is the y -intercept of $f(x)$? Of $g(x)$?
 - e. What is the x -intercept $f(x)$? Of $g(x)$?
 - f. Write $f(x) = |x + 2|$ as a split function with two linear components where neither component uses absolute value notation.
 - g. Write $g(x) = |x - 3|$ as a split function with two linear components where neither component uses absolute value notation.

2. Consider the functions $h(x) = |x| + 2$ and $i(x) = |x| - 3$
 - a. Sketch $h(x)$, $i(x)$ and $y = |x|$ on one set of axes, label each function graph clearly.
 - b. How do the graphs of $h(x)$ and $i(x)$ differ from the graph of $y = |x|$? Describe in terms of shifting or stretching.
 - c. What is the range of $h(x)$? Of $i(x)$?
 - d. What is the y -intercept of $h(x)$? Of $i(x)$?
 - e. What are the x -intercepts of $h(x)$? Of $i(x)$?
 - f. Write $h(x) = |x| + 2$ as a split function with two linear components where neither component uses absolute value notation.
 - g. Write $i(x) = |x| - 3$ as a split function with two linear components where neither component uses absolute value notation.

3. Consider the functions $j(x) = 2|x|$ and $k(x) = -3|x|$
- Sketch $j(x)$, $k(x)$ and $y = |x|$ on one set of axes, label each function graph clearly.
 - How do the graphs of $j(x)$ and $k(x)$ differ from the graph of $y = |x|$? Describe in terms of shifting or stretching.
 - What is the range of $j(x)$? Of $k(x)$?
 - What is the y -intercept of $j(x)$? Of $k(x)$?
 - What is the x -intercept of $j(x)$? Of $k(x)$?
 - Write $j(x) = 2|x|$ as a split function with two linear components where neither component uses absolute value notation.
 - Write $k(x) = -3|x|$ as a split function with two linear components where neither component uses absolute value notation.
4. Consider the function $y = -2|x+1|+4$.
- Sketch the function $y = -2|x+1|+4$ and $y = |x|$ on one set of axes, label each function graph clearly.
 - How does the graph of $y = -2|x+1|+4$ differ from the graph of $y = |x|$? Describe in terms of shifting or stretching.
 - What is the range of $y = -2|x+1|+4$?
 - What is the y -intercept of $y = -2|x+1|+4$? Show your work for determining this answer.
 - Write $y = -2|x+1|+4$ as a split function with two linear components where neither component uses absolute value notation.
 - Use the two linear functions to determine the x -intercepts of $y = -2|x+1|+4$. Show your work.

Activity Set 3.2

INTRODUCTION TO QUADRATIC FUNCTIONS

PURPOSE

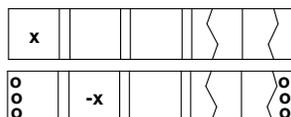
To learn how to analyze quadratic extended tile sequences and use $\pm x$ -squares while modeling these sequences. To learn how to distinguish quadratic extended tile sequences from linear extended tile sequences. To learn how to graph quadratic functions and note key features on the graphs such as turning points, x -intercepts and y -intercepts and to connect these ideas to the ranges of quadratic functions. To learn to use a graphing calculator to support t-table and graphing work.

MATERIALS

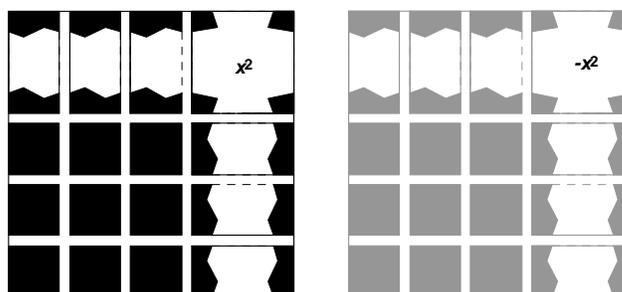
Black and red tiles



White and opposite white x -strips



Black and red x -squares

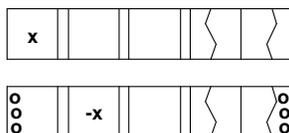


Graphing calculator with table functions (recommended)

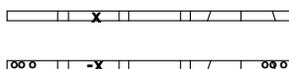
INTRODUCTION

White and Opposite White x -Strips and Edge Pieces

As noted in Activity Set 3.1, we will now use white and opposite white x -strips. We will use white and opposite white x -strips as we used white and opposite white n -strips, the only difference is that x is a real number. Previously we used n an integer. Label white and opposite white x -strips with x and $-x$ to indicate this change.

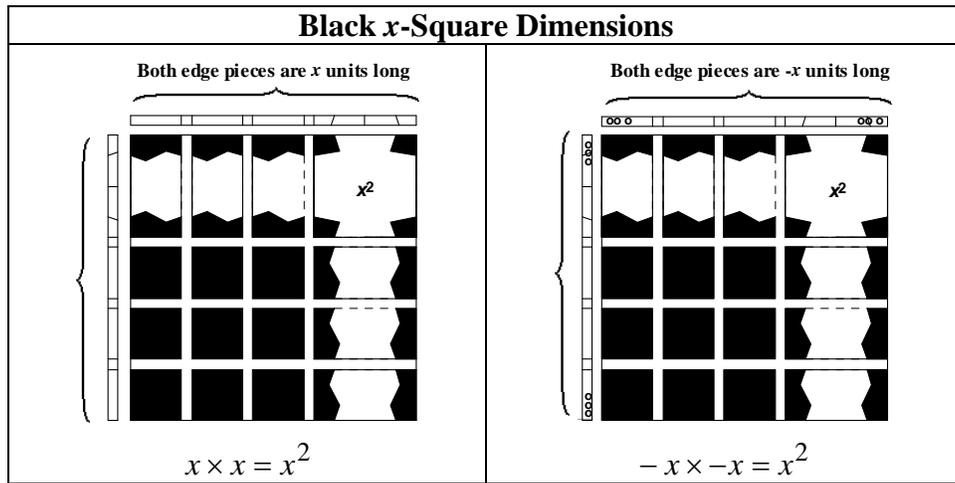


White and opposite white x -edge pieces will measure the lengths of x and $-x$. Label white and opposite white x -edge pieces with x and $-x$.



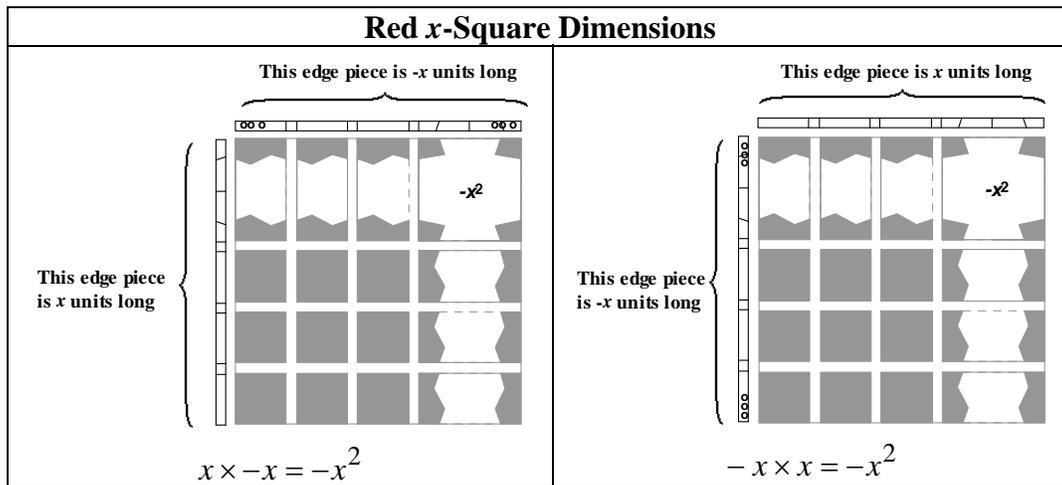
Black x -Square

A Black x -Square represents a square number, $x^2 = 1^2, 2^2, 3^2 \dots$. Both sides of the square are the same length as the long edge of the white and opposite white x -strips. Because the square is black, the dimensions of the square can be $x \times x$ or $-x \times -x$ as illustrated in the following diagram.



Red x -Square

A Red x -Square represents a negative square number, $-x^2 = -1^2, -2^2, -3^2 \dots$. Both sides of the square are the same length as the long edge of the white and opposite white x -strips. Because the square is red, the dimensions of the square can be $x \times -x$ or $-x \times x$.



Modeling with Black and Red x Squares—Example

An extended sequence of tile figures whose x th figure is modeled by a black x -square might look like the following extended sequence of tile figures*

x	-3	-2	-1	0	1	2	3	...	x
Size	-3×-3	-2×-2	-1×-1	0×0	1×1	2×2	3×3		$x \times x$

Black and red x -squares can be all or part of the x th figure for an extended tile sequence and can be combined with other black and red x -squares, white and opposite white x -strips and individual black and red tiles.

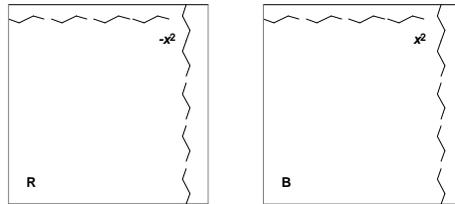
* Note: tile figures involving $\pm x^2$ components will usually be drawn at 70% of the usual scale

Function Terms

Polynomials, Coefficients, Leading Coefficient, Degree of a Polynomial, Parabolas, Quadratic Functions and Turning Points are defined in context; see activity 3

SKETCHING TIPS**Sketching x -Squares**

To sketch a black or red x -square, you may wish to simply sketch and label an outline of the square. Remember the length of the edge of the square should match the length of the long edge of an x -strip.

**TECHNOLOGY NOTES (graphing calculator models such as the TI-83 or 84 series)[†]****Using a Graphing Calculator to Display T-Tables****Step One**

Enter the formula for your function in the graphing menu (usually the $y =$ button)

Step Two

Open the TBL SET (Table Set) menu (usually above the WINDOW button)

There are two independent variable options—Auto and Ask, each are described separately.

AUTO DISPLAY LIST option

- I. Enter the first x -value from your t-table after TblStart =
- II. Enter the smallest difference between the x values in your t-table after Δ Tbl =.
 Δ stands for the Greek letter delta and means “change in.”
 For example, if $x = 1, 2, 3$, enter Δ Tbl = 1. If $x = 1, 1.5, 2, 2.5$, enter Δ Tbl = .5
- III. Set the Independent variable (Indpnt) to **Auto** by scrolling to **Auto** and hitting ENTER.
- IV. Open the TABLE window (usually above the GRAPH button) to see a filled out vertical t-table that you can use to see x values and their corresponding function values.

[†] Most newer graphing calculators have Table functions; see Table in your calculator guide index if the directions here do not match your calculator

- V. Use the up and down arrow keys to access entries before the first line and after the last line in the window.

EXAMPLE

y = menu	TBLSET menu
$y = x + 1$	TblStart = -2 Δ Tbl = .5 <i>Auto</i>

TABLE view	
x	Y1
-2	-1
-1.5	-.5
-1	0

Table continues on ...

ASK DISPLAY LIST option

- I. Set the Independent variable (Indpnt) to *Ask* by scrolling to *Ask* and hitting ENTER.
- II. Open the TABLE window to see a blank vertical t-table that you can use to enter x values and see corresponding function values (use the delete key to clear unwanted x - values).

EXAMPLE

y = menu	TBLSET menu
$y = x + 1$	<i>Ask</i>

TABLE view	
x	Y1
1.7	2.7
<i>Enter numbers here</i>	

Enter x values in each row; →
 use arrow buttons to scroll up and down →

To Change Functions and Display Multiple T-Tables**CHANGE Function**

Clear existing function and reenter a new function in the graphing menu (y = button)

ENTER Multiple Functions

Enter each function on separate line in the graphing menu (usually $y_1 =$, $y_2 =$, etc) under the y= button.

VIEWING Multiple T-Tables

To view t-tables for multiple functions already entered in the graphing menu, notice the function names (Y1, Y2, etc) will be displayed as the column headers in the TABLE window. To access columns off of the screen, use the right and left arrow keys on your calculator.

To View Function Graphs

- I. Enter the formula for your function in the graphing menu (the $y =$ button)
- II. Use the WINDOW button and enter in (axes) values for
Xmin = Ymin =
Xmax = Ymax =
- III. Press the GRAPH button to see a display of your function graph
- IV. Change the window settings and explore the ZOOM features to look at different views of your graph.

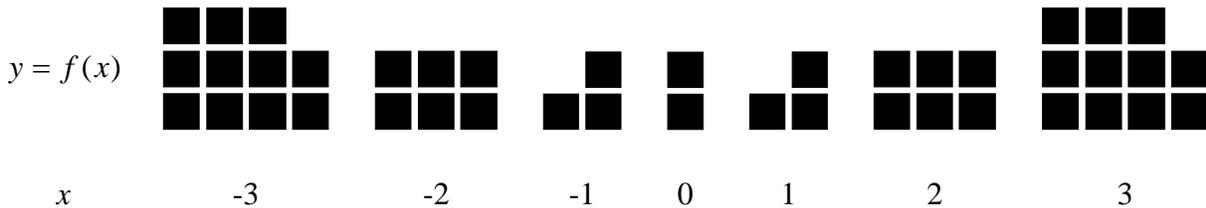
ZOOM Standard is often a good place to initially view the functions in this activity set

ZOOM Standard sets the WINDOW to:

Xmin = -10 Xmax = +10
Ymin = -10 Ymax = +10

1. Explain why we use black x -squares and red x -squares instead of white and opposite white x -squares.

2. (*) Consider the following extended sequence of tile figures, $y = f(x)$, with domain, \mathbf{R} .
 - a. Use clearly labeled looping to help determine the different components of each figure and analyze the extended tile sequence to find the x th figure and the symbolic form for $y = f(x)$.



Sketch the x th figure here; don't include sketches of edge pieces.

$$y = f(x) = \underline{\hspace{4cm}}$$

- b. Describe at least two features of the extended $y = f(x)$ tile sequence that help you see $y = f(x)$ is not a linear function.

- c. Describe the 100th $y = f(x)$ figure. What does it look like? What is $f(100)$?

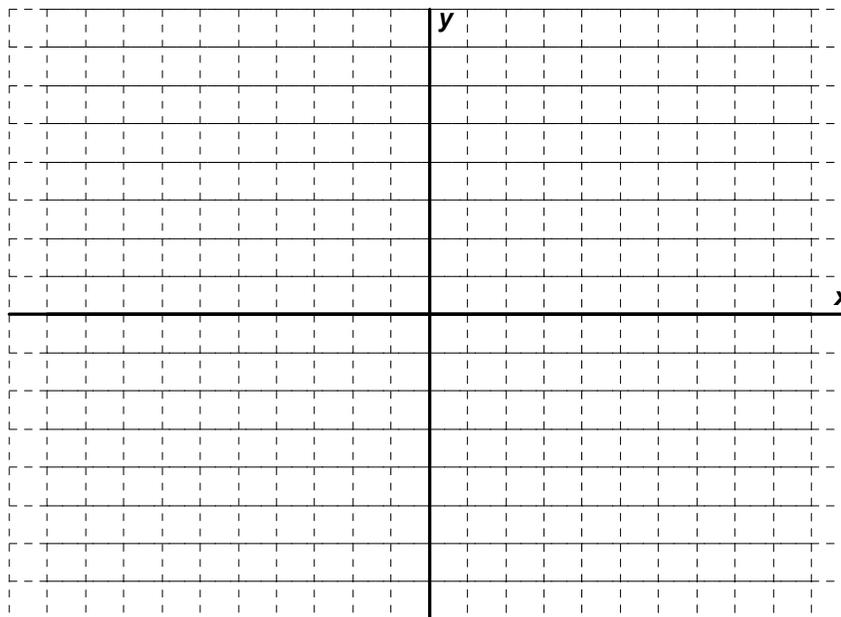
- d. For which x is $f(x) = 2502$? Is there more than one solution?

- e. Fill out the indicated function values in the following t-table (see “Technology Notes: Using a Graphing Calculator to Display T-Tables”).

x	-2	-1.5	-1	-.5	-.25	0	.25	.5	1	1.5	2	x
$y = f(x)$												

- f. By looking at the symbolic form of the function $y = f(x)$, what do you think the range of $y = f(x)$ is? Give this range on a number line and using inequality notation.

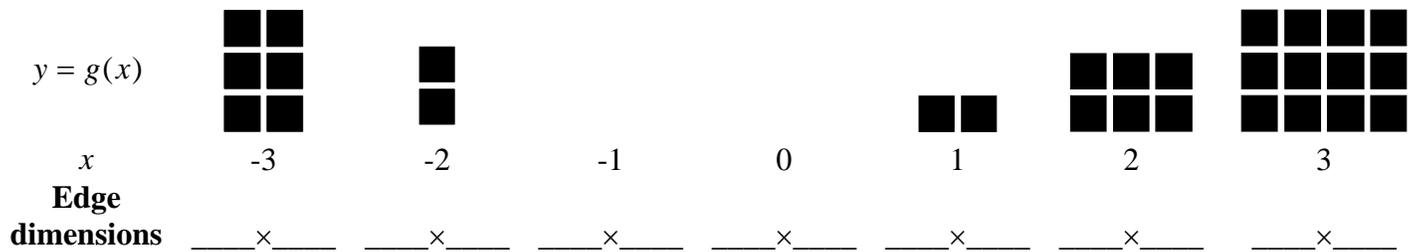
- g. Label the axes with appropriate numbers, plot ALL of the coordinate pairs from your t-table and sketch $y = f(x)$ (don't forget the end arrows). You may wish to view the graph of $y = f(x)$ on a graphing calculator as well (see “Technology Notes: To View Function Graphs”). Notice that, unlike the absolute value graphs, the “bottom” of this graph does not come to a sharp point.



- h. How does your answer for part f. about range show visually on the graph in part g.? Explain the connection.

6. Consider the following extended sequence of tile figures, $y = g(x)$, with domain, \mathbf{R} .
- a. Describe at least two features of the extended $y = g(x)$ tile sequence that help you see $y = g(x)$ is not a linear function.

- b. Using edge dimensions is an excellent technique for determining the algebraic structure of rectangular components of tile figures. Determine the edge dimensions for the following extended sequence of rectangular tile figures, $y = g(x)$. It may be easier to start with $x = 1, 2, 3$ and then consider the rest of the figures.

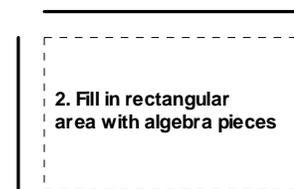


- c. What are the edge dimensions for the x th figure? _____ × _____
- d. Use the following tips to model and then sketch the x th $y = g(x)$ figure.

As illustrated in the diagram

1. Start by setting up the two edge lengths of the rectangular figure by using combinations of white $\pm x$ -edge pieces and ± 1 -edge pieces of the appropriate dimensions (label each edge piece).
2. Proceed by filling in the correct $\pm x$ -squares, $\pm x$ -strips, etc. to model the rectangle itself. Let the individual edge piece dimensions and colors guide you as you determine which pieces make up the whole x th figure.

1. Set up and label both edges

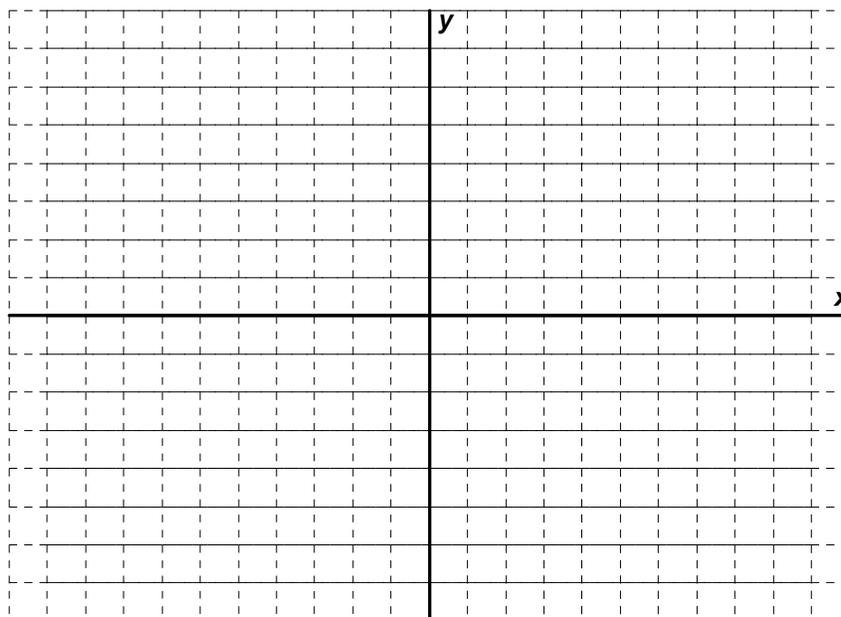


- e. What is $y = g(x)$? Remember, the edge sets are a guide, not part of the function.

- f. Fill out enough x values and corresponding $y = g(x)$ values in this t-table so that you can sketch the graph of $y = g(x)$.

x													
$y = g(x)$													

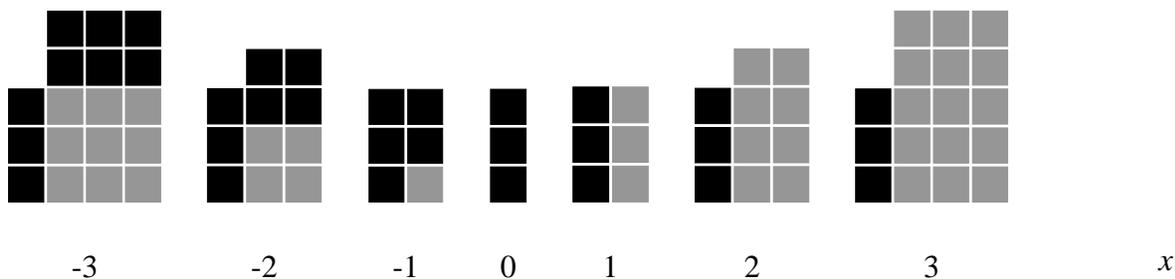
- g. What do you think the turning point is for $y = g(x)$? Double check that you have enough coordinate pairs close to this point to be sure there is no lower point.
- h. (*) Label the axes with appropriate numbers and sketch $y = g(x)$.



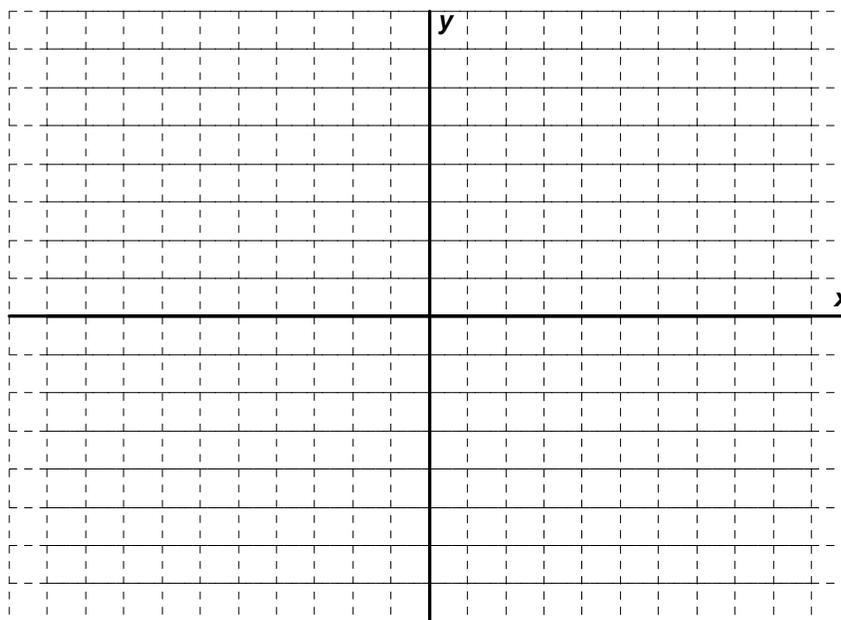
- i. (*) What are the coordinates for the turning point for $y = g(x)$? What is the range of $y = g(x)$? How are these two ideas connected?
- j. (*) All of the figures in the extended tile sequence are black, but there are points on the graph of $y = g(x)$ where $g(x) < 0$. Explain this.
- k. (*) What is the y -intercept for $y = g(x)$? Label this point on your graph.
- l. (*) What are the x -intercepts for $y = g(x)$? Label these points on your graph.

7. Consider the following two extended sequences of tile figures, each with domain, \mathbf{R} . For each extended sequence:
- Analyze the extended sequence (the x value is below each figure)
 - Model and sketch the x th figure and give the symbolic formula for the sequence (use edge pieces as a guide for modeling, but don't include them in your sketch).
 - Fill in a variety of useful points on the blank t-table, try to find the turning point.
 - Note the x and y -intercepts of the function
 - Sketch the graph, label the turning point, the x -intercepts and the y -intercept.
 - State the range of the function

a. $y = h(x)$

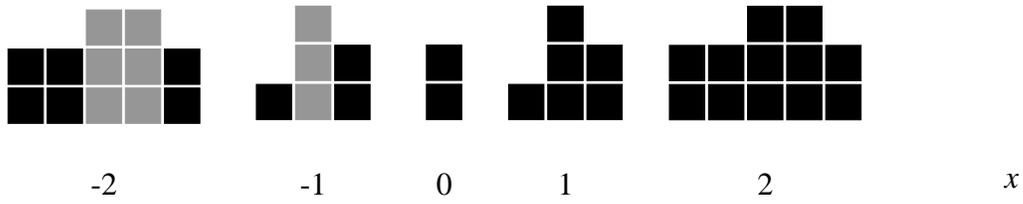


x												x
$y = h(x)$												

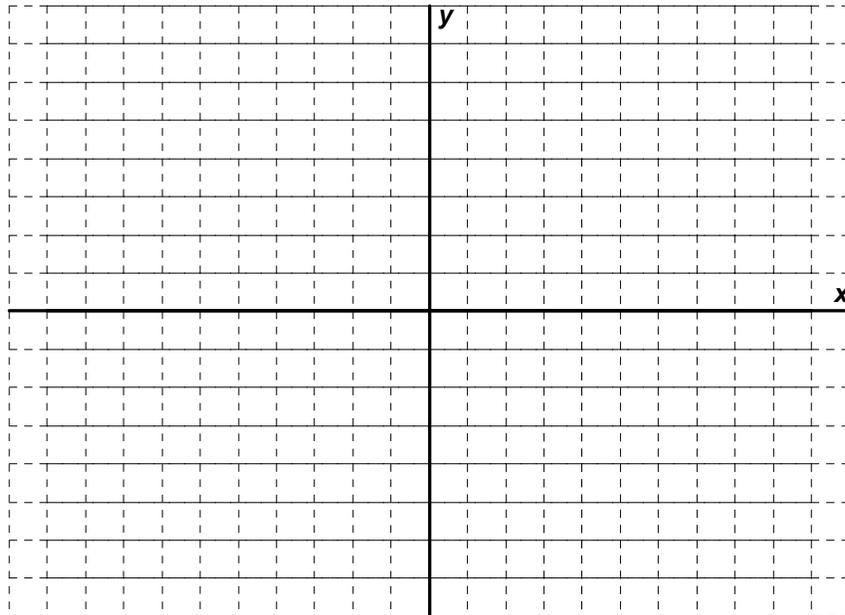


Range of $y = h(x)$:

b. $y = j(x)$



x													x
$y = j(x)$													



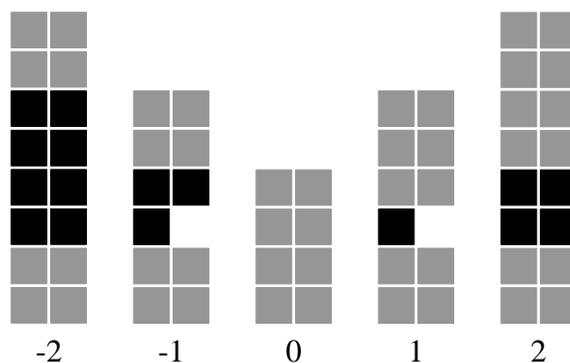
Range of $y = j(x)$:

Homework Questions 3.2

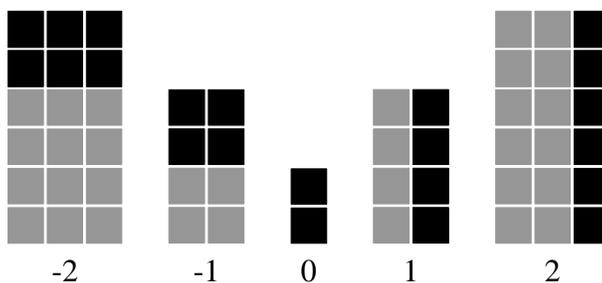
INTRODUCTION TO QUADRATIC FUNCTIONS

1. Assume that $y = b$ is in the range of a quadratic function. Are there always two distinct x values such that $f(x) = b$? Use a sketch to help explain this idea. Are there any special cases?
2. Where is the turning point of a parabola located relative to the x -intercepts? Use a sketch to help explain this idea.
3. For each of the following two extended sequences of tile figures, each with domain, \mathbf{R} :
 - i) Analyze the extended sequence (the x value is below each figure)
 - ii) Sketch the x th figure and give the symbolic formula for the sequence (use edge pieces as a guide for modeling, but don't include them in your sketch of the x th figure).
 - iii) Fill in a variety of useful points on a t-table; try to find the turning point and the x -intercepts.
 - iv) Sketch the graph; label the turning point, the x -intercepts and the y -intercept.
 - v) State the range of the function

a. $y = f(x)$



b. $y = g(x)$



Activity Set 3.3

ALGEBRA PIECES AND QUADRATIC FUNCTIONS

PURPOSE

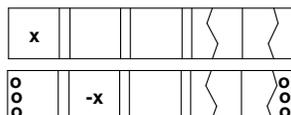
To learn to use algebra pieces to find the key features of quadratic graphs such as y -intercepts, x -intercepts, turning points and to find points of intersections with other quadratic and linear graphs. To learn to connect quadratic algebra piece work with the corresponding symbolic steps. To learn about the factored form of a quadratic function and its relationship to the x -intercepts of a parabola.

MATERIALS

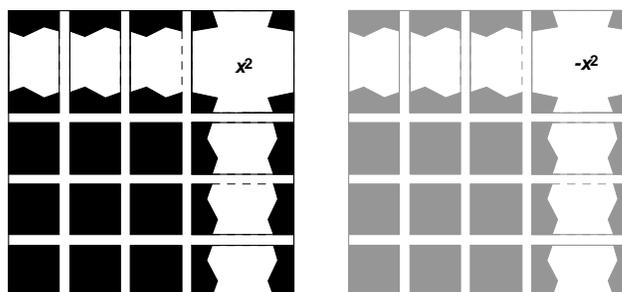
Black and red tiles



White and opposite white x -strips



Black and red x -squares



Graphing calculator with table functions (recommended)

INTRODUCTION

The General Form of a Quadratic Function

Since a quadratic function is a polynomial of degree two, we can think of all quadratic functions as having a form $y = ax^2 + bx + c$ where the coefficients a , b and c are real numbers and a is nonzero.

Finding Important Features of Quadratic Functions with Algebra Piece Models

You may have noticed that all of the important features of the parabolas graphed in Activity Set 3.2 were easy to find. For many quadratic functions, the important features are not so obvious.

Let's consider the extended tile sequence with rectangle tile figures $g(x) = x(x+1)$ from Activity Set 3.2

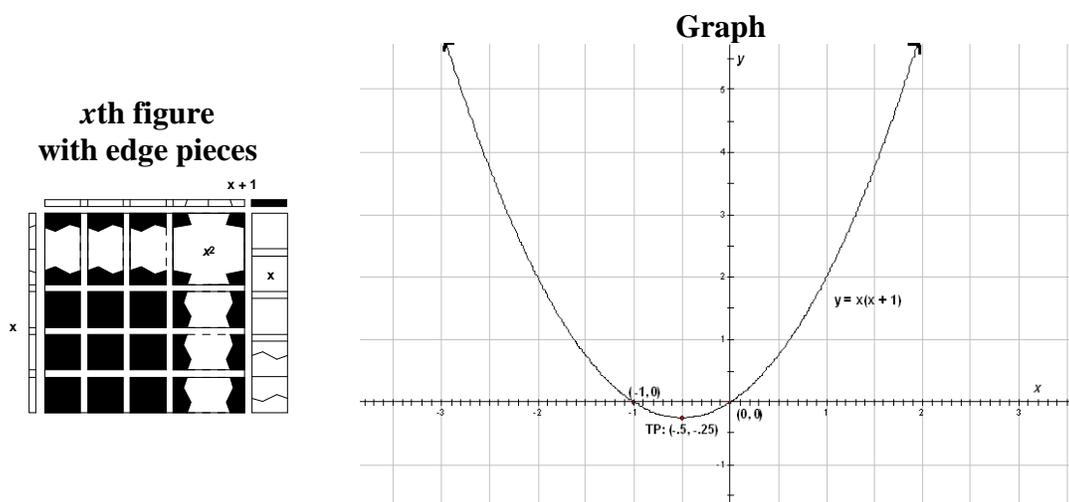


It turns out we can use the x th figure of this sequence to easily determine the y -intercept (this is always true), and in this case, the x -intercepts. Once we have found the x -intercepts, it is relatively easy to find the turning point of any quadratic function.

Finding x -Intercepts with Edge Sets

The x -intercepts on any graph are the points where the graph crosses the x -axis and the y -value is 0. We determined the symbolic formula $g(x) = x(x+1)$ by looking at the edge sets of the x th figure of this sequence. We can also use these edge sets to find the x -intercepts. The first step for finding the x -intercepts of any function is to set the function equal to zero. In this case; $y = g(x) = 0$. For this function, this is $x(x+1) = 0$.

In Chapter 1 we learned that an array of black and red tiles has net value 0, only if at least one of the edge sets of the array also has net value 0. The x th figure of $g(x)$ is rectangular, and therefore, it is an array. The array representing $g(x)$ can only be equal to 0 (have net value 0) if one or both edge sets has net value 0. Therefore, to find the x -intercepts of $g(x)$, we set each edge set = 0 and solve. $x = 0$ or $x+1 = 0$ yields $x = 0$ and $x = -1$ as the x values for the x -intercepts of $g(x) = x(x+1)$.



Must Be Array = 0

You can *ONLY* find x -intercepts (and solve equations) using the edge sets of a rectangle if you have an array set equal to zero. For example, if an algebra piece model is organized as: $x^2 + 6x = 7$, you would first need to move all of the pieces to one side of the equal sign before proceeding. In this case, this would look like: $x^2 + 6x - 7 = 0$.

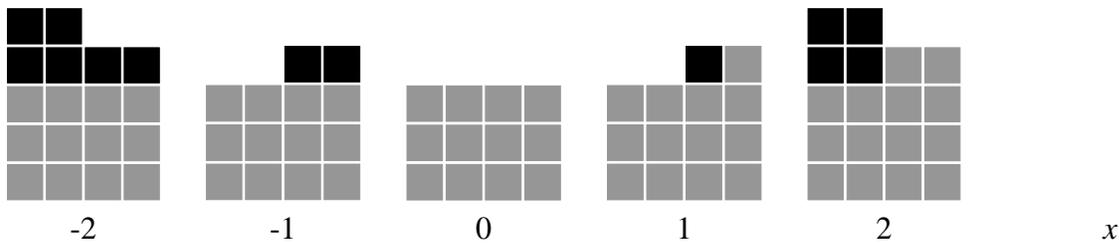
Factored Form of a Quadratic

In the previous example, the function $y = g(x)$ can be written in two ways: 1) $g(x) = x^2 + 1$ and 2) $g(x) = x(x+1)$. The second method is called the Factored Form of $g(x)$. The ideas of factoring a quadratic function and finding the x -intercepts of a quadratic function go hand and hand. Can you see why?

Working with Non-Rectangular x th Figures

Not all x th figures are rectangular. However, one can often add zero pairs of white and opposite white x -strips to create a rectangular shape and we will explore this idea in this activity set.

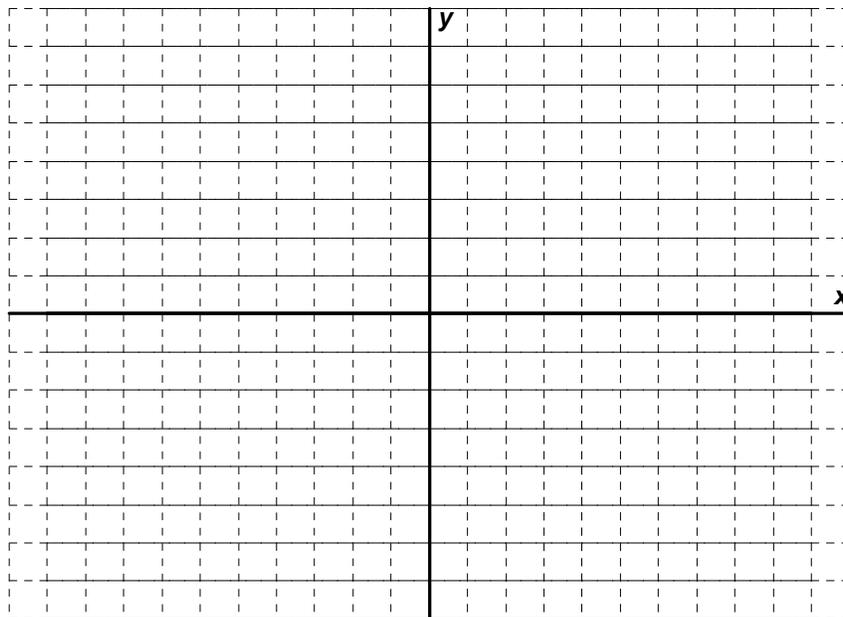
1. For the extended sequences of tile figures, $y = f(x)$, with domain, \mathbf{R} :
- a. Analyze the extended sequence (the x value is below each figure), sketch the x th figure and give the symbolic formula for the sequence simplified into a $y = ax^2 + bx + c$ form.



$$y = f(x) = \underline{\hspace{4cm}}$$

- b. What is the y -intercept for $y = f(x)$? How can you look at an extended tile sequence and quickly tell the y -intercept?
- c. (*) Use algebra pieces to model setting the x th figure of $y = f(x)$ equal to 0. The x th figure of $y = f(x)$ is not rectangular. Add zero pairs (one pair at a time) of white and opposite white x -strips to the x th figure of $y = f(x)$ until you *can* form a set of algebra pieces with the same net value as $y = f(x)$ that can be made into a rectangle. Lay out the edges of the rectangle and sketch the edge and rectangle model here.
- d. (*) Use the edge sets from part c. to determine the factored form of $y = f(x)$.

- e. (*) Use the edge sets from part c. to determine the x -intercepts of $y = f(x)$. Show your work.
- f. (*) Use the x -intercepts from part d. to determine the turning point for $y = f(x)$. Show your work.
- g. Plot the x -intercepts, the y -intercept and the turning point for $y = f(x)$, if necessary, plot a few more coordinate pairs for $y = f(x)$ and then sketch the entire graph of $y = f(x)$.



- h. What is the range of $y = f(x)$?

2. Use Figure 1 as a guide while answering the questions in parts a – d.

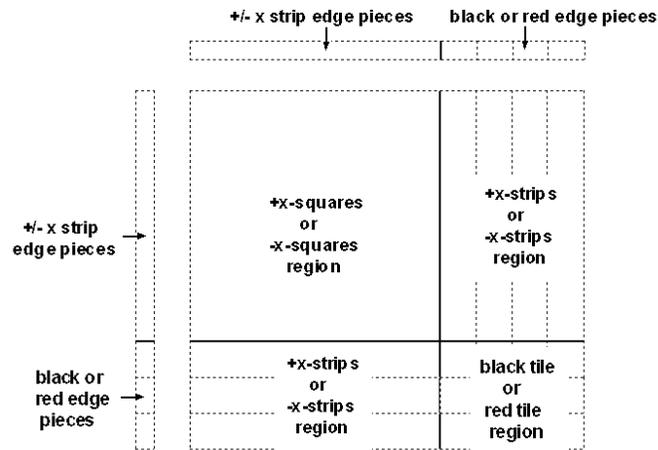
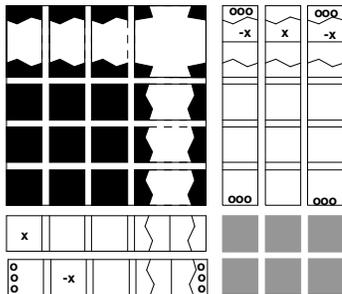


Figure 1

- a. Arrange two black x -squares and one red x -square, three white $+x$ -strips and two black tiles into a rectangle. What edge sets correspond to this rectangle? Now remove one black and one red x -square and arrange the remaining collection into a new rectangle. What edge sets correspond to this new rectangle? Why is the second rectangle more efficient? Think of this example and answer the question: Why does Figure 1 say $+x$ -squares OR $-x$ -squares region?

- b. Arrange one black x -square, three white $+x$ -strips, four black tiles and two red tiles into a rectangle. What edge sets correspond to this rectangle? Remove two black and two red tiles and form a new rectangle. What edge sets correspond to this new rectangle? Why is the second rectangle more efficient? Think of this example and answer the question: Why does Figure 1 say black tile OR red tile region?

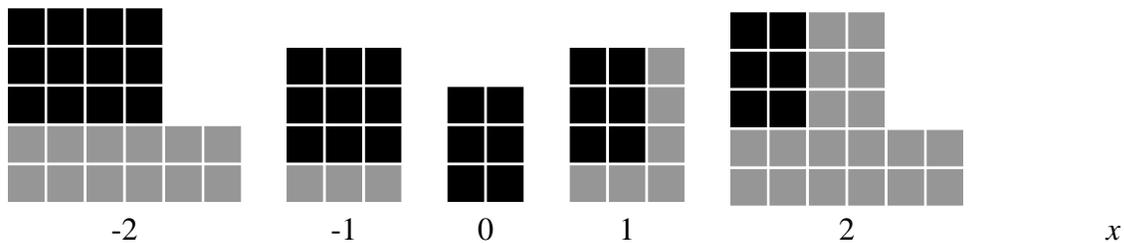
- c. Evaluate the following rectangle. Set up the edge sets, do they work? Is there a better rectangle formed with this same set of pieces? If so, sketch it. Think of this example and answer the question: Why does Figure 1 have regions labeled $+x$ -strips OR $-x$ -strips?



- d. Is the statement “You should use a minimal collection of algebra pieces when forming a quadratic rectangle” true? Explain why or why not.

3. (*) For the extended sequences of tile figures, $y = g(x)$, with domain, \mathbf{R} :

a. Analyze the extended sequence, give the symbolic formula for the sequence simplified into a $y = ax^2 + bx + c$ form and sketch the x th figure.

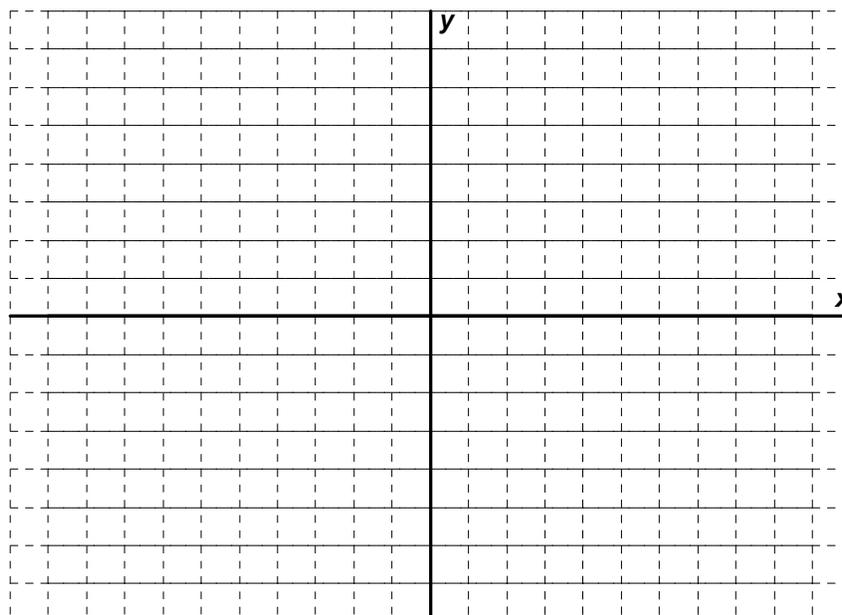


$y = g(x) = \underline{\hspace{4cm}}$

b. Find each of the following (if they exist). Sketch your models and show your work.

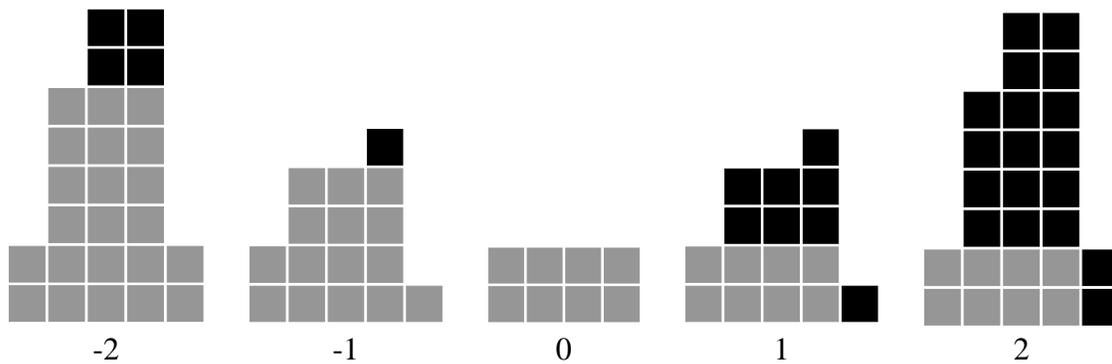
y-intercept	x-intercepts	Turning Point	Range	Factored Form

c. Plot the x -intercepts, the y -intercept and the turning point for $y = g(x)$, if necessary, plot a few more coordinate pairs for $y = g(x)$ and then sketch the entire graph of $y = g(x)$.



4. Consider the following two extended sequences of tile figures, each with domain, \mathbf{R} :

a. Analyze the extended sequence $y = h(x)$.



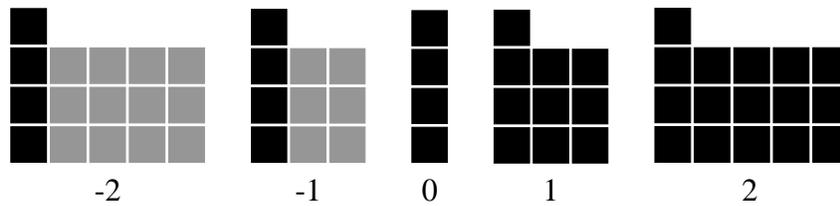
b. Sketch the x th figure for $y = h(x)$ and give the symbolic formula for the sequence simplified into a $y = ax^2 + bx + c$ form.

c. Find each of the following (if they exist). Sketch your models and show your work.

y-intercept	x-intercepts	Turning Point	Range	Factored Form

Sketches and work

- d. Analyze the extended sequence $y = j(x)$.



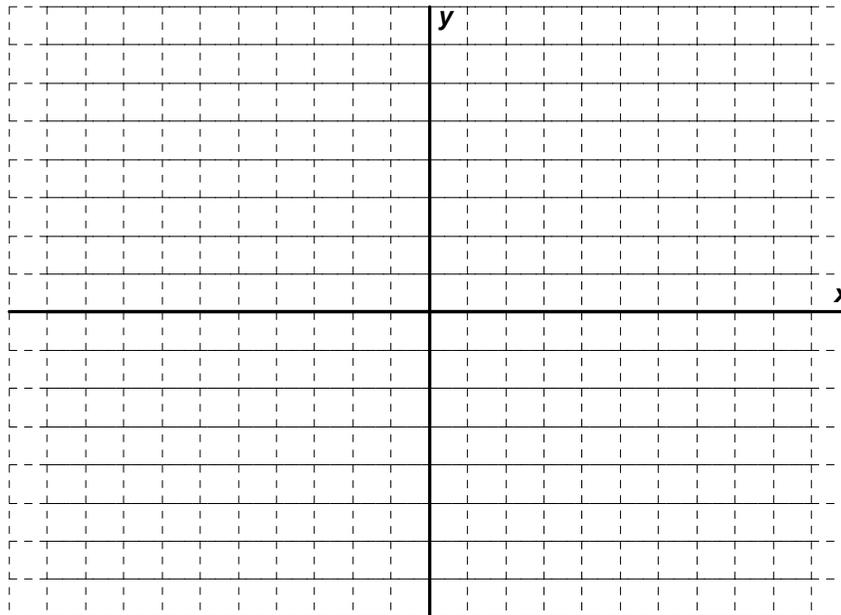
- e. Sketch the x th figure for $y = j(x)$ and give the symbolic formula for the sequence in a simplified form.

- f. Find each of the following (if they exist). Sketch your models and show your work.

y-intercept	x -intercepts	Turning Point	Range	Factored Form

Sketches and work

- g. Plot the key points for $y = h(x)$ and $y = j(x)$ and sketch both graphs.

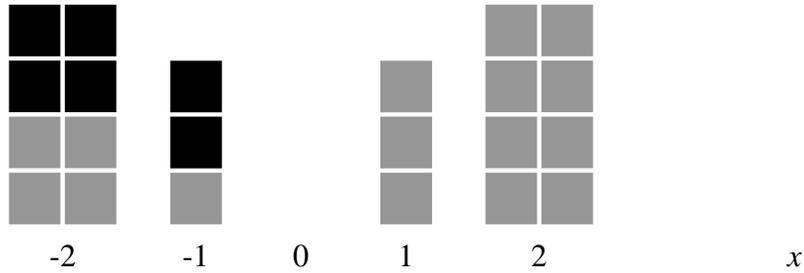


- h. You can probably tell from your graph where $y = h(x)$ and $y = j(x)$ intersect. However, you can use the “rectangle technique” to find these two points symbolically. Using your algebra pieces, set the x th figure for $y = h(x)$ equal to x th figure for $y = j(x)$. Arrange the pieces so they are all on one side of the equal sign and use the “form a rectangle and measure the edge sets” technique to determine the solutions to the resulting equation. These solutions are the x values for the intersections of $y = h(x)$ and $y = j(x)$. Show your work in the following two column table. Use the functions to determine the corresponding y values for the two points where the two functions intersect.

ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK

5. Analyze the following two extended sequences, each with domain, \mathbf{R} . Sketch the x th figures and give the symbolic formula for each sequence simplified into a $y = ax^2 + bx + c$ form. Find the key features (if they exist). Sketch your models and show your work.

a. $y = k(x)$.

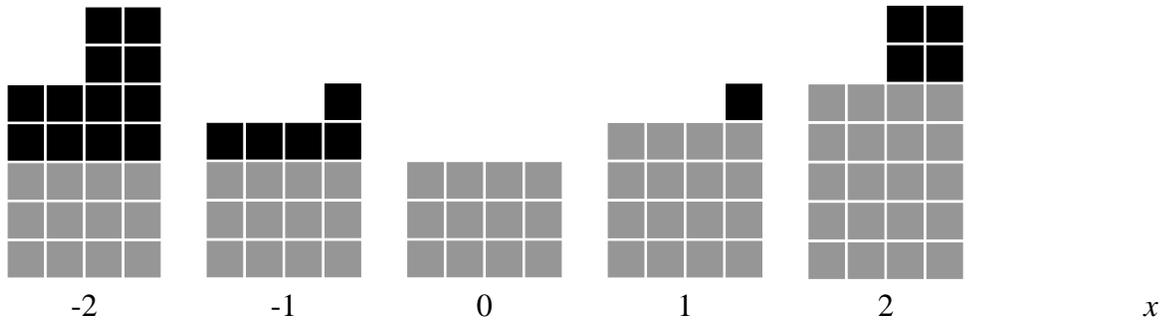


$y = k(x) = \underline{\hspace{2cm}}$

y-intercept	x-intercepts	Turning Point	Range	Factored Form

Sketches and work

b. $y = m(x)$.



$y = m(x) = \underline{\hspace{2cm}}$

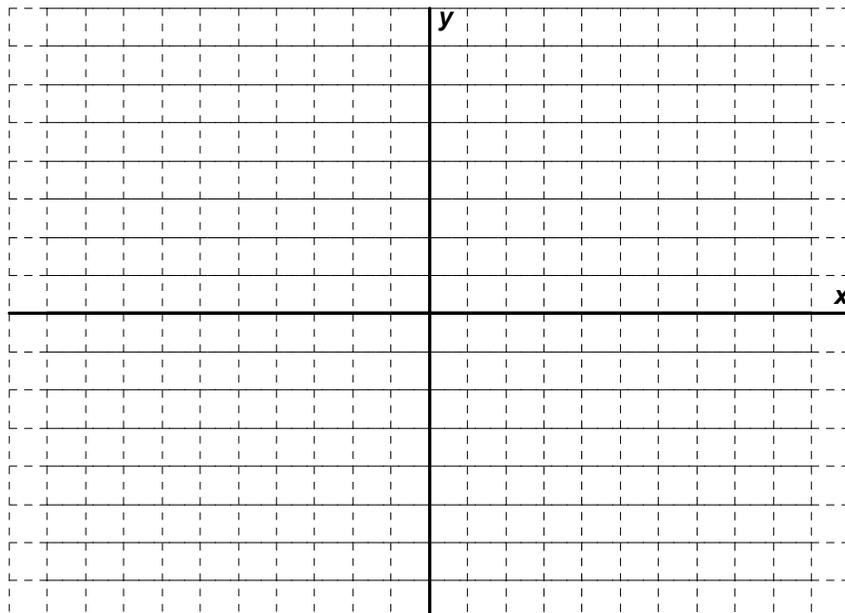
y-intercept	x-intercepts	Turning Point	Range	Factored Form

Sketches and work

- c. Use algebra pieces to determine where $y = k(x)$ and $y = m(x)$ intersect. Show your work in the two column table. Find the x and the y values of these points.

ALGEBRA PIECE WORK with notes	CORRESPONDING SYMBOLIC WORK

- d. Graph $y = k(x)$ and $y = m(x)$. Label all the key points on each parabola and the points where the two graphs intersect.



Homework Questions 3.3

ALGEBRA PIECES AND QUADRATIC FUNCTIONS

1. The graph of a function is Concave Up when it is cup shaped and can “hold water” at that location. In what circumstances are parabolas concave up, i.e., when is the turning point is the lowest point on the graph? Describe the circumstances in terms of algebra pieces and in terms of the symbolic form, $y = ax^2 + bx + c$
2. The graph of a function is Concave Down when it like an upside down cup and cannot “hold water” at that location. In what circumstances are parabolas concave down, i.e., when is the turning point the highest point on the graph? Describe the circumstances in terms of algebra pieces and in terms of the symbolic form, $y = ax^2 + bx + c$
3. Sketch diagrams of lines crossing parabolas and create each of the following. Don't worry about finding the symbolic equation or the algebra piece models for the lines or for the parabolas, just give sketches.
 - a. A line that intersects a parabola 0 times. Give several examples. Are there any criteria for such a circumstance?
 - b. A line that intersects a parabola exactly 1 time. Give at least two examples. Are there any criteria for such a circumstance?
 - c. A line that intersects a parabola exactly 2 times.
 - d. Can a line cross a parabola 3 or more times? Explain why or why not.
4. Sketch diagrams of parabolas and create each of the following. Don't worry about finding the symbolic equation or the algebra piece model for the parabolas, just give sketches.
 - a. Two parabolas that intersect 0 times. Give several examples.
 - b. Two parabolas that intersect exactly 1 time. Give at least two examples.
 - c. Two parabolas that intersect exactly 2 times.
 - e. Can two parabolas intersect exactly 3 or more times? Explain why or why not.
5.
 - a. Create an algebra piece model for $y = x^2 + 9x + 18$. Use the model to analyze the function and find the y -intercepts, x -intercepts, turning points, range and factored form of the quadratic function.
 - b. Create an algebra piece model for $y = -x^2 - 3x + 18$. Use the model to analyze the function and find the y -intercepts, x -intercepts, turning points, range and factored form of the quadratic function.

- c. Use a two column table to show the algebra piece work and corresponding symbolic work for determining the points of intersection for the parabolas in part a. and part b.
 - d. Sketch the parabolas in part a. and part b. together. Label each graph, mark all of the key features and label the points of intersection of the two parabolas.
6. Factor each of the following quadratic functions and determine the x -intercepts of the function. You may wish to use algebra pieces as you determine the factors, but you do not need to sketch the algebra piece models or the graphs for these questions.
- a. $y = x^2 + 10x + 24$
 - b. $y = x^2 - 3x - 10$
 - c. $y = -2x^2 - 6x$
 - d. $y = x^2 - 6x + 9$
 - e. $y = -x^2 - 5x + 6$
 - f. $y = x^2 - 16$
 - g. $y = 2x^2 - x - 1$
 - h. $y = 3x^2 - 8x + 4$
7. How can you compare the white and opposite white x -strips and the black and red tiles in the algebra piece model for a quadratic function to determine if the “rectangle” technique for factoring the quadratic will work? Explain and give several examples to support your explanation.

Activity Set 3.4

COMPLETING THE SQUARE, THE QUADRATIC FORMULA AND QUADRATIC GRAPHS

PURPOSE

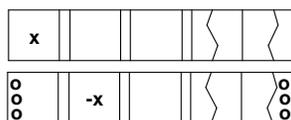
To learn how to use squares and square roots while solving quadratic equations. To learn how to complete the square to find the quadratic formula and the $y = a(x-h)^2 + k$ form of a quadratic function. To learn how the graphs of general quadratic functions differ from the graph of the simplest parabola: $y = x^2$. To be able to analyze any quadratic function.

MATERIALS

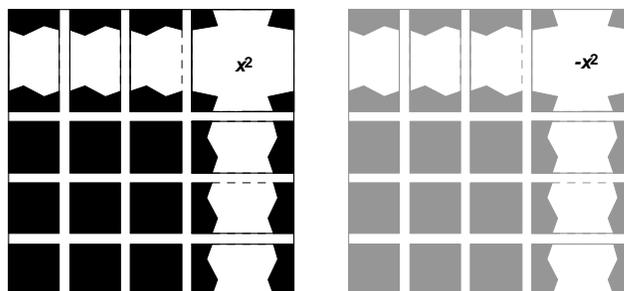
Black and red tiles



White and opposite white x -strips



Black and red x -squares



Graphing calculator with table functions (recommended)

INTRODUCTION

Squares and Square Roots

In our previous work, we noticed that equations such as $x^2 + 6x = 7$ were difficult to solve. However if we “set everything equal to zero,” this allowed us to use quadratic rectangle arrays to solve such equations. It turns out, if components of quadratic equations are *square* rectangular arrays, we can use additional techniques to solve these equations.

Suppose we wish to solve an equation such as $x^2 = 4$. Previously we would approach this by setting everything equal to zero:

$$x^2 - 4 = 0$$

and then factoring:

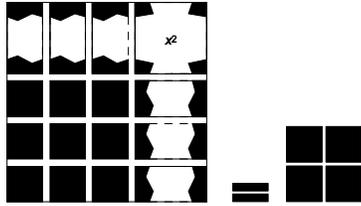
$$(x - 2)(x + 2) = 0$$

$$(x - 2)(x + 2) = 0 \text{ if } x - 2 = 0 \text{ or } x + 2 = 0.$$

$$x^2 - 4 = 0 \text{ if } x = \pm 2.$$

However, because everything on the left side of the equation (x^2) is *square* and everything on the right side of the equation is a *number*, we can use an additional, and in this case, faster technique for determining the solutions to $x^2 = 4$. We can think of the new technique in terms of algebra pieces and we can think of the new technique graphically.

ALGEBRA PIECES for $x^2 = 4$

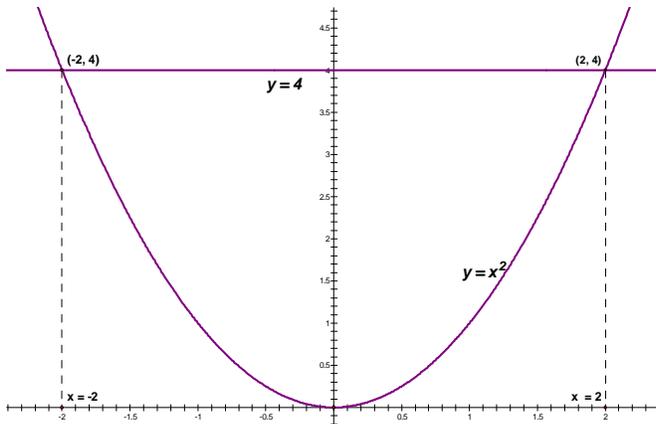


The dimensions of the black x -square must be

$$2 \times 2 \text{ or } -2 \times -2$$

$$\text{If } x^2 = 4, \text{ then } x = \pm 2.$$

GRAPHING for $x^2 = 4$

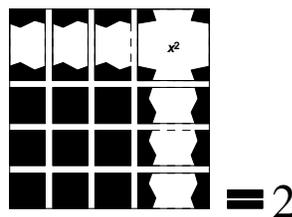


To solve $x^2 = 4$, we can also think of the *intersection of two functions*: $y = x^2$ and $y = 4$.

It is easy to see graphically the two intersection points are $(-2, 4)$ and $(2, 4)$.

$x^2 = 4$ is a particularly easy example. Both the left side and the right side of the equation are already square. Suppose we wish to solve an equation such as $x^2 = 2$. In this case we cannot make 2 black tiles into a square array shape. Let's look at this new equation using algebra pieces and using a graph.

ALGEBRA PIECES for $x^2 = 2$

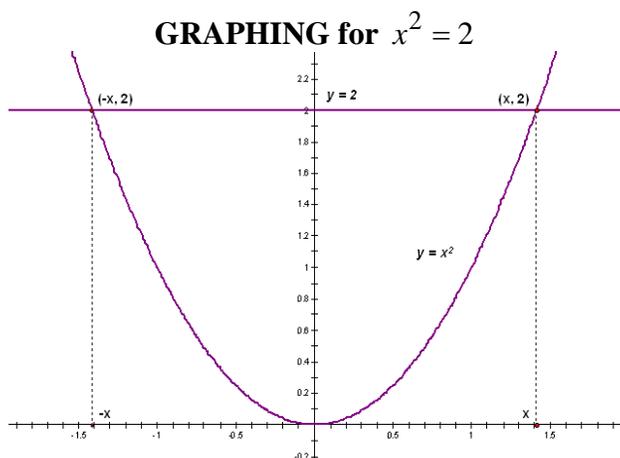


The dimensions of the black x -square are $x \times x$ or $-x \times -x$. Thus, if we can find a number whose square is 2, the opposite of that number should also have a square equal to 2.

By *definition*, the square root of 2 ($\sqrt{2}$) is the number whose square is 2. $\sqrt{2} \times \sqrt{2} = 2$

Thus, according to the algebra piece model, the dimensions of the black x -square must be $\sqrt{2} \times \sqrt{2}$ or $-\sqrt{2} \times -\sqrt{2}$.

To solve $x^2 = 2$, we “take the square root of both sides” and keep in mind that we should determine both the positive and the negative answer. If $x^2 = 2$ then $x = \pm\sqrt{2}$.



To solve $x^2 = 2$, we can also think of the *intersection of two functions*: $y = x^2$ and $y = 2$.

It is easy to see graphically if x is the x -value of an intersection point of $y = x^2$ and $y = 2$, then so is $-x$. This parallels our algebra piece work: If $x^2 = 2$, then $x = \pm\sqrt{2}$.

Notice this new technique “take the square root of both sides” is really all we did for $x^2 = 4$. Since $\sqrt{4} = 2$, if $x^2 = 4$, then $x = \pm\sqrt{4} = \pm 2$.

The two examples we have looked at both have a simple x^2 on the one side of the equation. However, our new technique: **Taking the Square Root of Both Sides** works if one side is any *square* and the other side is any *positive number* (why does the number have to be positive?).

Suppose we wish to solve $(x+1)^2 = 2$. Using our previous technique, “take the square root of both sides” yields: $\sqrt{(x+1)^2} = \pm\sqrt{2}$ (it is redundant to write \pm on both sides; why?).

This simplifies to $x+1 = \pm\sqrt{2}$. This splits into two solutions: $x = -1 - \sqrt{2}$ and $x = -1 + \sqrt{2}$. Since $\sqrt{2} \approx 1.41$, the two solutions to $(x+1)^2 = 2$ are: $x = -2.4$ and $x = .4$ (approximately).

The Quadratic Formula

You may have noticed that all of the quadratic functions in Activity Sets 3.2 and 3.3 factored and that all of the corresponding x -intercepts and intersection points had integers or simple fraction x -values. Of course there are many quadratic functions where x -intercept or intersection x -values are not integers or simple fractions. For equations such as $y = x^2 - 2$; we can find the x -intercepts by solving the equations such as $x^2 = 2$, but to solve quadratic equations such as $x^2 + 2x - 1 = 0$, we need a more powerful technique.

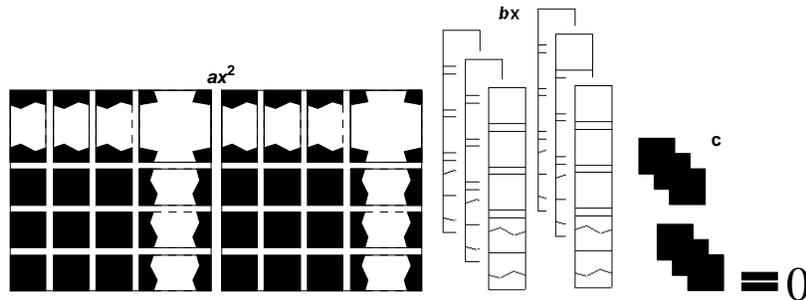
We will start with a general quadratic equation ($ax^2 + bx + c = 0$, $a \neq 0$) and manipulative the equation until we create a *square* left side set equal to a *number* right side. After taking square roots, we will have derived a formula (the **Quadratic Formula**) we can use to solve any quadratic equation. The technique we will use for creating a square left side is called: **Completing the Square**.

COMPLETING THE SQUARE FOR

$$ax^2 + bx + c = 0, a \neq 0$$

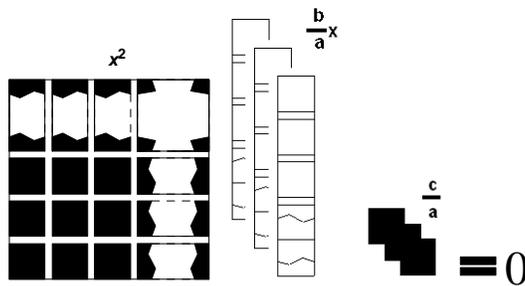
The values of a , b and c are not fixed. The sizes and colors on the diagrams are just for illustration

This is what we wish to solve



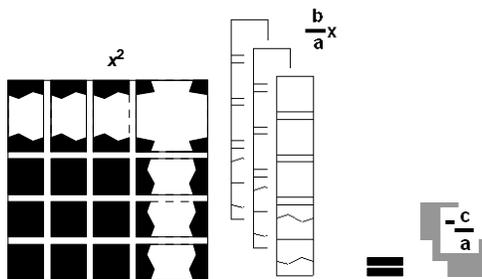
$$ax^2 + bx + c = 0$$

This is simpler if there is only one black x -square, divide everything by a (remember $a \neq 0$). Unlike the rest of the “just for illustration” components of the diagram; the remaining x -square will actually be black.



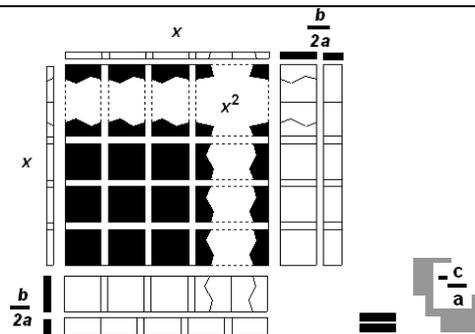
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

To facilitate making a *square* left side and a *number* right side; move $\frac{c}{a}$ to the right side.



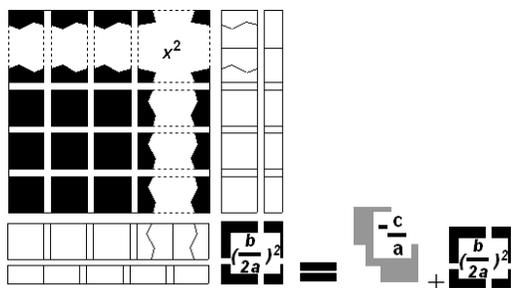
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Our goal is to make a *square* on the left. If we ignore the lack of black or red tiles on the left for the moment, we can divide the $\frac{b}{a}$ x -strips into two equal piles, each of size $\frac{b}{2a}$ and form the following partial square on the left.



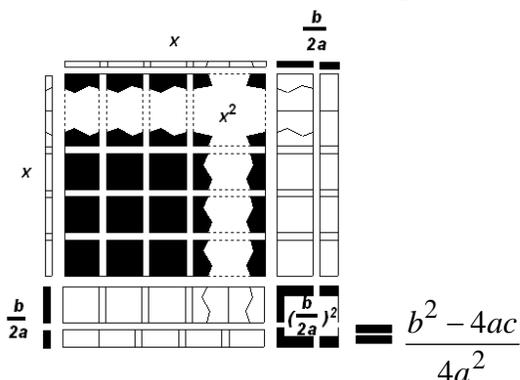
We can see that in order to *Complete the Square*, we must add $(\frac{b}{2a})^2$ black tiles on the left (and hence to both side). Note: $\frac{b}{2a}$ may, in fact, be negative.

Add $(\frac{b}{2a})^2$ black tiles to both sides



$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$$

The symbolic formula looks complicated, but we can use the edge set dimensions to *factor* the left side of the symbolic equation. We can also add the fractional right hand side.



The factors for the left side are:

$$(x + \frac{b}{2a})(x + \frac{b}{2a})$$

Adding the fractions on the right side gives:

$$-\frac{4ac}{4a^2} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

Altogether this is:

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

Since the left side of our equation is a *square* and the right side of our equation is a *number*, we can “take the square root of both sides” and see the following:

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \sqrt{\frac{b^2 - 4ac}{(2a)^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

We solve for x and we find the Quadratic Formula.

$$\text{If } ax^2 + bx + c = 0 \text{ and } a \neq 0, \text{ then } x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula Example

Suppose we wish to find the x -intercepts for $y = x^2 + 2x - 1$. First we set the function equal to zero and then we can use the quadratic formula.

$$x^2 + 2x - 1 = 0. \text{ Here } a = 1, b = 2 \text{ and } c = -1.$$

Using the quadratic formula:

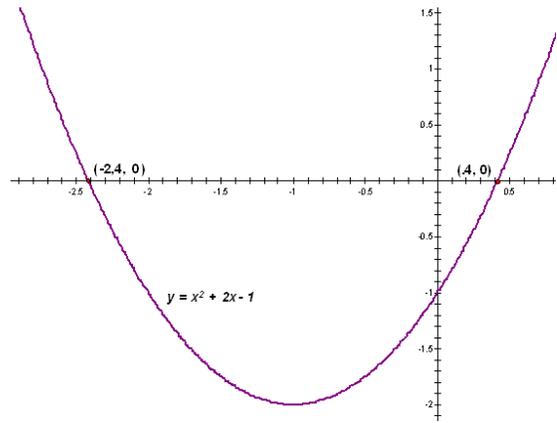
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = -\frac{2}{2(1)} \pm \frac{\sqrt{(2)^2 - 4(1)(-1)}}{2(1)} = -1 \pm \frac{\sqrt{8}}{2} = -1 \pm \frac{2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

The two x -intercepts are (approximately): $(-2.4, 0)$ and $(.4, 0)$.

Notice if we complete the square directly for $x^2 + 2x - 1 = 0$, we would have:

$$\begin{aligned} x^2 + 2x &= 1 \\ x^2 + 2x + 1 &= 1 + 1 \\ (x + 1)^2 &= 2 \end{aligned}$$

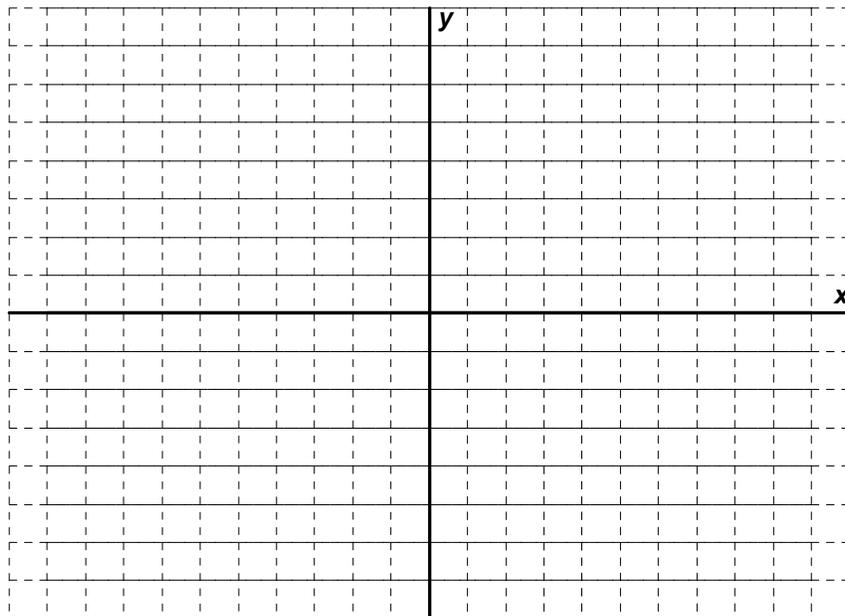
(which was a previous example)



1. Does the quadratic formula always work and does it always give two real number solutions?

a. Analyze $y = x^2 + 2$ and find each of the following (if they exist). Graph $y = x^2 + 2$; label all of the key points.

y-intercept	x-intercepts	Turning Point	Range	Factored Form

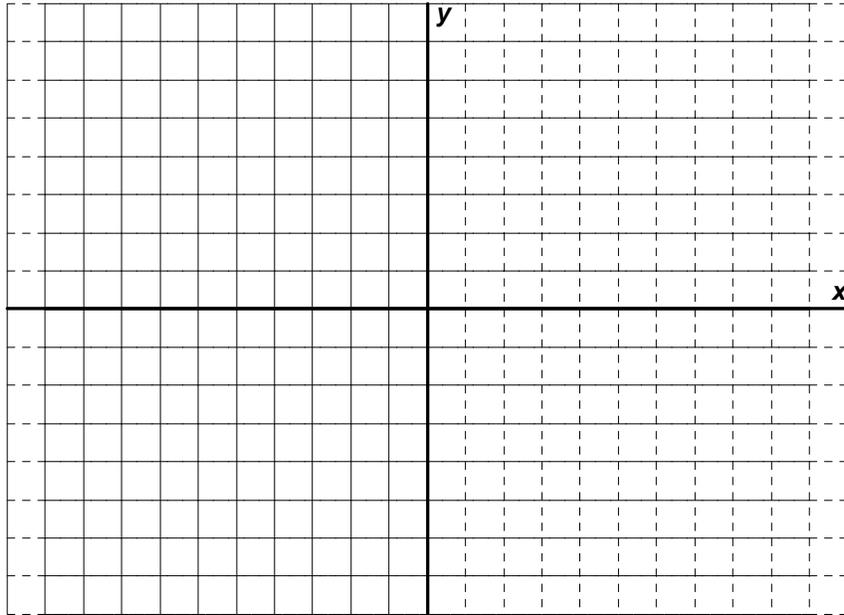


Sketches and work

b. Use the quadratic formula to find the x -intercepts for $y = x^2 + 2$ (note $b = 0$). What happens? How does this relate to the graph of $y = x^2 + 2$?

- c. Analyze $y = x^2 + 4x + 4$ and find each of the following (if they exist). Graph $y = x^2 + 4x + 4$; label all of the key points.

y-intercept	x-intercepts	Turning Point	Range	Factored Form

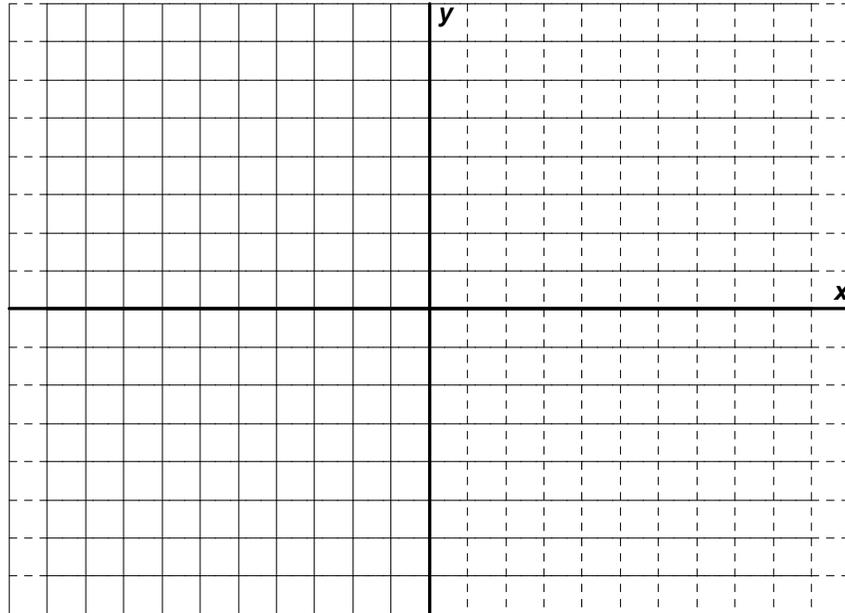


Sketches and work

- d. Use the quadratic formula to find the x -intercepts for $y = x^2 + 4x + 4$. What happens?
 How does this relate to the graph of $y = x^2 + 4x + 4$?

- e. Analyze $y = x^2 + x - 6$ and find each of the following (if they exist). Graph $y = x^2 + x - 6$; label all of the key points.

y-intercept	x-intercepts	Turning Point	Range	Factored Form



Sketches and work

- f. Use the quadratic formula to find the x -intercepts for $y = x^2 + x - 6$. What happens? How does this relate to the graph of $y = x^2 + x - 6$?

5. (*) Graph Shifting III

- a. How does the graph of $y = 2x^2$ differ from the graph of $y = x^2$?

- b. How does the graph of $y = -2x^2$ differ from the graph of $y = x^2$?

- c. Let a be any nonzero real number. How does the graph of $y = ax^2$ differ from the graph of $y = x^2$? Does it matter if a is positive or negative?

- d. What is the turning point and the range of $y = ax^2$? Does it matter if a is positive or negative?

- e. What is the y -intercept of $y = ax^2$?

- f. What are the x -intercepts of $y = ax^2$?

6. Let a , h and k be any nonzero real numbers.
- In the graph of $y = a(x - h)^2 + k$, what role does k play? How does it relate to the range of the function?
 - In the graph of $y = a(x - h)^2 + k$, what role does h play? How does it relate to the turning point of the function? What is the turning point?
 - In the graph of $y = a(x - h)^2 + k$, what role does a play?

7. Let's analyze $y = x^2 + 2x - 6$.

- a. We can use the techniques from completing the square to rewrite $y = x^2 + 2x - 6$ in a $y = a(x - h)^2 + k$ form. Use the right column to explain each step. It will probably help to look at all of the steps before writing out the explanations.

SYMBOLIC STEPS	EXPLANATION
$y = [x^2 + 2x] - 6$	
$y = [x^2 + 2x + 1] - 1 - 6$	
$y = [x^2 + 2x + 1] - 7$	
$y = (x + 1)^2 - 7$	

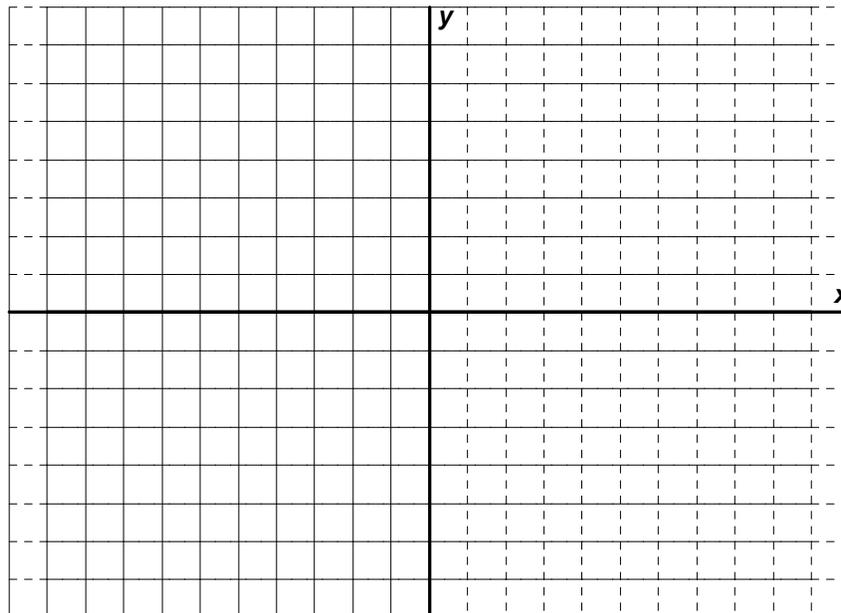
- b. How does $y = x^2 + 2x - 6$ differ from $y = x^2$? Is it easier to use the $y = x^2 + 2x - 6$ form or the $y = (x + 1)^2 - 7$ form to see this?
- c. What is the y -intercept for $y = x^2 + 2x - 6$? Is it easier to use the (general) $y = x^2 + 2x - 6$ form or the (shifted) $y = (x + 1)^2 - 7$ form to see this?

d. What are the x -intercepts for $y = x^2 + 2x - 6$?

e. What is the turning point of $y = x^2 + 2x - 6$? Is it easier to use the $y = x^2 + 2x - 6$ form or the $y = (x+1)^2 - 7$ form to see this?

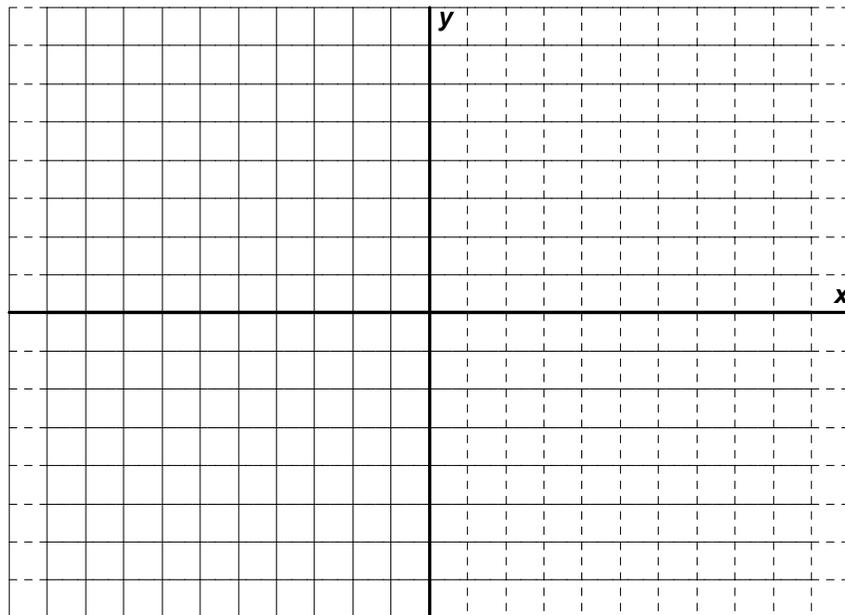
f. What is the range of $y = x^2 + 2x - 6$? Is it easier to use the $y = x^2 + 2x - 6$ form or the $y = (x+1)^2 - 7$ form to see this?

g. Graph $y = x^2 + 2x - 6$; label all of the key points.



8. Analyze $y = x^2 + 8x + 12$ and find each of the following (if they exist). Graph $y = x^2 + 8x + 12$; label all of the key points.

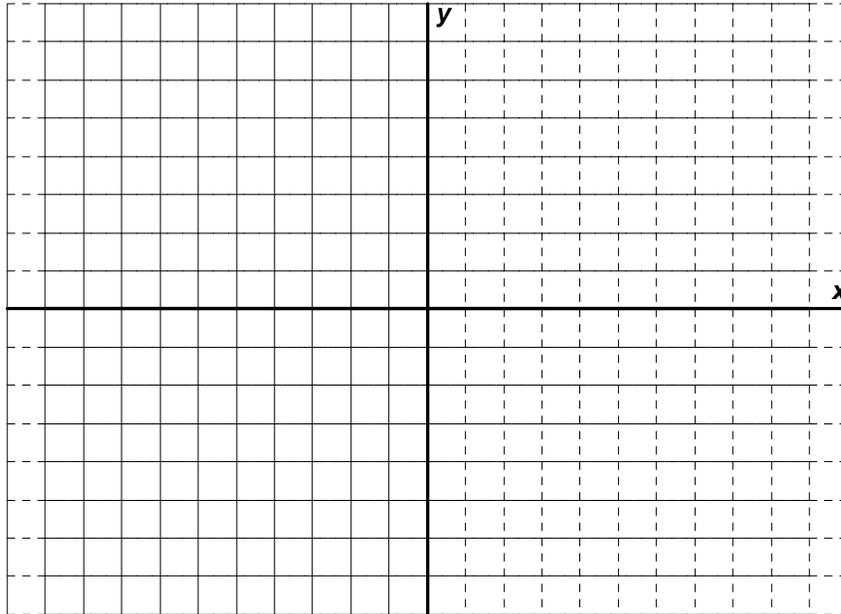
y-intercept	x-intercepts	Turning Point	Range
Factored Form		$y = a(x - h)^2 + k$ Form	



Sketches and work

9. Analyze $y = x^2 - x - 1$ and find each of the following (if they exist). Graph $y = x^2 - x - 1$; label all of the key points.

y-intercept	x-intercepts	Turning Point	Range
Factored Form		$y = a(x - h)^2 + k$ Form	

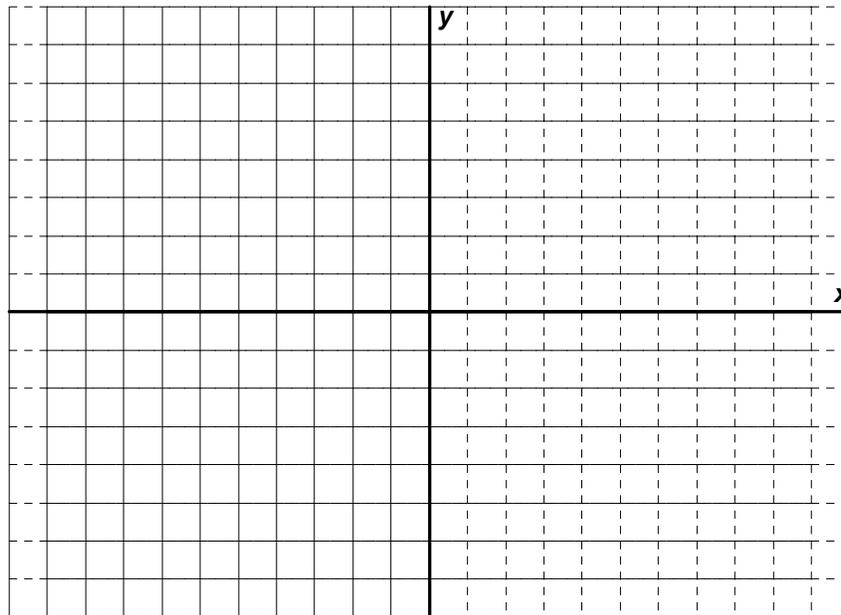


Sketches and work

10. Analyze $y = 2x^2 + 4x + 5$ and find each of the following (if they exist). Graph $y = 2x^2 + 4x + 5$; label all of the key points.

Hint: In order to find the $y = a(x - h)^2 + k$ form of $y = 2x^2 + 4x + 5$, you must first factor out the 2 from the x^2 and x terms. Start with $y = 2[x^2 + 2x] + 5$ and be careful about which number you subtract from 5.

y-intercept	x-intercepts	Turning Point	Range
Factored Form		$y = a(x - h)^2 + k$ Form	



Sketches and work

Homework Questions 3.4

COMPLETING THE SQUARE, THE QUADRATIC FORMULA AND QUADRATIC GRAPHS

1. Using the $y = a(x - h)^2 + k$ form, the $y = ax^2 + bx + c$ form or the factored form, whichever is best for the given parameters; create an example of each of the following types of quadratic functions. For each quadratic function that you create:
 - i) Give the $y = ax^2 + bx + c$ form, the $y = a(x - h)^2 + k$ form (do this by completing the square, show your work) and the factored form (if it exists).
 - ii) Give the x -intercepts, if they exist.
 - iii) Give the y -intercept, the turning point (show your work for finding this) and the range.
 - iv) Sketch a graph of the parabola. Label the key points.
 - a. Concave up, two x -intercepts.
 - b. Concave up, one x -intercept.
 - c. Concave up, zero x -intercepts.
 - d. Concave down, two x -intercepts.
 - e. Concave down, one x -intercept.
 - f. Concave down, zero x -intercepts
 - g. Turning point (2, 4)
 - h. Concave up, with x -intercepts (-2, 0) and (3, 0)
 - i. Concave down, with y -intercept (0, 4)
 - j. Turning point (1, -32) and y -intercept (0, -30)
2. True or False: All quadratics that factor have two x -intercepts. Justify your conclusion.
3. True or False: All quadratics that do not factor with algebra pieces have zero x -intercepts. Justify your conclusion.
4. What is the x -value for the turning point for a quadratic of the form $y = ax^2 + bx + c$?
Hint: Look at the derivation of the quadratic formula.
5. Analyze $y = 3x^2 - 12x + 12$ and find each of the following. Graph the function; label all of the key points.
 - i) y -intercept
 - ii) x -intercepts (if they exist)
 - iii) Turning Point
 - iv) Range

- v) Factored form (if it exists)
 - vi) $y = a(x - h)^2 + k$ form (do this by completing the square, show your work)
6. Analyze $y = -x^2 - 2x + 2$ and find each of the following. Graph the function; label all of the key points.
- i) y -intercept
 - ii) x -intercepts (if they exist)
 - iii) Turning Point
 - iv) Range
 - v) Factored form (if it exists)
 - vi) $y = a(x - h)^2 + k$ form (do this by completing the square, show your work)
7. Analyze $y = -2x^2 - 2x - 2$ and find each of the following. Graph the function; label all of the key points.
- i) y -intercept
 - ii) x -intercepts (if they exist)
 - iii) Turning Point
 - iv) Range
 - v) Factored form (if it exists)
 - vi) $y = a(x - h)^2 + k$ form (do this by completing the square, show your work)

Activity Set 3.5 INEQUALITIES

PURPOSE

To learn to use graphing and algebra together to find solutions to function inequality statements. To learn to how to explain the rule: “When you multiply or divide by a negative number, you switch the inequality sign.” To solve inequalities involving linear and quadratic terms and connect these solutions to corresponding function graphs.

MATERIALS

Graphing Calculator (recommended)

INTRODUCTION

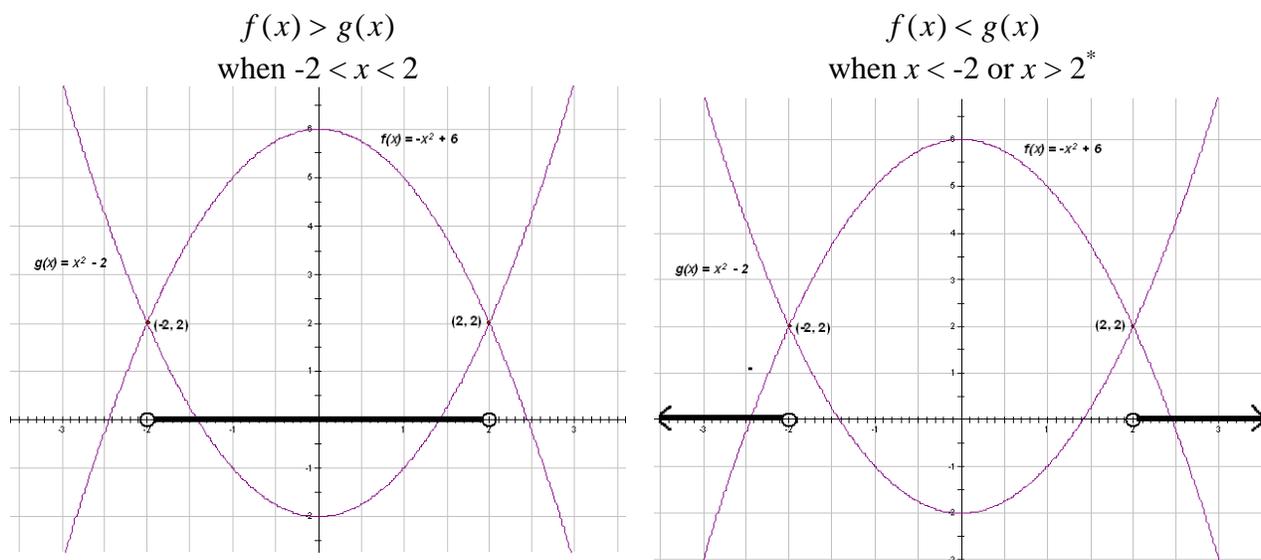
Function Inequalities

In this activity set you will be asked questions about where two functions are equal (a familiar concept) as well as where one function is less than or greater than another function. Statements about function inequalities are based on comparing function (output) values and are easy to see on a graph. A function $f(x)$ is less than a function $g(x)$ on an interval if $f(x) < g(x)$ for every x in that interval. Visually, $f(x) < g(x)$ when the graph of $f(x)$ is below the graph of $g(x)$. Solutions to function inequality statements are given as x -value inequality statements.

Inequality Example

$$f(x) = -x^2 + 6 \qquad g(x) = x^2 - 2$$

$$f(x) = g(x) \text{ at } (-2, 2) \text{ and } (2, 2)$$



*Note: A common error is to write $2 < x < -2$ instead of $x < -2$ or $x > 2$. However, $2 < x < -2$ reads “2 is less than x which is less than -2”. The combined statement, “2 is less than -2”, is incorrect. We say $x < -2$ **OR** $x > 2$ instead of $x < -2$ **AND** $x > 2$ for a similar reason. x cannot be both less than -2 **AND** greater than 2 at the same time.

TECHNOLOGY NOTES (graphing calculator models such as the TI-83 or 84 series)**Using a Graphing Calculator to Find the Intersection of Two Graphs****Step One**

Enter the formula for both of the functions in the GRAPHING menu

Step Two

Adjust the WINDOW and display the graphs of the two functions. Make sure the locations where the two functions cross (intersect) show.

Step Three

Select 2nd CALC (above Trace) and select 5. INTERSECT

The graphing menu will appear, the cursor will be on one curve and the display at the bottom of the screen will read “First curve?” Select ENTER

The graphing menu will still show, the cursor will be move to the second curve and the display at the bottom of the screen will read “Second curve?” Select ENTER

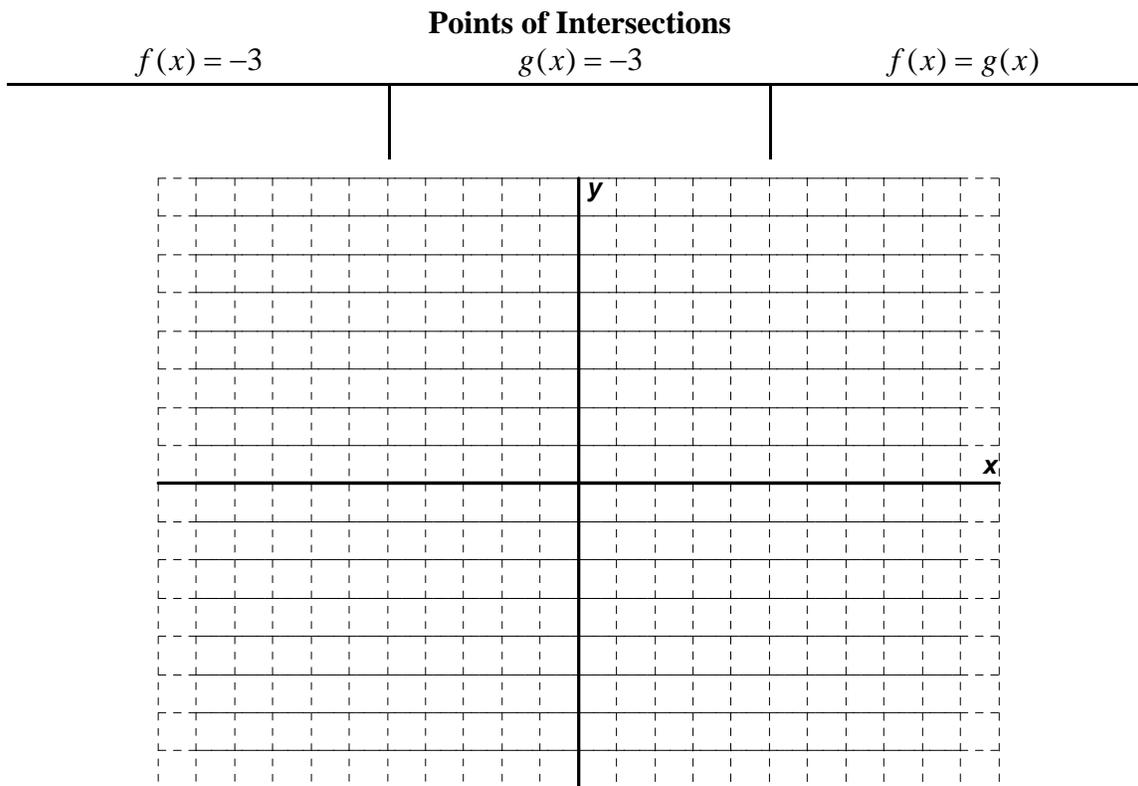
The graphing menu will still show, and the display at the bottom of the screen will read “Guess?” Select ENTER and the x and y coordinates of the point of intersection will be displayed.

Step Four

To find a second point of intersection for two functions, repeat Step Three, but use the arrow keys to move the cursor close to the second point of intersection before selecting ENTER after “First curve?” appears.

1. (*)

- a. Graph $y = -3$, $f(x) = -3x + 4$ and $g(x) = 2x - 1$ on the same grid. Find and label the indicated points of intersection. Show your work in the workspace given below.



Workspace

For each of the following, give the x -values where:

b. $f(x) < -3$

$f(x) > -3$

c. $g(x) < -3$

$g(x) > -3$

d. $f(x) < g(x)$

$f(x) > g(x)$

2.

- a. Graph $h(x) = -2x^2 + 6x + 4$, $j(x) = 3x - 5$ and $y = -5$ on the same grid. Use the table to list the y -intercept, x -intercepts, etc. for $h(x)$. Find and label the indicated points of intersection. Show your work in the workspace given below.

Points of Intersections				
$h(x) = -5$	$j(x) = -5$	$h(x) = j(x)$		
<hr/>				
$h(x) = -2x^2 + 6x + 4$				
y-intercept	x-intercepts	Turning Point	Range	$y = a(x - h)^2 + k$
<hr/>	<hr/>	<hr/>	<hr/>	<hr/>

Workspace

For each of the following, give the x -values where:

b. $h(x) < -5$

$h(x) > -5$

c. $j(x) < -5$

$j(x) > -5$

d. $h(x) < j(x)$

$h(x) > j(x)$

3.

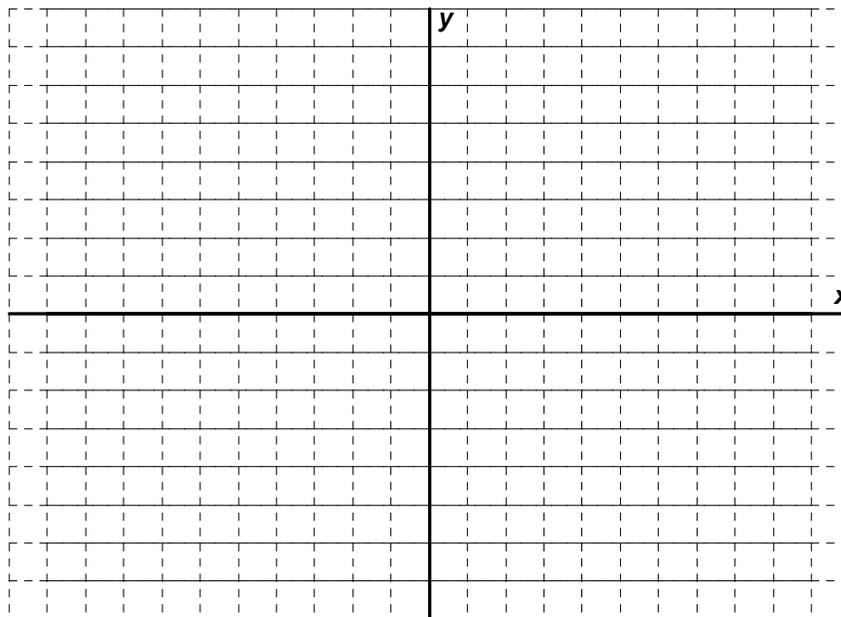
- a. Graph $h(x) = -2x^2 + 6x + 4$ and $k(x) = \frac{1}{2}x^2 - 2x + 3$ on the same grid. Use the table to list the y -intercept, x -intercepts, etc. for $k(x)$. Find and label the indicated points of intersection. Show your work in the workspace given below.

Points of Intersections

$$h(x) = k(x)$$

$$k(x) = \frac{1}{2}x^2 - 2x + 3$$

y-intercept	x-intercepts	Turning Point	Range	$y = a(x - h)^2 + k$



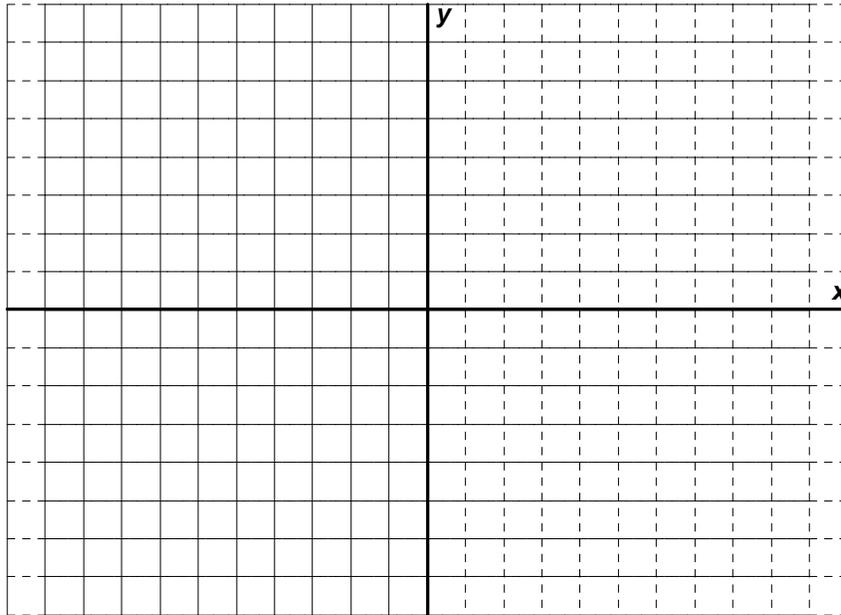
Workspace

For each of the following, give the x -values where:

b. $h(x) < k(x)$

$h(x) > k(x)$

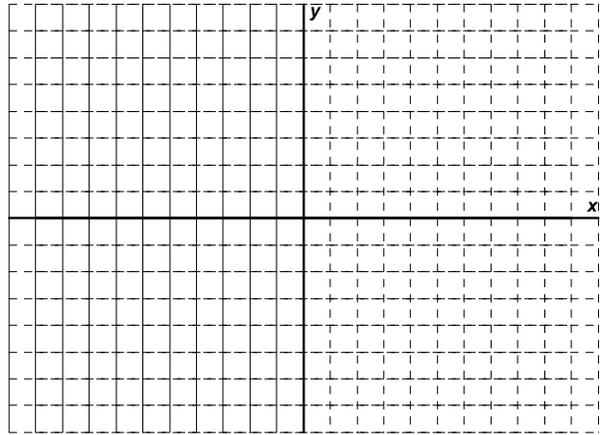
4. You may recall the inequality rule; “when you multiply or divide by a negative number, you switch the inequality sign.”
- a. Use a graphical technique to find the solution to: $-2x < 4$



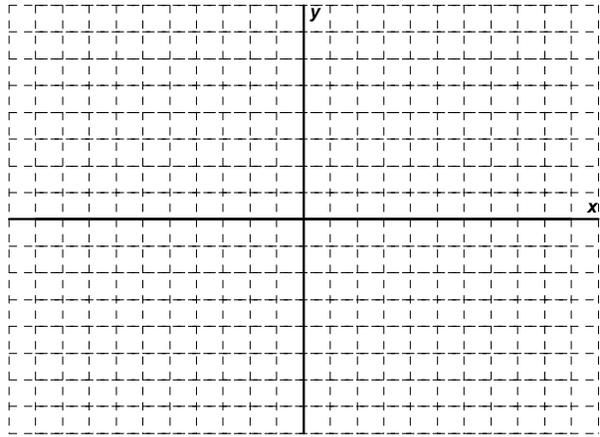
- b. Use addition (“on both sides”) to change $-2x < 4$ to an $ax < b$ form where $a > 0$ and b is any real number. Algebraically solve the new inequality.
- c. Explain, as you would explain it to a new learner, why the following statement is true: “When you multiply or divide by a negative number, you switch the inequality sign”

5. Solve each of the following inequalities, show your work and show the solution graphically.

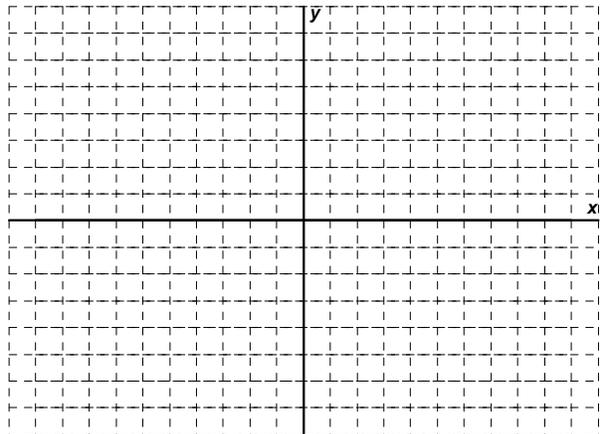
a. $5x + 4 > 13$



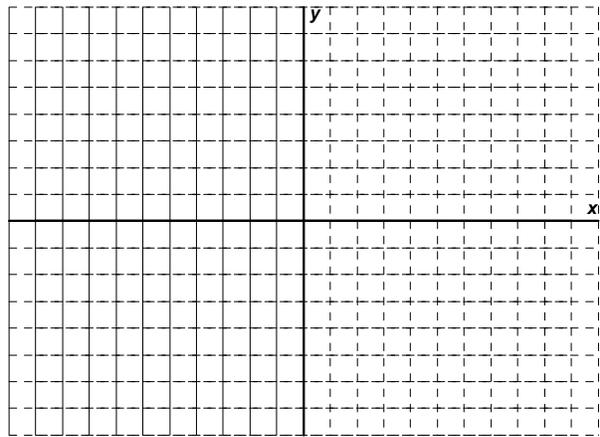
b. $5x + 4 \leq -6x + 7$



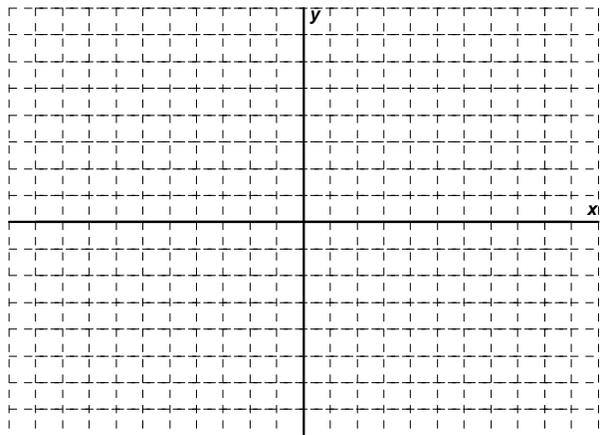
c. $3x^2 + 4x - 6 < -6x + 7$



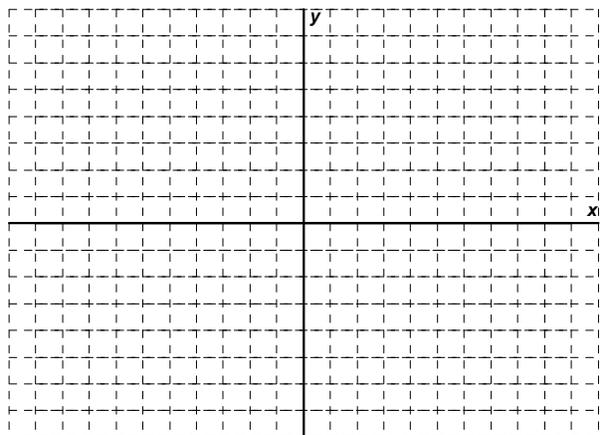
d. $3x^2 + 4x - 6 \geq -x^2 + 3x - 3$



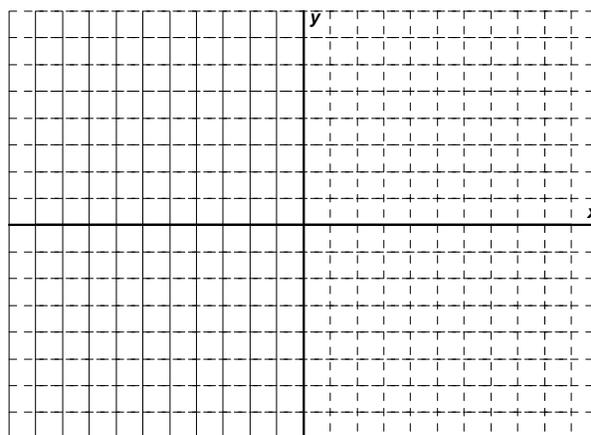
e. $2x^2 - 4x + 5 > -\frac{1}{2}x^2 + 3x + 6$



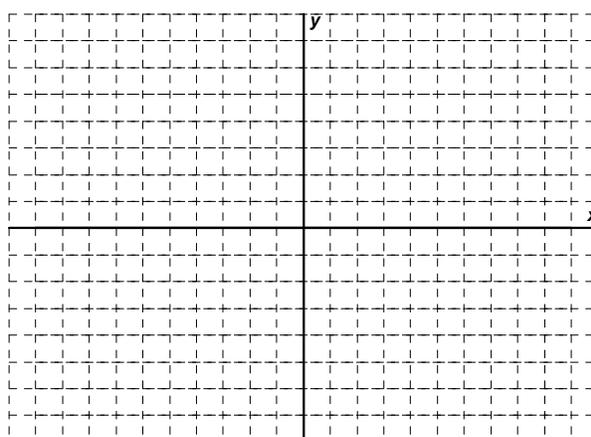
f. $x^2 + 2x + 2 \leq -x^2 - 2x$



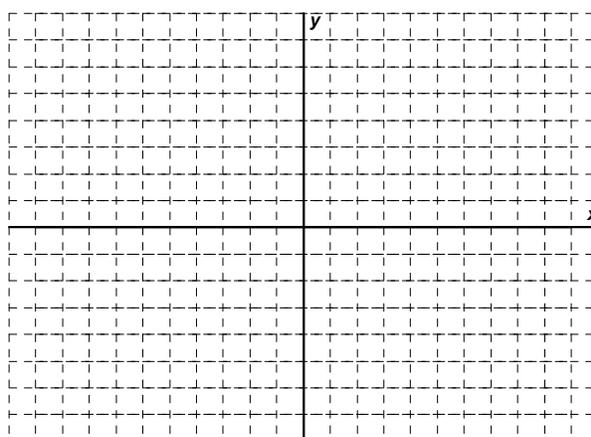
g. $x^2 + 2x + 2 \geq -x^2 - 2x$



h. $x^2 + 2x + 2 < -x^2 - 2x$



i. $x^2 + 2x + 2 > -x^2 - 2x$



Homework Questions 3.5

INEQUALITIES

1. Graph $y = -4$, $f(x) = 2x - 4$ and $g(x) = -x + 3$ on the same grid. Find all points of intersection and determine the x -values for each of the following. Show your work.
 - a. $f(x) < -4$ when: $f(x) > -4$ when:
 - b. $g(x) < -4$ when: $g(x) > -4$ when:
 - c. $f(x) < g(x)$ when: $f(x) > g(x)$ when:

2. Consider $f(x) = x^2 - 2x - 8$ and $g(x) = x^2 + x - 12$.
 - a. Determine the x -intercepts, the y -intercepts, the turning points and the ranges for $f(x)$ and $g(x)$. Show your work.
 - b. Graph $f(x) = x^2 - 2x - 8$ and $g(x) = x^2 + x - 12$ on the same grid.
 - c. Determine where $f(x) = x^2 - 2x - 8$ and $g(x) = x^2 + x - 12$ intersect. Show your work.
 - d. Give the x -values where $f(x) < g(x)$ and where $f(x) > g(x)$ when:

3. For each part; create an example of two linear functions, $f(x)$ and $g(x)$ that satisfy the given condition and answer the question. In each case, graph the functions, give all key points of each function and note where $f(x) = g(x)$, $f(x) < g(x)$ and $f(x) > g(x)$. Show your work.
 - a. $f(x)$ and $g(x)$ have one point of intersection. In general, what feature of $f(x)$ and $g(x)$ assures these two linear functions will intersect?
 - b. $f(x)$ and $g(x)$ do not intersect. In general, what feature of $f(x)$ and $g(x)$ assures these two linear functions will not intersect?

4. For each part; create an example of a linear function, $f(x)$, and a quadratic function $g(x)$ that satisfy the given condition (use $y = ax^2 + bx + c$, $y = a(x - h)^2 + k$ or the factored form of $g(x)$; whichever you prefer). In each case, graph the functions, give all key points of each function and note where $f(x) = g(x)$, $f(x) < g(x)$ and $f(x) > g(x)$. Show your work.
 - a. $f(x)$ and $g(x)$ have two points of intersection.
 - b. $f(x)$ and $g(x)$ have exactly one point of intersection.

- c. $f(x)$ and $g(x)$ do not intersect.
5. For each part; create an example of two quadratic functions $f(x)$ and $g(x)$ that satisfy the given condition (use $y = ax^2 + bx + c$, $y = a(x - h)^2 + k$ or factored forms; whichever you prefer). In each case, graph the functions, give all key points of each function and note where $f(x) = g(x)$, $f(x) < g(x)$ and $f(x) > g(x)$. Show your work.
- a. $f(x)$ and $g(x)$ have two points of intersection.
- b. $f(x)$ and $g(x)$ have exactly one point of intersection.
- c. $f(x)$ and $g(x)$ do not intersect.

CHAPTER 3 VOCABULARY AND REVIEW TOPICS

VOCABULARY

Activity Set 3.1

1. Domain (function)
2. Range (function)
3. Real Numbers
4. Rational Numbers
5. Irrational Numbers

Activity Set 3.2

6. White x -Strip
7. Opposite White x -Strip
8. White and Opposite White Edge Pieces
9. Black x -Square
10. Red x -Square
11. Polynomials
12. Coefficients
13. Leading Coefficient
14. Degree of a Polynomial
15. Parabolas
16. Quadratic Functions
17. Turning Points

Activity Set 3.3

18. Quadratic function general form

Homework 3.3

19. Concave Up
20. Concave Down

SKILLS AND CONCEPTS

Activity Set 3.1

- A. Graphing and modeling “all” of the points by extending function domains to all real numbers and finding non-integer data points on graphs.
- B. Using x as the independent variable and y as the dependent variable;
- C. Finding function ranges and describing subsets of real numbers using Number Line and Inequality notations.
- D. Working with extended tile figures sequences, graphs and the symbolic forms of absolute value functions.
- E. Writing absolute value functions as a split function with two linear components.

Homework 3.1

- F. Shifting and stretching of absolute value graphs

Activity Set 3.2

- G. Using a graphing calculator to display t-tables
- H. Using a graphing calculator to view function graphs
- I. Modeling, analyzing and sketching graphs corresponding to extended tiles sequences with x -square components.
- J. Modeling quadratic functions and analyzing their graphs (parabolas)
- K. Finding turning points for quadratic functions and connecting this idea to the range of the function.

Activity Set 3.3

- L. Finding important features of quadratic functions with algebra piece models
- M. Factoring quadratic functions with algebra pieces and edge sets
- N. Finding x -intercepts with algebra pieces and edge sets
- O. Connecting x -intercepts to the factored form of quadratic functions
- P. Using algebra pieces to determine where the graphs of two functions intersect

SKILLS AND CONCEPTS**Activity Set 3.4**

- Q. Working with square roots, taking the square root of “both sides” of an equation.
- R. Using algebra pieces to understand completing the square
- S. Using completing the square to find the quadratic formula.
- T. Use the quadratic formula to find x -intercepts.
- U. Shifting and stretching of quadratic function graphs
- V. Completing the square to find shifted forms of a quadratic function: $y = a(x - h)^2 + k$.

Activity Set 3.5

- W. Finding where collections of linear and quadratic graphs intersect.
- X. Solving function inequality statements through graphing.
- Y. Understanding function inequality rules.

CHAPTER 3 PRACTICE EXAM

1. Analyze the following extended sequence of tile figures, $y = f(x)$, with domain, \mathbf{R} and sketch figures corresponding to $x = -1.5$ and $x = 2.5$. What function is this?

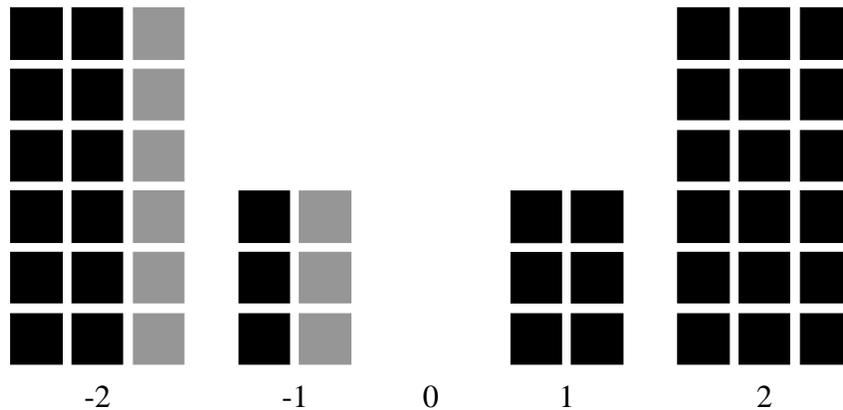
$y = f(x)$							
x	-2	-1.5	-1	0	1	2	2.5

2. Analyze and sketch the graph corresponding to the following extended sequence of tile figures with domain, \mathbf{R} . Give the symbolic rule for the function $y = g(x)$. What is the range of $y = g(x)$?

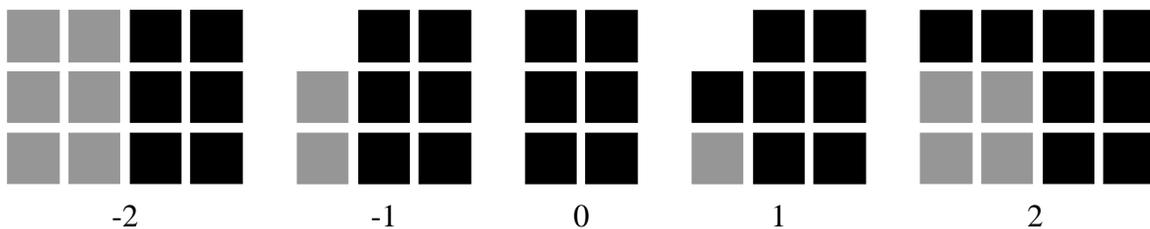
$y = g(x)$							
x	-3	-2	-1	0	1	2	3

3. Define an absolute value function that goes through the points $(0, -2)$ and $(2, 0)$ where one of these points is the “v” point on the function. Give both the absolute value function and the split linear function that matches the absolute value function. Is there more than one solution? If so, give both solutions.
4. Sketch the function $y = 4|x - 2| + 1$ on graph paper. Label the axes appropriately.
 - a. What is the range of $y = 4|x - 2| + 1$? Explain how you came to this answer.
 - b. What is the y -intercept of $y = 4|x - 2| + 1$?
 - c. What are the x -intercepts of $y = 4|x - 2| + 1$?
5. Write $y = 5|x - 2| + 9$ as a split function with two linear components where neither component uses absolute value notation.
6. If the two x -intercepts of a parabola are at $(2, 0)$ and $(4, 0)$, what is the x -value for the turning point of this parabola? Is there enough information to determine the y -value for the turning point of this parabola? If yes, sketch the parabola and give the y -value, if no, sketch two different examples of parabolas that satisfy these conditions.

7. For the following extended sequence of tile figures, with domain, \mathbf{R} : Analyze the extended sequence, sketch the x th figure that matches the figures and give the symbolic formula corresponding to the extended sequence of tile figures.

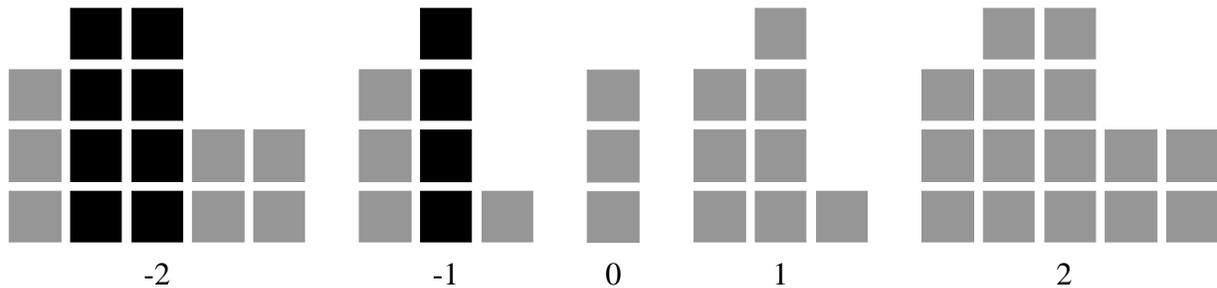


8. For the following extended sequence of tile figures, with domain, \mathbf{R} . Analyze the extended sequence, give the symbolic rule for the corresponding function, sketch the graph of the function; label the turning point, the x -intercepts and the y -intercept. State the range of the function



9. Sketch examples of a quadratic functions where $y = 2$:
- Occurs twice in the range of the function
 - Occurs once in the range of the function
 - Is not in the range of the function
- 10.
- Sketch figures for $x = 0, \pm 1$ and ± 2 for the extended tile sequence corresponding to $y = (x+3)(x-1)$.
 - Sketch the x th figure that visually matches your figures.
 - What minimal collection of algebra pieces does your x th figure reduce to?

11. Analyze the following extended sequence of tile figures, sketch the x th figure and give the symbolic formula for the sequence. Use algebra pieces to find the x -intercepts for the function. Show your algebra piece and corresponding symbolic work in a two column table. What are the y -intercept and the turning point for this function?



12. Use algebra pieces to factor $y = x^2 + x - 12$. Sketch the pieces and explain each step.
13. Factor each of the following quadratic functions and determine the x -intercepts of the function. You may wish to use algebra pieces as you determine the factors, but you do not need to sketch the algebra piece models or the graphs for these questions.

a. $y = x^2 + 6x + 5$

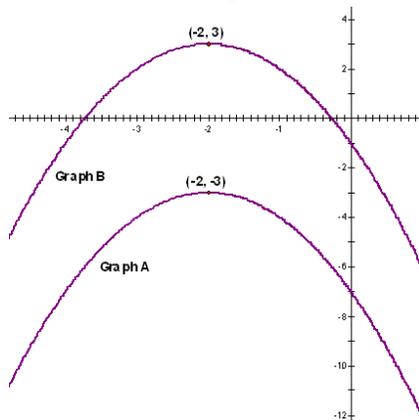
b. $y = x^2 - 25$

c. $y = 2x^2 - 6x - 20$

d. $y = x^2 + 8x + 16$

14. Use algebra pieces to find where $y = x^2 - 3x - 4$ and $y = -4x - 2$ intersect. Use two column tables to show your algebra piece work and corresponding symbolic work. Sketch both functions and label the key points on each function and the points of intersection that you found.
15. Determine where $y = x^2 - 6x + 5$ and $y = -x^2 + 4x + 5$ intersect. Sketch both functions and label the key points on each function and the points of intersection that you found. Show your work.
16. Give the equation of the quadratic function with x -intercepts $(-2, 0)$ and $(3, 0)$ and y -intercept $(0, 12)$. Graph the function; label all of the key points and find:
- The Turning Point
 - The Range
 - The Factored Form (if it exists)
 - The $y = a(x - h)^2 + k$ Form

17. Determine the equation of Graph A and the equation of Graph B. Describe each graph in terms of shifting $y = x^2$. Label all of the key points on each graph.



18. Analyze $y = -3x^2 - 12x - 9$ and find each of the following. Graph the function; label all of the key points.
- y-intercept
 - x-intercepts (if they exist)
 - Turning Point
 - Range
 - Factored Form (if it exists)
 - $y = a(x - h)^2 + k$ Form

19. Using differences; is this number pattern linear or quadratic? Find the function that matches this number pattern. Show your work.

x	-3	-2	-1	0	1	2	3
y	-11	-9	-7	-5	-3	-1	1

20. Using differences; is this number pattern linear or quadratic? Find the function that matches this number pattern. Show your work.

x	-3	-2	-1	0	1	2	3
y	22	9	2	1	6	17	34

21. Graph $f(x) = -4x + 3$ and $g(x) = 5x - 3$ on the same grid. Determine where $f(x) = g(x)$, $f(x) < g(x)$ and $f(x) > g(x)$. Show your work and show your solutions graphically.

22. Graph $f(x) = -x^2 + x + 18$ and $g(x) = 5x - 3$ on the same grid. Determine where $f(x) = g(x)$, $f(x) < g(x)$ and $f(x) > g(x)$. Show your work and show your solutions graphically.
23. Graph $f(x) = -x^2 + x + 18$ and $g(x) = x^2 - x - 6$ on the same grid. Determine where $f(x) = g(x)$, $f(x) < g(x)$ and $f(x) > g(x)$. Show your work and show your solutions graphically.
24. Create an example of two quadratic functions, $f(x)$ and $g(x)$, each of the form $y = a(x - h)^2 + k$ where $a < 0$ for $f(x)$ and $a > 0$ for $g(x)$ (the a s, are not necessarily the same) where $f(x)$ and $g(x)$ have no points of intersection. Sketch $f(x)$ and $g(x)$ together and label all key points.
25. Create an example of two quadratic functions, $f(x)$ and $g(x)$, each of the form $y = a(x - h)^2 + k$ where $a < 0$ for $f(x)$ and $a > 0$ for $g(x)$ (the a s, are not necessarily the same) where $f(x)$ and $g(x)$ have two points of intersection. Sketch $f(x)$ and $g(x)$ together and label all key points. Show your work for finding the points of intersection.

CHAPTER FOUR

POLYNOMIALS
AND
COMPLEX NUMBERS

Activity Set 4.1

INTRODUCTION TO HIGHER DEGREE POLYNOMIALS

PURPOSE

To learn to use graphing and algebra together to analyze higher degree polynomial functions. To use a graphing calculator to find local minimum and local maximum values of a polynomial function. To use a graphing calculator to find the x -intercepts of a polynomial function. To use x -intercepts to help factor polynomial functions.

MATERIALS

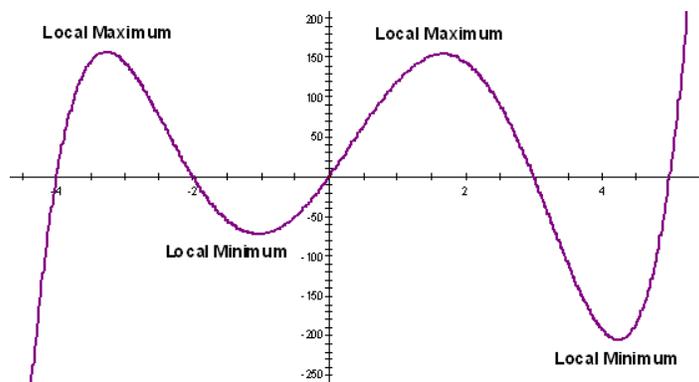
Graphing Calculator (required)

INTRODUCTION

We have studied degree one polynomials, linear functions, of the form $y = mx + b$ and degree two polynomials, quadratic functions, of the form $y = ax^2 + bx + c$. In this activity set we will look at higher degree polynomials such as cubic (degree three) and quartic (degree four) functions. Remember, a polynomial of degree n is a function of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$ where the coefficients (the numbers in front of the variables) a_0, \dots, a_n , ($a_n \neq 0$) are real numbers and the x powers (the x degrees) $1, \dots, n$ are positive counting numbers. The degree of a polynomial is its highest x power.

Local Minimums and Maximums

Many graphs have points that are lower than all of the other nearby points; such a point is called a Local Minimum. Points that are higher than all of the other nearby points are called Local Maximums.



If a point on a graph is lower than all of the other points on the graph (not just nearby points), then the point is an Absolute Minimum. Similarly, if a point on a graph is higher than all of the other points on the graph (not just nearby points), then the point is an Absolute Maximum. Absolute minimums and maximums are also considered local minimums and maximums.

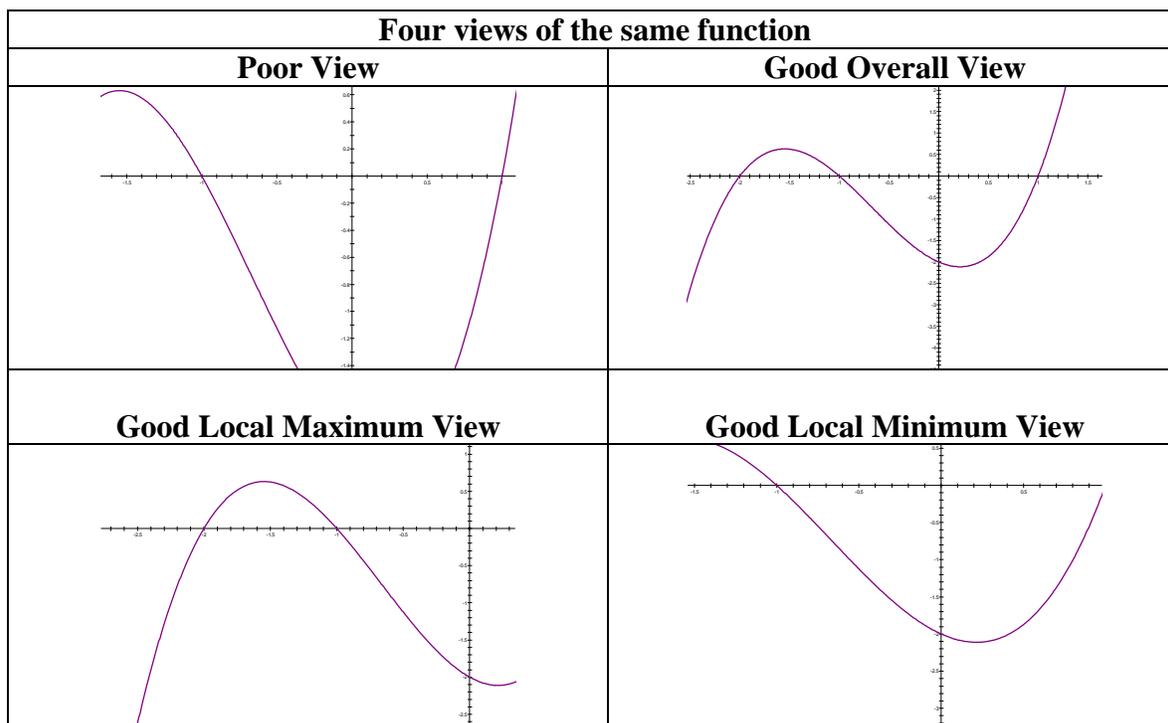
All quadratic functions have either one absolute minimum or one absolute maximum. Many graphs, however, bend back and forth creating local minimums and local maximums that are not the absolute lowest or absolute highest points on the graph. The above graph is a good example of a graph with local minimums, local maximums and no absolute minimums or absolute maximums (assuming the graph continues over a range of all real numbers).

TECHNOLOGY NOTES (graphing calculator models such as the TI-83 or 84 series)**Using a Graphing Calculator to find Local Minimum and Local Maximum Values*****Step One**

Enter the formula for your function in the graphing menu

Step Two

Set the x and y ranges under WINDOW and view your function using GRAPH. Adjust the x and y ranges as needed until your entire graph (or portion you are analyzing) shows clearly.

**Step Three**

- I. Select the CALC menu (usually 2nd CALC—CALC above TRACE). Select **Minimum** or **Maximum**.
- II. The calculator should show a screen with a cursor at a random spot on the graph. The graph will say: **Left Bound?** Hit ENTER to choose this location for the left side (close to, but to the left of) of the point you are seeking or use the arrow keys to move the cursor to a new spot before hitting ENTER. On some calculators you can enter the numerical value of your chosen **Left Bound?** at this stage.
- III. Once the **Left Bound?** is entered, the graph will show **Right Bound?** Repeat to select the right side (close to, but to the right of) of the point you are seeking to find.

* Most symbolic methods for determining local maximum and local minimum values are a topic covered in calculus courses and are beyond the scope of these materials.

IV. The screen will now show **Guess?** Hit ENTER to see the coordinates for the local maximum or minimum point you have been looking for.

Using a Graphing Calculator to find x -Intercepts

Step One

Enter the formula for your function in the graphing menu

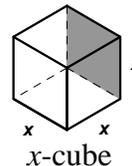
Step Two

Set the x and y ranges under WINDOW and view your function using GRAPH. Adjust the x and y ranges as needed until your graph shows clearly.

Step Three

- I. Select the CALC menu item **Zero**. (Note: The x -intercepts of functions are also referred to as “zeros” since the function evaluated at any x -intercept will equal 0.)
- II. Repeat the left and right bound steps outlined in finding local maximum and minimum values; this time, however, you are marking the left and right side of the x -intercept you are looking at.
- III. When the screen shows **Guess?** hit ENTER to see the coordinates for the x -intercept you are looking for.

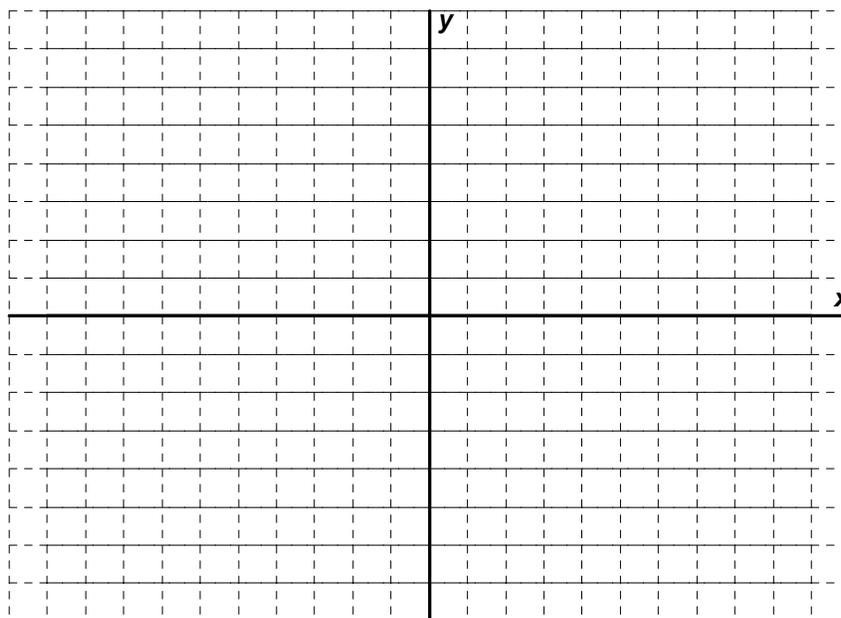
1. Let's explore the basic *cubic function*: $y = x^3$. Because this is a polynomial of degree three, we cannot model this function with algebra pieces. Algebra pieces only contain degree two ($\pm x$ -squares) and degree one ($\pm x$ -strip) pieces. To model a degree three component, we would need 3-D x -cubes.



- a. (*) To start your exploration of $y = x^3$, fill out the function values in the following table.

x	-2	-1.5	-1	-0.5	0	1	1.5	2	2.5
$y = x^3$									

- b. By looking at the symbolic form of the function, what is the range of $y = x^3$?
- c. Label the axes appropriately, graph coordinate pairs and sketch $y = x^3$. You may want to look at the graph of $y = x^3$ on a graphing calculator as well.



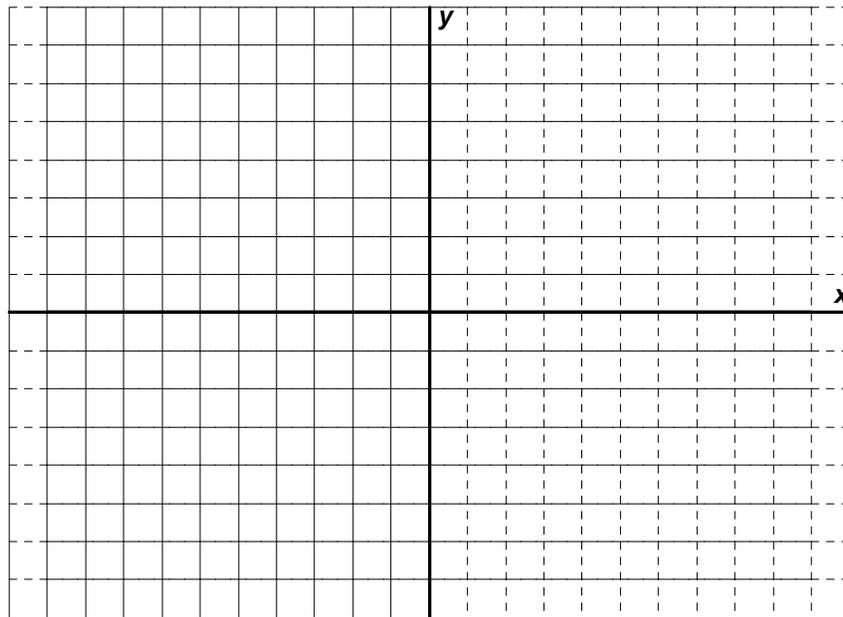
- d. How does your answer about range show visually on the graph?
- e. What is the y -intercept for $y = x^3$?

f. What is the x -intercept for $y = x^3$? How do you know there are not more?

g. Are there any turning points on the graph of $y = x^3$?

h. Are there any local maximums or minimums on the graph of $y = x^3$?

2. Graph $y = -x^3$.



a. How are $y = -x^3$ and $y = x^3$ similar?

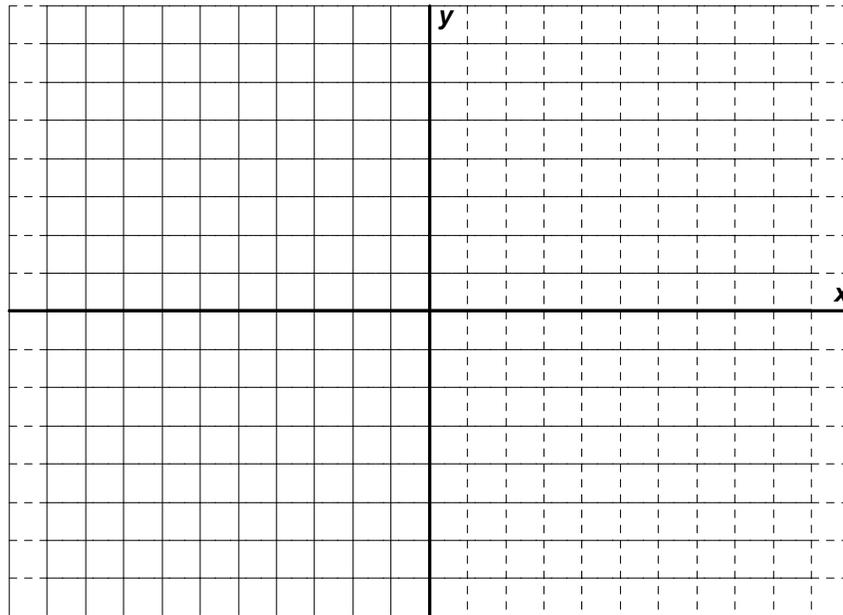
b. How does $y = -x^3$ differ from $y = x^3$?

3. Let's explore a different form of a cubic function: $f(x) = x(x-2)(x+2)$.
- Using the symbolic form, what are the x -intercepts of $f(x) = x(x-2)(x+2)$?
 - Using the symbolic form, what is the y -intercept of $f(x) = x(x-2)(x+2)$?
 - Use your graphing calculator to graph $f(x) = x(x-2)(x+2)$ and use the table feature to help fill out the function values in the following t-table. The blank cells are for part d).

	Local max			Local min					
x	-3	-2		-1	0	1		2	3
$f(x)$									

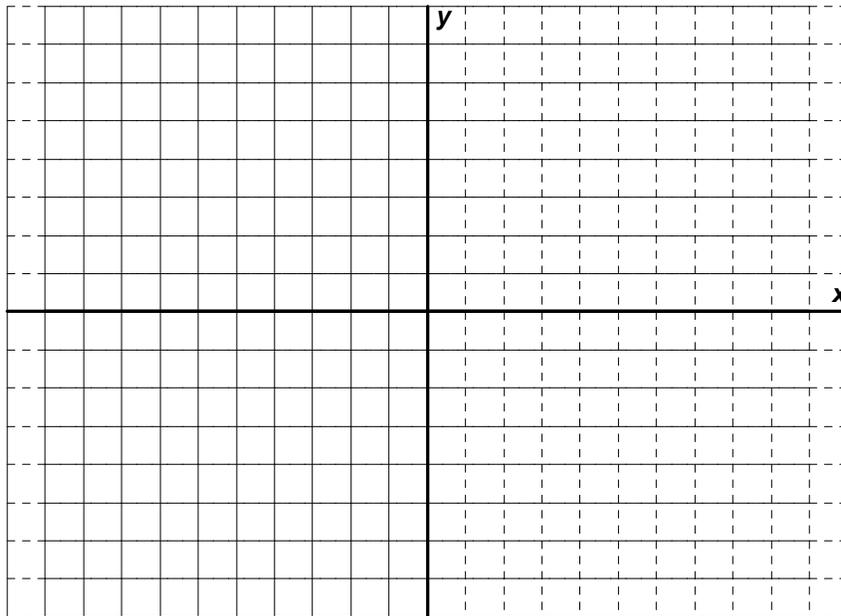
- Use your graphing calculator and find the locations of the local minimum and maximum points for $f(x) = x(x-2)(x+2)$. Use these points to fill out the appropriate columns in the t-table.

- e. Label the axes appropriately, graph coordinate pairs and sketch $f(x) = x(x-2)(x+2)$. You may want to look at the graph of $f(x) = x(x-2)(x+2)$ on a graphing calculator as well. Label the intercepts and the local maximum and local minimum points with their coordinates.



- f. What is the range of $f(x) = x(x-2)(x+2)$?
- g. For a quadratic function, the local minimum or local maximum is always on the vertical line that vertically divides the parabola symmetrically into two pieces. For quadratic functions with x -intercepts, the turning point is exactly half way between the two x -intercepts. Is this the case with cubic functions? Are the local minimum and maximum points halfway between the pairs of x -intercepts of $f(x) = x(x-2)(x+2)$?

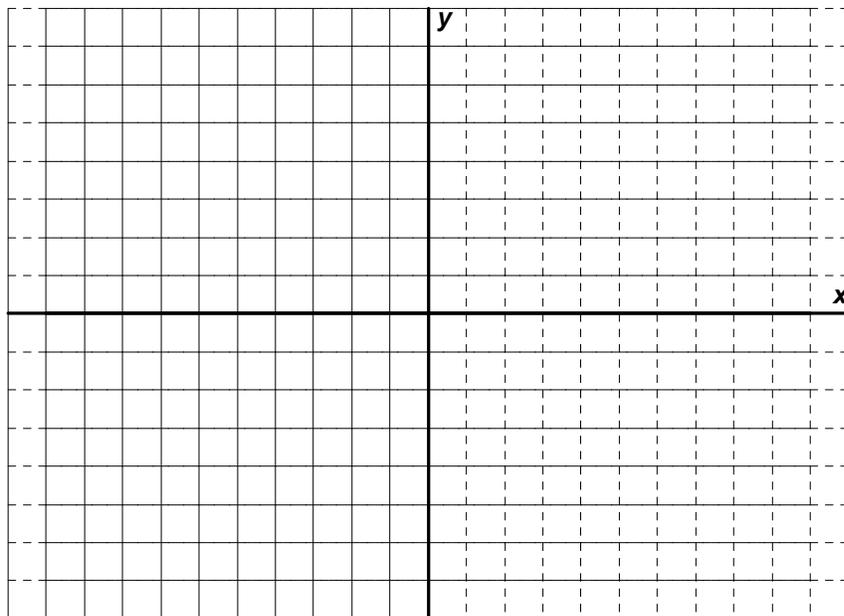
4. Determine the key points and graph $g(x) = -3x(x - 2)(x + 2)$. Label the key points including the local maximum and local minimum points.



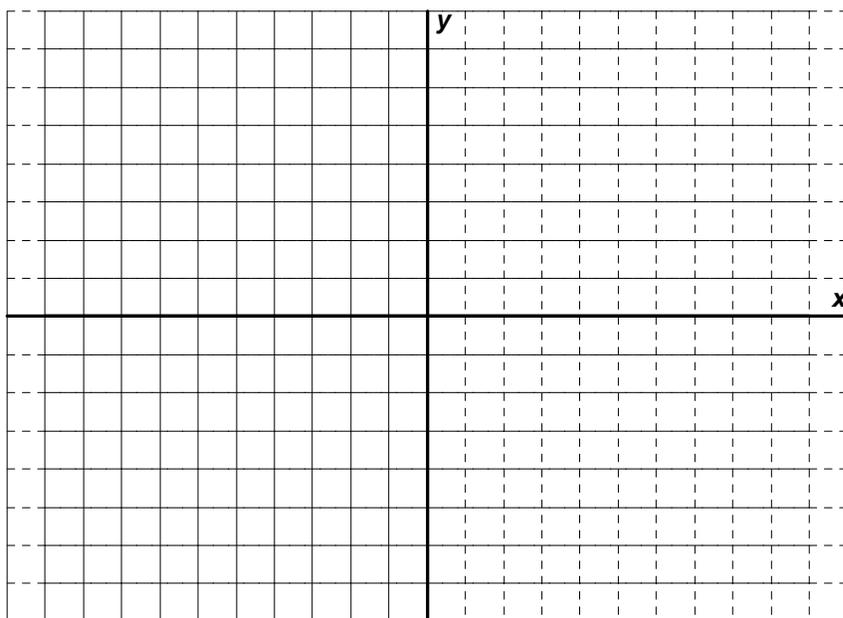
Workspace

- a. How does multiplying $f(x) = x(x - 2)(x + 2)$ by -3 to get $g(x)$ change $f(x)$? What changes? x -intercepts? The y -intercept? The function range? The values of the local maximum and local minimum points?

5. Let's explore the function $h(x) = x^3 + 2x^2 - 5x - 6$. The degree of this polynomial is three, so you might expect that the function looks somewhat similar to $f(x)$ and $g(x)$ from the previous questions (in its general shape).
- Use the graphing calculator CALC menu to find all three x-intercepts for $h(x) = x^3 + 2x^2 - 5x - 6$. List them.
 - Based on the answer to part a; what is your guess about the factored form of $h(x)$, $h(x) = (x+a)(x+b)(x+c)$ where $a, b, c \in \mathbf{R}$? Write $h(x)$ in this factored form and then graph both the original ($h(x) = x^3 + 2x^2 - 5x - 6$) and the factored form of $h(x)$ to double check your work (they should be the same graph!).
 - Find the local maximum and minimum values for $h(x) = x^3 + 2x^2 - 5x - 6$.
 - Graph $h(x) = x^3 + 2x^2 - 5x - 6$, label all of the key points.

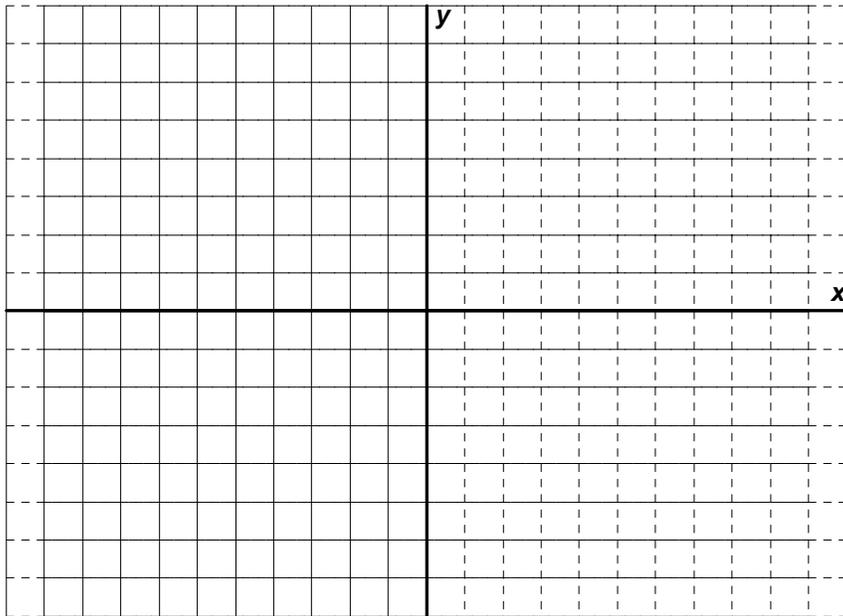


6. How does the graphing calculator technique for factoring work when the leading coefficient (the coefficient of the highest degree term) is not 1? Let's explore the function $k(x) = -3x^3 + 6x^2 + 9x$. Notice that all three terms of $k(x)$ have a factor of -3 and so it makes sense, to start, to factor this out: $k(x) = -3(x^3 - 2x^2 - 3x)$
- Use factoring, the graphing calculator CALC or the graphing calculator table menu to find all three x-intercepts for $k(x) = -3(x^3 - 2x^2 - 3x)$. List them.
 - Based on the answer to part a; what is your guess about the factored form of $k(x)$, $k(x) = -3(x+a)(x+b)(x+c)$ where $a, b, c \in \mathbf{R}$? Write $k(x)$ in this factored form and then graph both the original $k(x) = -3(x^3 - 2x^2 - 3x)$ and the factored form of $k(x)$ to double check your work.
 - Find the local maximum and minimum values for $k(x) = -3x^3 + 6x^2 - 9x$.
 - Graph $k(x) = -3x^3 + 6x^2 - 9x$, label all of the key points.



7. What is the general shape and range of higher degree polynomials? For example, how does the quadratic function $y = x^2$ behave? Graph $y = x^2$, $y = x^4$ and $y = x^6$ together. What observations do you note about this family of functions?

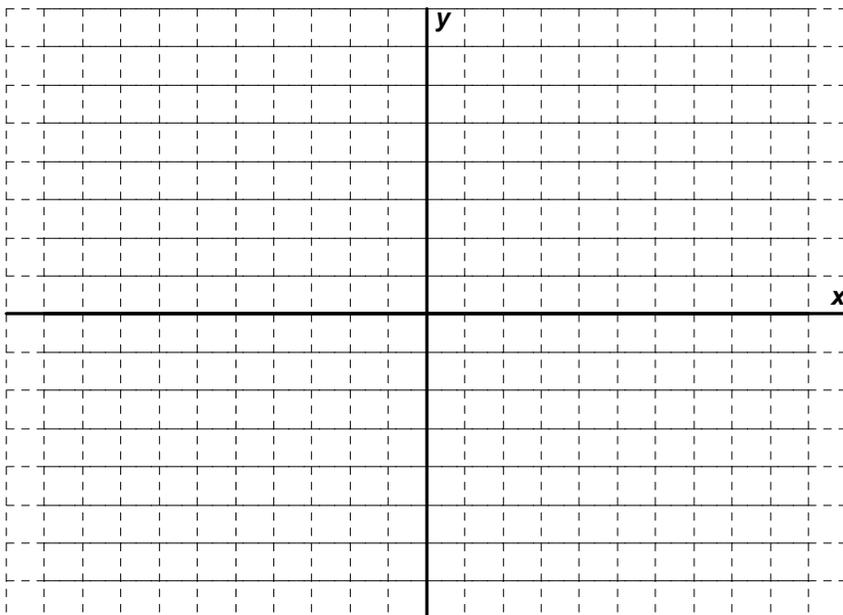
Workspace:



Observations:

8. Graph $y = x^3$, $y = x^5$ and $y = x^7$ together. What observations do you note about this family of functions?

Workspace:



Observations:

- 9.

10. If you had to guess at the shape and range of $y = x^{12}$, what would you say?

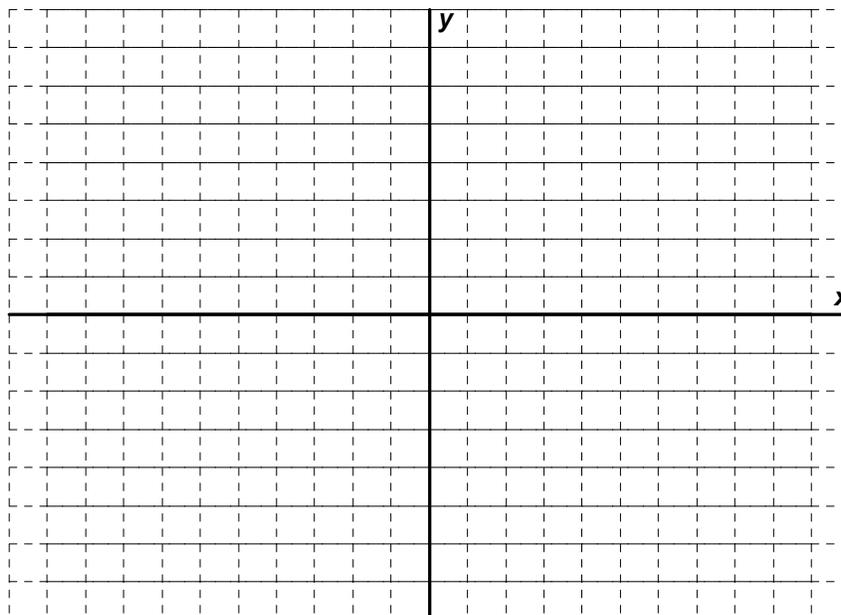
11. If you had to guess at the shape and range of $y = x^{17}$, what would you say?

12. The techniques we used to factor cubic polynomial functions can work for some factorable higher degree polynomials. For each of the following functions:

- i) Determine the x -intercepts and the y -intercept for the function.
- ii) Write out the factored form of the polynomial (if appropriate). Check your work.
- iii) Find the local maximum, the local minimum and the range of the polynomial function and graph the function. Label all key points

a. $y = \frac{1}{2}x^4 - 5x^2 + \frac{9}{2}$

Workspace:



Homework Questions 4.1

INTRODUCTION TO HIGHER DEGREE POLYNOMIALS

1. Let $a \in \mathbf{R}$.
 - a. What role does a play in $y = ax^n$ if n is odd? Give a few example sketches, the range and any local or absolute minimums or maximums of such polynomial functions. What are the x -intercepts and y -intercepts for these functions?
 - b. What role does a play in $y = ax^n$ if n is even? Give a few example sketches, the range and any local or absolute minimums or maximums of such polynomial functions. What are the x -intercepts and y -intercepts for these functions?
2. For each part; find the x -intercepts of the function using the CALC menu and factor the function. Graph the function, give the range and label all key points (intercepts, local and absolute minimums and maximums). Show your work.
 - a. $h(x) = x^4 + 9x^3 + 26x^2 + 24x$
 - b. $h(x) = 4x^3 + 16x^2 + 4x - 24$
3. For each part; create an example of a polynomial function that satisfies the given conditions, and graph the function; give the range and label all key points (intercepts, local and absolute minimums and maximums). Show your work.
 - a. The function has two local maximums; one of which is also an absolute maximum.
 - b. The function has two local minimums and two local maximums but no absolute minimum or absolute maximum.

Activity Set 4.2

SPECIAL POLYNOMIAL FACTORS, FOIL AND POLYNOMIAL DIVISION

PURPOSE

To learn about special polynomial factors such as the *Difference of Perfect Squares* and the *Sum and Difference of Perfect Cubes* and how to factor these special polynomials. To learn about Polynomial Division and how to use polynomial division with a linear factor to find other factors of a polynomial function.

MATERIALS

Graphing Calculator (required)

INTRODUCTION

Difference of Perfect Squares

We have seen that a quadratic such as $y = x^2 - 4$ can be factored as $y = x^2 - 4 = (x + 2)(x - 2)$. The form of $y = x^2 - 4$ is called a Difference of Perfect Squares since both x^2 and $4 = 2^2$ are perfect squares.

Difference of Perfect Cubes and Polynomial Division

Since the Difference of Perfect Squares has such a neat factorization, it makes sense to wonder about the Difference of Perfect Cubes. For example, does $y = x^3 - 1 = x^3 - 1^3$ also have a nice factorization? It is easy to see that since $y = x^3 - 1$ is just $y = x^3$ shifted down by 1, the only x -intercept for $y = x^3 - 1$ is $x = 1$. Thus, we know that $(x - 1)$ is a factor of $y = x^3 - 1$.

To find the other factor, we use a technique called Polynomial Division. Polynomial division works just like regular long division (but with polynomials); as long as you keep track of the variable degrees (like place values) as you divide. This is shown in the following example.

Polynomial Division Example $(x^3 - 1) \div (x - 1)$

$x^3 - 1$ has been rewritten as $x^3 + 0x^2 + 0x - 1$ to keep track of the variable degree place values.

$$\begin{array}{r}
 \overline{) x^3 + 0x^2 + 0x - 1} \\
 \underline{x^3 - x^2} \\
 0 + x^2 - 0x \\
 \underline{x^2 - x} \\
 0 + x - 1 \\
 \underline{x - 1} \\
 0
 \end{array}$$

Therefore $x^3 - 1 = (x - 1)(x^2 + x + 1)$

Substitution Factoring Techniques

Polynomial division is straightforward, but it takes time to write out the steps. For some polynomial functions we can adjust known facts to produce additional (and often faster) factorization techniques. In time you may be able to see the substitution “in your head” and just factor directly.

Example

We know that $y = x^2 + 4x + 4 = (x + 2)(x + 2)$, therefore, it also makes sense that $y = x^4 + 4x^2 + 4 = (x^2 + 2)(x^2 + 2)$.

The details of this example of Substitution Factoring are worked out in the following steps:

I. We see that $y = x^4 + 4x^2 + 4$ might follow an x^2 type of factoring pattern, but perhaps, we don't see the factors right off.

II. We substitute a new variable B^* for x^2 . Let $B = x^2$ and then

$$y = x^4 + 4x^2 + 4 = (x^2)^2 + 4x^2 + 4 = (B)^2 + 4B + 4 = B^2 + 4B + 4$$

III. Since we are familiar with the quadratic form $y = B^2 + 4B + 4$, we can factor:

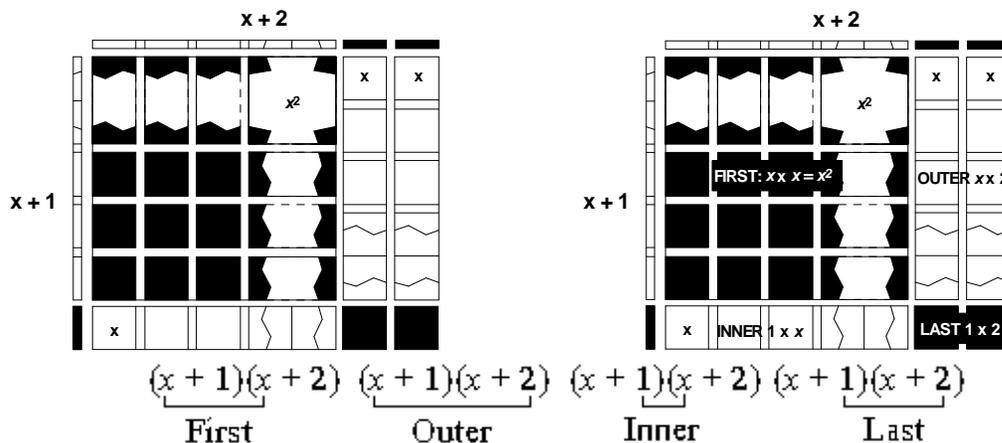
$$y = B^2 + 4B + 4 = (B + 2)(B + 2).$$

IV. B is just a placeholder, so it is important to substitute x^2 back in for B to yield

$$y = x^4 + 4x^2 + 4 = (x^2 + 2)(x^2 + 2).$$

FOIL

To successfully factor polynomial functions, we must be experts at re-multiplying the factors to check our work. In Chapter 3, we looked at many functions such as $y = (x + 1)(x + 2)$. Using algebra pieces (Figure 1), it is easy to see that $y = (x + 1)(x + 2) = x^2 + 2x + 1x + 2 = x^2 + 3x + 2$. Traditionally, two Binomials (polynomials with two terms) are multiplied out using the FOIL Method. FOIL stands for First – Outer – Inner – Last (Figure 2) and these are just the components of the four region algebra piece model we are already comfortable using.



* Letters other than B can also be used. The choice of the letter B is not important.

Multiplying Polynomials Beyond FOIL

To multiply polynomials, the key is to multiply each term in each polynomial factor by the each term in each other polynomial factor.

Suppose you wish to multiply out $(x+1)(x+2)(x+3)$. In general, we multiply polynomial factors two at a time, and then the result by the other terms.

Example: Polynomial Multiplication

$$(x+1)(x+2)(x+3) =$$

$$[(x+1)(x+2)] \times (x+3) = [[(x+1)(x+2)] \times x] + [[(x+1)(x+2)] \times 3].$$

Therefore:

$$(x+1)(x+2)(x+3) = [(x^2 + 3x + 2) \times x] + [(x^2 + 3x + 2) \times 3] =$$

$$(x^3 + 3x^2 + 2x) + (3x^2 + 9x + 6)$$

And altogether

$$(x+1)(x+2)(x+3) = x^3 + 6x^2 + 11x + 6$$

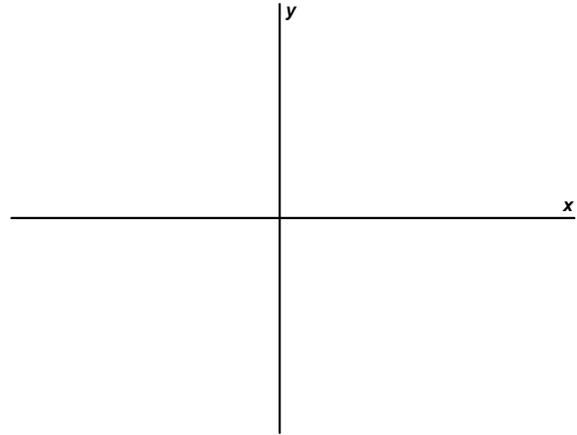
For the following questions; assume a is any real number.

1. Consider $y = x^2 - a^2$. Factor this polynomial to create a Difference of Perfect Squares Template.
2. (*) Using ideas from graphing, explain why $y = x^2 + a^2$ does not factor.
3. (*) Consider $y = x^3 - a^3$. Find one x -intercept and then use polynomial division to create a Difference of Perfect Cubes Template.
4. Consider $y = x^3 + a^3$. Use ideas from graphing to explain why $y = x^3 + a^3$ should factor. Find one x -intercept for $y = x^3 + a^3$ and then use polynomial division to create a Sum of Perfect Cubes Template.

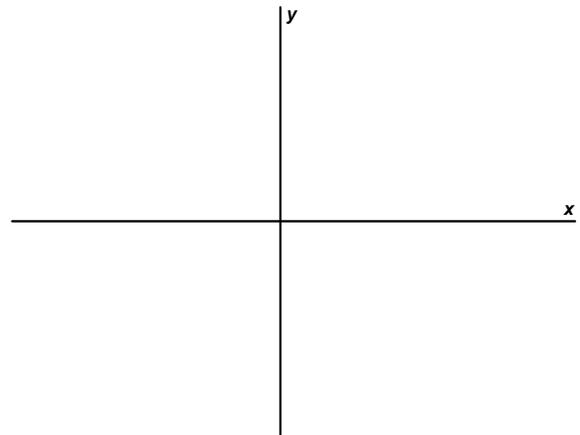
5. Consider $y = x^4 - a^4$. Use the placeholder technique ($A = x^2$) to factor $y = x^4 - a^4$ into two quadratic factors and then use the difference of perfect squares template to further factor $y = x^4 - a^4$ and create a Difference of Perfect Quartics Template.
6. Using ideas from graphing, explain why $y = x^n + a^n$ does not factor when n is even.
7. (*) Consider $y = x^5 - a^5$. Find one x -intercept and then use polynomial division to create a Difference of Perfect Quintics Template.
8. Based on your previous work; what do you think the factorization of $y = x^7 - a^7$ is?

Factor each of the following polynomial functions and sketch a simple graph of each function. Label the x -intercepts of the functions with their x -values. Show your work.

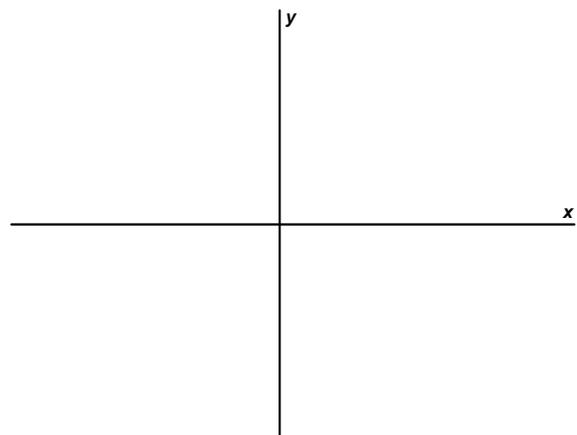
9. $y = x^3 + 64$



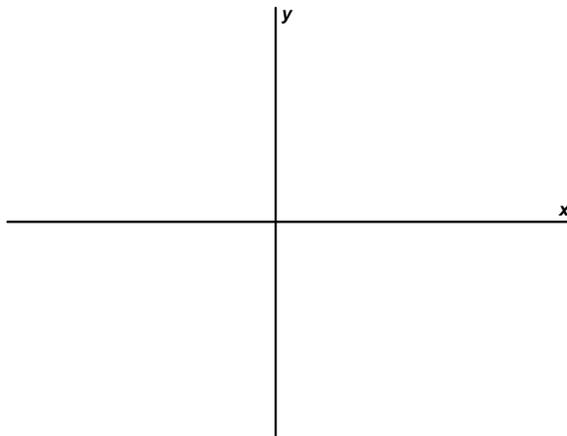
10. $y = x^6 - 1$ Hint: Think of $y = x^6 - 1$ as either $y = (x^3)^2 - (1^3)^2$ or $y = (x^2)^3 - (1^2)^3$



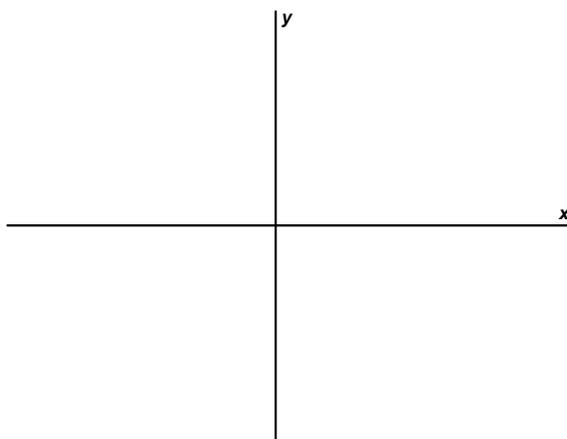
11. $y = x^4 - x^2 - 6$. Hint $3 = (\sqrt{3})^2$



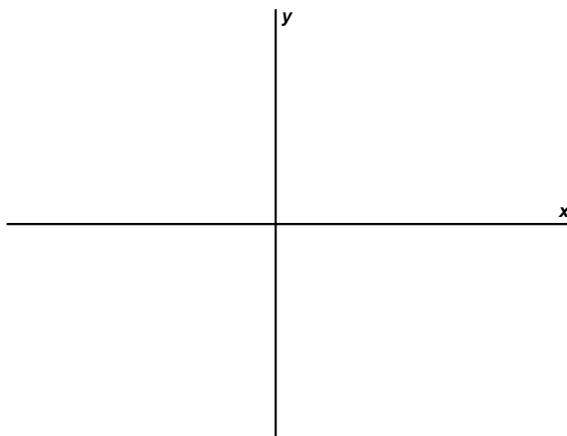
12. $y = x^4 - 16$



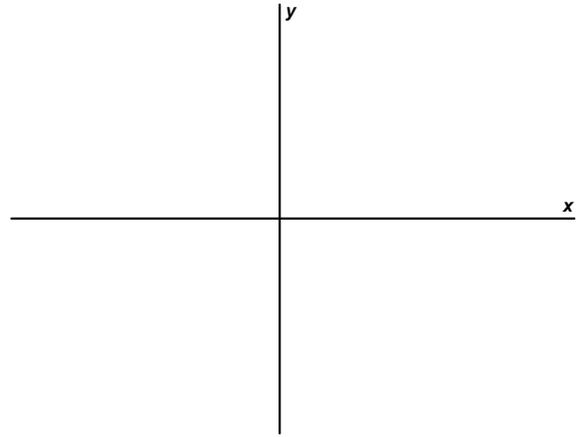
13. $y = x^5 - 32$



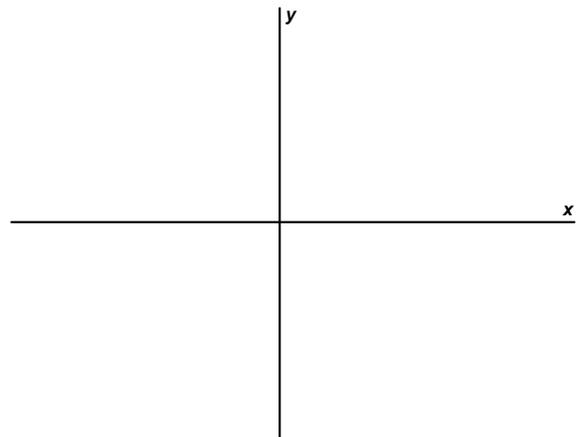
14. (*) $y = x^6 + 7x^3 + 12$



15. $y = x^3 - 2x^2 + x - 2$



16. $y = x^7 - x^4 + x^3 - 1$



Homework Questions 4.2

SPECIAL POLYNOMIALS FACTORS, FOIL AND POLYNOMIAL DIVISION

Factor each polynomial function and sketch a simple graph of each function. Label the x -intercepts of the functions with their x -values. Show your work.

1. $y = x^2 - 625$

2. $y = 8x^3 - 27$. Hint $8x^3 = (2x)^3$

3. $y = 16x^4 - 81$

4. $y = x^8 + 5x^4 + 6$

5. $y = x^5 + 32$

6. $y = 2x^{10} + 6x^5 + 4$

7. $y = 2x^3 + 4x^2 + x + 2$

8. $y = x^4 - 2x^3 + 2x^2 - 2x + 1$

Activity Set 4.3

INTRODUCTION TO COMPLEX NUMBERS

PURPOSE

To learn about the special number i and the set of complex numbers, \mathbf{C} . To learn how to model complex numbers and complex number operations (addition, subtraction, multiplication and division) using black, red, green and yellow tiles.

MATERIALS

Black, red, green and yellow tiles
and edge pieces



INTRODUCTION

Zeros of Polynomials

We have seen in Chapter 3 and in Activity Sets 4.1 and 4.2 that polynomials have less than or equal to the same number of x -intercepts as the degree of the polynomial. The word, x -intercept, is in fact a special example of the idea of a zero of a polynomial. A zero for a polynomial $p(x)$ is a number a such that $p(a) = 0$. The x -intercepts we have been looking at are, in fact, real polynomial zeros. We know these x -values are real numbers since the x -axis is a copy of the real number line, and any number a on the line is a real number. When solving an equation, we use the term root. Functions have zeros and equations have roots. For example, $p(x) = x - 1$ has one real zero of $x = 1$ and the equation $x - 1 = 0$ has one real root of $x = 1$. Thus the terms zero and root are often used to express the same idea.

You might have imagined that since every quadratic is a polynomial of degree two, such a quadratic, $q(x)$, ought to have some special set of two values, say a_1 and a_2 such that $q(a_1) = 0$ and $q(a_2) = 0$. In this case, a_1 and a_2 would each be a zero or root of $q(x)$. Similarly, you might have imagined that every polynomial of degree n ought to have a set of n polynomial roots.

In fact it is true that every polynomial of degree n has a set of n polynomial zeros^{*}, however, it is not always the case that these are *real zeros*. To find all of the zeros for a polynomial function, we need to extend the real number system to include a new number i . i is called the imaginary number and is an extremely special number in mathematics.

The imaginary number, i , is defined to equal $\sqrt{-1}$. That is: $i = \sqrt{-1}$. It is important to remember that $\sqrt{-1}$ makes no sense in the real number system and has no graphical counterpart in the real number system. The extension of the real numbers that includes i is called the Set of Complex Numbers, \mathbf{C} . The set of complex numbers is the set of all possible additive and multiplicative combinations of real numbers and the imaginary number, i as follows:

$$\mathbf{C} = \{a + bi \mid a, b \in \mathbf{R}, i = \sqrt{-1}\}$$

The set of all numbers of the form $a + bi$ where a, b are real numbers and $i = \sqrt{-1}$

* Note: Zeros such as $x = 1$ for $p(x) = (x - 1)^2$ are called zeros of multiplicity two or double roots. They count twice (as two zeros or roots) for $p(x)$.

Note that any real number a can be written as $a + 0i$, so the set of real numbers is a subset of the set of complex numbers. $\mathbf{R} \subset \mathbf{C}$.

In addition to the construction of a basic complex number, $a + bi$, the complex number system will only make sense as a useful number system for us if we can carry out all of the usual operations, $+$, $-$, \times and \div on this set. In fact, in order to effectively work with complex numbers, we need the set of complex numbers to be *closed* under each of these operations and we will explore this in the activities in this activity set.

Closure

A set of numbers is said to be Closed (have closure) under an operation if the combination of any two numbers from the set under the operation results in another number from the set.

Closure Example

It is easy to see the set of *Even Integers* is closed under the operation of addition. If we pick two arbitrary even integers, $2n$ and $2m$, $n, m \in \mathbf{Z}$, then $2n + 2m = 2(n + m)$ which is also an even integer since $n + m \in \mathbf{Z}$. On the other hand, the set of *Odd Integers* is not closed under the operation of addition. We can show this by merely exhibiting a *single example* that does not work. $3 + 5 = 8$. Since 8 is even, not odd, we can conclude the set of *Odd Integers* is not closed under the operation of addition.

Green and Yellow Tiles

We will be using green and yellow tiles to model the imaginary (often called the complex) part of complex numbers. One green tile has value $+i$ and one yellow tile has value $-i$. Green and yellow are opposite colors, just like black and red are opposite colors. All of the language that we used for black and red tile collections can be used for green and yellow tile collections.

One green tile = $+i$



One yellow tile = $-i$



Complex Number Examples Using Black, Red, Green and Yellow Tiles

  	  	  	  
$1 + 2i$	$-1 - 2i$	$0 + 3i$	$3 + 0i$
A complex number	A complex number	An all imaginary complex number	An all real complex number

Complex Number Terms

Complex Conjugate is defined in context; see activity 5

Green and Yellow Tiles and Edge Pieces

To denote $\pm i$ edge values, green and yellow edge pieces will be used in the same fashion that black and red edge pieces have been used. However, there is a new relationship between green and yellow tiles and black, red, green and yellow edge pieces; this will be explored in the activities in this activity set.

SKETCHING TIPS

Sketching Tiles for Addition and Subtraction

While sketching complex number addition and subtraction in this activity set, you may wish to denote black tiles by B, red tiles by R, green tiles by G and yellow tiles by Y rather than sketching and coloring square tiles.

Sketching Tiles for Multiplication and Division

While sketching complex number multiplication and division in this activity set, you will need to show the square shape of the tiles. Because there are four different tiles, it is easiest to sketch square tiles and label them with B, R, G and Y as needed.



Sketching Edges for Multiplication and Division

While sketching complex number multiplication and division in this activity set, label the edges clearly with the value of the edge component. For example:



1. Take a dozen or so black, red, green and yellow tiles, toss them on your table and fill out row 1 (Toss 1). Repeat as you fill out each row in the table; remove all matching black and red pairs and all matching green and yellow pairs before filling out the last column. As you toss, discuss your results. What observations can you make about black, red, green and yellow tiles and complex numbers? List all of your observations.

Collection	Number of Tiles					Complex Number
	Total	Black	Red	Green	Yellow	
Toss 1						
Toss 2						
Toss 3						
Toss 4						
Toss 5						
Toss 6						
Toss 7						
Toss 8						

Observations

Complex Number Addition and Subtraction

2. Use black, red, green and yellow tiles to model each of the following questions. Note that complex number addition can be done with any real number coefficients, but we will use only integer number coefficients to match our tile models.

a. $(*) (-2 + 3i) + (1 - 4i) = ?$

b. $(*) (-2 + 3i) - (1 - 4i) = ?$

c. $(*) (-2 + 3i) + (-2 - 3i) = ?$

d. (*) $(2 + 3i) - (2 - 3i) = ?$

e. Use the ideas you have developed from the tile models to answer this question:
What is the general “rule” for adding two complex numbers: $(a + bi) + (c + di)$ where $a, b, c, d \in \mathbf{R}$?

f. Use the ideas you have developed from the tile models to answer this question:
What is the general “rule” for subtracting two complex numbers: $(a + bi) - (c + di)$ where $a, b, c, d \in \mathbf{R}$?

3. Is the set of complex numbers closed under addition? Explain why or why not.

4. Is the set of complex numbers closed under subtraction? Explain why or why not.

Relationship between Black, Red, Green and Yellow Tiles

5. Before we can multiply complex numbers (such as $(2 + 3i) \times (1 - 4i)$) we need to understand the relationship between the basic components of the complex number set: 1 , -1 , i and $-i$. For each part, use your black, red, green and yellow tiles and edge pieces to create the given model and write in the tile name that matches the given edge dimensions. Complete the color and number statements.

Model
Example



Color Statement

black \times black = black

Number Statement

$1 \times 1 = 1$

	Model	Color Statement	Number Statement
a.		black \times green = _____	$1 \times i =$ _____
b.		black \times yellow = _____	$1 \times \bar{i} =$ _____
c.		red \times green = _____	$\bar{1} \times i =$ _____
d.		red \times yellow = _____	$\bar{1} \times \bar{i} =$ _____
e.		green \times green = _____	$i \times i =$ _____
f.		green \times yellow = _____	$i \times \bar{i} =$ _____
g.		yellow \times yellow = _____	$\bar{i} \times \bar{i} =$ _____

6. Summarize the multiplicative and color relationships between 1 , $\bar{1}$, i and \bar{i} by filling out the following two charts

\times	B	R	G	Y
B				
R				
G				
Y				

\times	1	$\bar{1}$	i	\bar{i}
1				
$\bar{1}$				
i				
\bar{i}				

Complex Number Multiplication

7. Use black, red, green and yellow tiles and edge pieces to model each of the following questions. Set the edge pieces up carefully and then fill in the tiles. Sketch your work here; label the edge piece values on your sketches. Simplify your final numerical solutions to an $(a + bi)$ form.

a. (*) $(2 + 3i) \times (1 + 4i) = ?$

$$(2 + 3i) \times (1 + 4i) = \underline{\hspace{2cm}}$$

b. $(-2 + 3i) \times (1 - 4i) = ?$

$$(-2 + 3i) \times (1 - 4i) = \underline{\hspace{2cm}}$$

c. $(2 + 3i) \times (2 - 3i) = ?$

$$(2 + 3i) \times (2 - 3i) = \underline{\hspace{2cm}}$$

d. Use the ideas you have developed from the tile models to answer this question:
What is the general “rule” for multiplying two complex numbers: $(a + bi) \times (c + di)$ where $a, b, c, d \in \mathbf{R}$? (Hint: FOIL). Give the rule in an $(a + bi)$ form.

8. Is the set of complex numbers closed under multiplication? Explain why or why not.

Complex Conjugate

9. The multiplication expression $(a + bi) \times (a - bi)$ where $a, b \in \mathbf{R}$ expresses a very special relationship. $(a - bi)$ is called the Complex Conjugate of $(a + bi)$.

What is $(a + bi) \times (a - bi) = ?$

Complex Number Division with Arrays

10. Just like with black and red tiles, we can use the array model for division to determine the solution to complex number division questions. For each of the following, set up the divisor as the left side of an array, fill in a model of the dividend as the array and then complete the division question by filling in the quotient as the top edge of the array. Sketch your models, clearly label the edges and give the final numerical solutions in an $(a + bi)$ form.

a. $6i \div 3 = ?$

$$6i \div 3 = \underline{\hspace{2cm}}$$

b. $6i \div -3 = ?$

$$6i \div -3 = \underline{\hspace{2cm}}$$

c. $6i \div 3i = ?$

$6i \div 3i = \underline{\hspace{2cm}}$

d. $\bar{6}i \div 3i = ?$

$\bar{6}i \div 3i = \underline{\hspace{2cm}}$

e. $\bar{6}i \div \bar{3}i = ?$

$\bar{6}i \div \bar{3}i = \underline{\hspace{2cm}}$

f. (*) $5 \div (1 + 2i) = ?$ Hint: Use zero pairs of green and yellow tiles to create a square array

$5 \div (1 + 2i) = \underline{\hspace{2cm}}$

- g. $5i \div (1 + 2i) = ?$ Hint: Use zero pairs of black and red tiles to create a rectangular array

$$5i \div (1 + 2i) = \underline{\hspace{2cm}}$$

- h. $(16 + 2i) \div (2 - 3i) = ?$ Hint: Use zero pairs of green and yellow tiles to create a rectangular array

$$(16 + 2i) \div (2 - 3i) = \underline{\hspace{2cm}}$$

Numerical Complex Number Division

11. Now think of $5i \div (1 + 2i)$ as $\frac{5i}{1 + 2i}$.

- a. What complex number can you multiply $(1 + 2i)$ by to create an all real number?

- b. Using the number you determined in part a), multiply both the numerator and the denominator of $\frac{5i}{1+2i}$ by this number to determine $5i \div (1 + 2i)$ numerically. Simplify to a final $(a + bi)$ form.

- c. In general: Think of $(a + bi) \div (c + di)$ as $\frac{a + bi}{c + di}$; $a, b, c, d \in \mathbf{R}$, not both c and $d = 0$.

What number can you multiply the numerator and the denominator by to “clear” the i from the denominator? Using this numerical division method, what is the simplified form of $\frac{a + bi}{c + di}$ with no complex numbers in the denominator? Give the final answer in an $(a + bi)$ form.

12. Is the set of complex numbers closed under division? Explain why or why not.

Homework Questions 4.3

INTRODUCTION TO COMPLEX NUMBERS

1. For the following addition and subtraction questions, use black, red, green and yellow tiles to model each addend (or subtrahend and minuend) and then the sum or difference.. Sketch and label your work. Be sure to carry out the whole operation; don't short cut by changing signs. In each case, give the completed addition or subtraction sentence.
 - a. $(2 + i) + (5 - 3i) = ?$
 - b. $(4 - 3i) - (5 - 3i) = ?$
 - c. $(-3 + i) - (-5 - 4i) = ?$
 - d. $(-3 + i) - (-5 + 3i) = ?$

2. We know with integers that “you can change subtracting a negative to adding a positive.” Is there a similar relationship with complex numbers? Use tile models and your work in question 1 to support your answer.

3. For the following multiplication questions, use black, red, green and yellow tiles and edge pieces to model a sequence of minimal arrays showing the multiplication steps. Sketch and label your work; label the net values of each edge set for each array and also the net value of the product on the last array. Briefly explain each step. Identify factors and products. In each case, give the completed multiplication sentence the array and edge sets shows.
 - a. $(-3 + i) \times (-1 - 4i) = ?$
 - b. $(-3 - i) \times (2 + 3i) = ?$

4. For the following division questions, use black, red, green and yellow tiles and edge pieces to model a sequence of minimal arrays showing the division steps. Sketch and label your work; label the net values of each edge set for each array and also the net value of the quotient in the last step. Briefly explain each step. Identify dividend, divisor and quotient. In each case, give the completed division sentence the array and edge sets shows as well as the corresponding multiplication sentence the array and edge sets could show.
 - a. $(6 + 2i) \div 2 = ?$
 - b. $(6 + 2i) \div 2i = ?$
 - c. $-13 \div (2 + 3i) = ?$

5. Use the complex conjugate numerical division shortcut to find the quotient for each part in question 4. Show your work.

Activity Set 4.4

WORKING WITH COMPLEX NUMBERS AND POLYNOMIALS ROOTS

PURPOSE

To learn about using complex numbers to work with square roots of negative numbers. To find all the polynomial roots of various quadratic, cubic, quartic and quintic polynomials and relate these roots to the polynomial factors and the graphs of the polynomial functions.

MATERIALS

Graphing Calculator (required)

INTRODUCTION

Working with Square Roots

Now that we have moved to working with complex numbers, we have a new ability—we can take the square root of a negative number. It is important to keep in mind that the square root of a negative number will be a complex number, not a real number.

Before we give an example, let's review a few ideas about square roots. One useful feature of square roots is this: As long as the individual factors make sense, we can split *the square root of a product* into *the product of square roots*. That means

$$\sqrt{A \times B} = \sqrt{A} \times \sqrt{B} \text{ as long as both } \sqrt{A} \text{ and } \sqrt{B} \text{ are defined.}$$

Similarly, *the square root of a quotient* is *the quotient of square roots*, as long as both the numerator and the denominators make sense, and, of course, as long as the denominator is not zero.

$$\sqrt{\frac{A}{B}} = \frac{\sqrt{A}}{\sqrt{B}} \text{ as long as both } \sqrt{A} \text{ and } \sqrt{B} \text{ are defined and } B, \sqrt{B} \neq 0$$

Square Root Examples

$$\sqrt{44} = \sqrt{4 \times 11} = \sqrt{2^2 \times 11} = \sqrt{2^2} \times \sqrt{11} = 2 \times \sqrt{11} = 2\sqrt{11}$$

$$\sqrt{\frac{108}{25}} = \sqrt{\frac{3 \times 6^2}{5^2}} = \frac{\sqrt{3 \times 6^2}}{\sqrt{5^2}} = \frac{\sqrt{3} \times \sqrt{6^2}}{\sqrt{5^2}} = \frac{\sqrt{3} \times 6}{5} = \frac{6\sqrt{3}}{5}$$

Note: Expressions such as $\sqrt{3} \times 6$ are usually written as $6\sqrt{3}$ to avoid the confusion of having $\sqrt{36}$ look like $\sqrt{36}$

Square Roots of Negative Numbers

Taking the square root of negative numbers involves the same ideas as the above examples of working with square roots, and additionally, uses the fact that $i = \sqrt{-1}$. The use of $i = \sqrt{-1}$ is shown in each of the following examples.

Complex Number Square Root Examples

$$\sqrt{-36} = \sqrt{-1 \times 6^2} = \sqrt{-1} \times \sqrt{6^2} = i \times 6 = 6i$$

$$\sqrt{-44} = \sqrt{-1 \times 4 \times 11} = \sqrt{-1 \times 2^2 \times 11} = \sqrt{-1} \times \sqrt{2^2} \times \sqrt{11} = i \times 2 \times \sqrt{11} = 2i\sqrt{11}$$

$$\sqrt{\frac{-36}{25}} = \frac{\sqrt{-1 \times 6^2}}{\sqrt{5^2}} = \frac{\sqrt{-1} \times \sqrt{6^2}}{\sqrt{5^2}} = \frac{6i}{5}$$

$$\sqrt{\frac{-108}{25}} = \sqrt{\frac{-1 \times 3 \times 6^2}{5^2}} = \frac{\sqrt{-1 \times 3 \times 6^2}}{\sqrt{5^2}} = \frac{\sqrt{-1} \times \sqrt{3} \times \sqrt{6^2}}{\sqrt{5^2}} = \frac{i \times \sqrt{3} \times 6}{5} = \frac{6i\sqrt{3}}{5}$$

Writing numbers using the square root notation (the radical notation) instead of a decimal calculator approximation is called using the Radical Form of a number.

Using Complex Numbers and Square Roots in Equations—Example

Suppose you wished to solve $x^2 = -4$. To solve $x^2 = -4$ you would just take the square root of both sides, and the same idea works here. The difference between $x^2 = -4$ and $x^2 = -4$ is that $x^2 = -4$ is solved using real numbers and $x^2 = -4$ is solved using complex numbers.

$$x^2 = -4; \quad x = \pm\sqrt{4} = \pm 2 \quad \text{and} \quad x^2 = -4; \quad x = \pm\sqrt{-4} = \pm 2i$$

Complex Number Factors Example

For a polynomial function of degree three, $p(x)$, with leading coefficient $a = 1$, if the polynomial roots for $p(x)$ are $x = 1$ and $x = \pm 2i$, then $p(x) = (x - 1)(x + 2i)(x - 2i)$.

Notice that since $(x + 2i)(x - 2i)$ is the factored form of the difference of perfect squares

$$p(x) = (x - 1)(x + 2i)(x - 2i) = (x - 1)(x^2 - (2i)^2) = (x - 1)(x^2 + 4) = x^3 - x^2 + 4x - 4$$

In this case $p(x)$ has all real number coefficients.

On the other hand, if $f(x)$ is a degree three polynomial with leading coefficient $a = 1$ and the polynomial roots for $f(x)$ are $x = 1$, $x = 2i$ and $x = -3i$ then

$$\begin{aligned} f(x) &= (x - 1)(x - 2i)(x + 3i) = (x - 1)(x^2 + ix + 6) = \\ &= x^3 + ix^2 + 6x - x^2 - ix - 6 = x^3 + (-1 + i)x^2 + (6 - i)x - 6 \end{aligned}$$

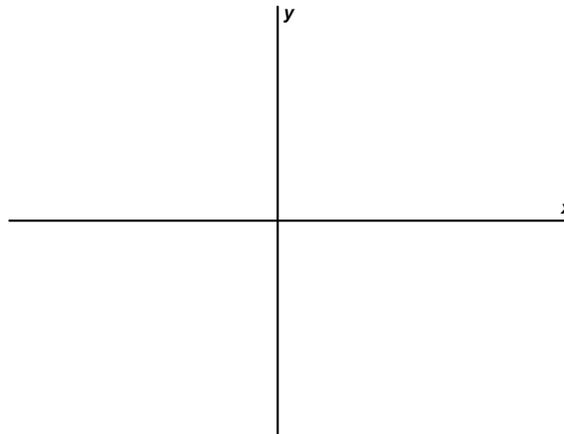
In this case $p(x)$ have complex number coefficients.

- ✓ For this course we will only work with polynomials with all real number coefficients that we can also graph.

1. A certain quadratic has a factored form of: $q(x) = (x - i)(x + i)$.

a. Multiply $q(x)$ out to its $y = ax^2 + bx + c$ form.

b. Graph $q(x)$; find and label all of the key points for $q(x)$; including any minimum or maximum values.

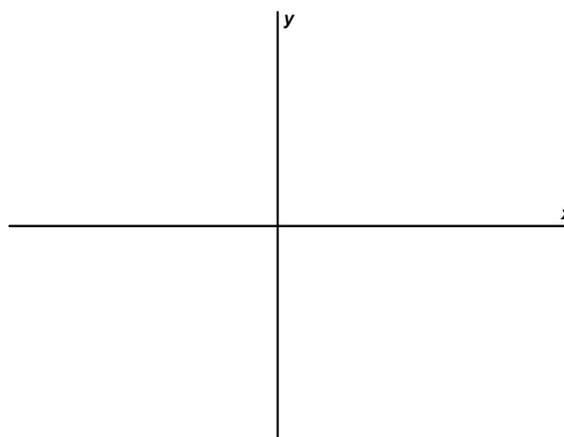


c. How do the factors of $q(x)$ relate to the graph of $q(x)$?

2. (*) $p(x)$ has a factored form of: $p(x) = (x - 2)(x - 3i)(x + 3i)$.

a. Multiply $p(x)$ out to its $y = ax^3 + bx^2 + cx + d$ form.

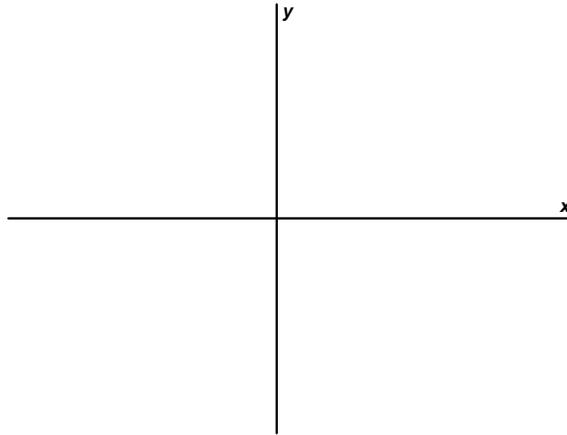
b. Graph $p(x)$; find and label all of the key points for $p(x)$; including any minimum or maximum values.



c. How do the factors of $p(x)$ relate to the graph of $p(x)$?

3. Consider $y = (x+1)^2 + 1$

a. Graph $y = (x+1)^2 + 1$



b. What is the y-intercept for $y = (x+1)^2 + 1$? Label it on your graph.

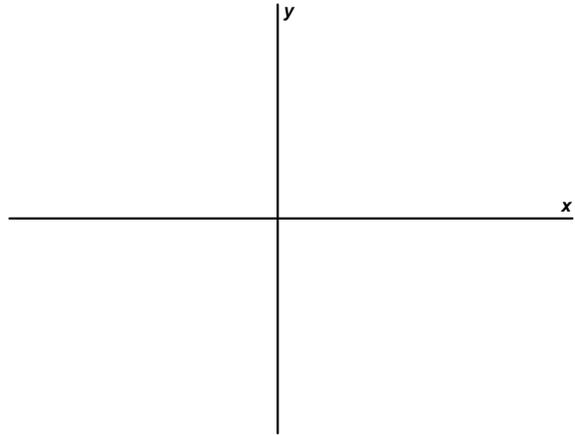
c. Are there any x-intercepts for $y = (x+1)^2 + 1$? How does this relate to the graph of the function?

d. Does $y = (x+1)^2 + 1$ have any real roots? How does this relate to the graph of the function?

e. Does $y = (x+1)^2 + 1$ have any complex roots? Set $(x+1)^2 + 1 = 0$ and solve for x . Note—you can set the equation up, solve for $(x+1)^2$ and then take the square root of both sides.

For each of the following, determine the key points that you need to graph the function and graph the function. Answer the given questions. To find the polynomial roots use any method—solving equations, factoring, graphing calculator or the quadratic formula. For each function, the number of polynomial roots (real and complex combined) is the same as the degree of the function and you should have the tools to find each of these polynomial roots.

4. $y = (x-3)^2 - 2$

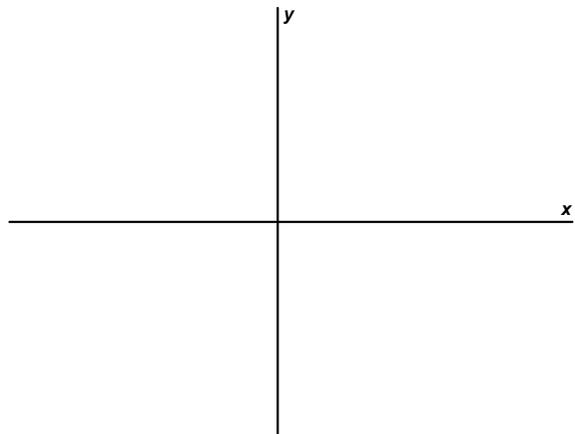
Workspace

x values for roots are: _____

real roots: _____ # complex roots: _____

Graph crosses x -axis _____ times

5. (*) $y = x^2 + 4x + 7$

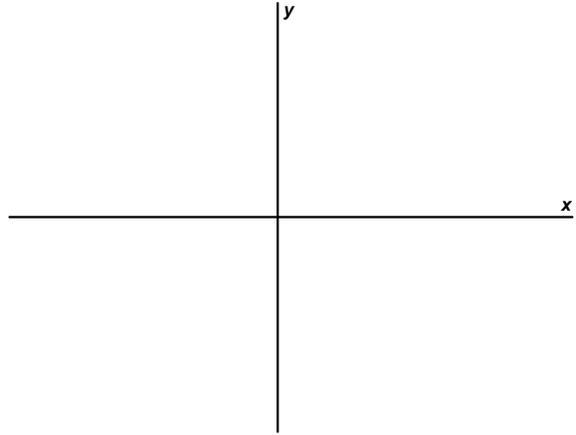
Workspace

x values for roots are: _____

real roots: _____ # complex roots: _____

Graph crosses x -axis _____ times

6. $y = x^2 + 6x + 9$

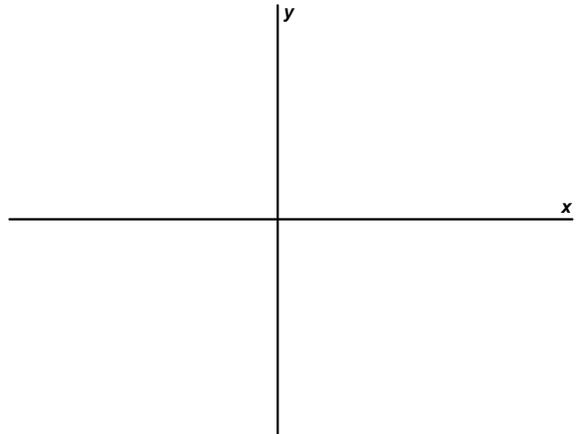
Workspace

x values for roots are: _____

real roots: _____ # complex roots: _____

Graph crosses x -axis _____ times

7. $y = x^2 + 6x + 5$

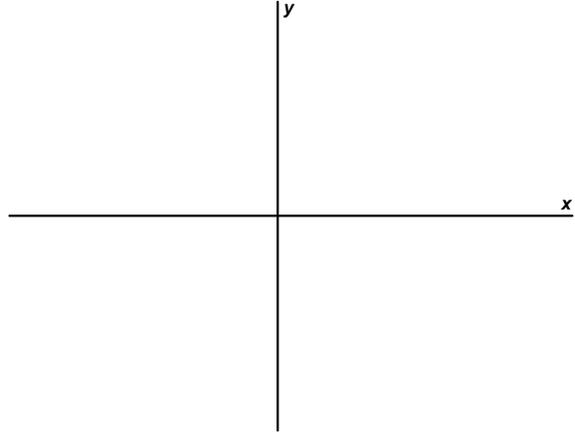
Workspace

x values for roots are: _____

real roots: _____ # complex roots: _____

Graph crosses x -axis _____ times

8. $y = x^2 + 4$

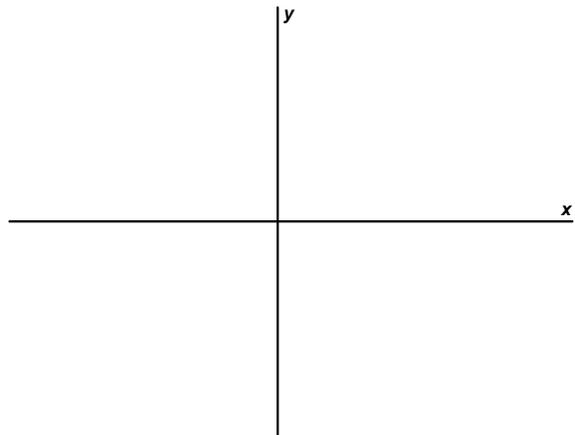
Workspace

x values for roots are: _____

real roots: _____ # complex roots: _____

Graph crosses x -axis _____ times

9. $y = x^2 - 5x + 7$

Workspace

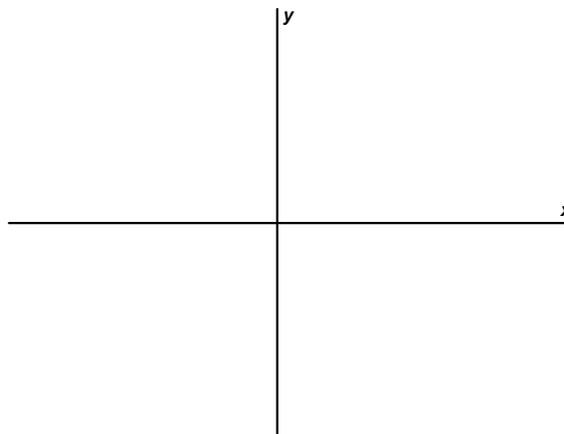
x values for roots are: _____

real roots: _____ # complex roots: _____

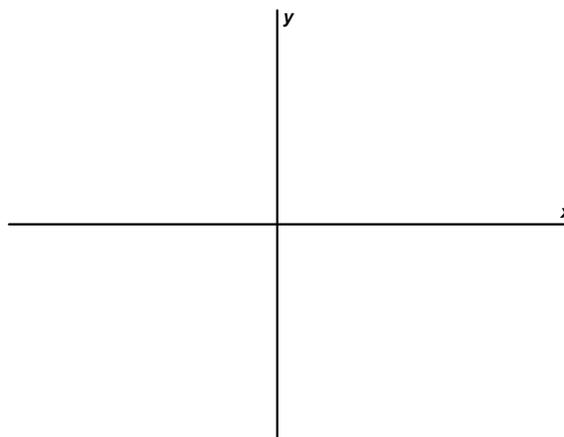
Graph crosses x -axis _____ times

10. For each of the following, give the symbolic rule for a quadratic function that satisfies the given conditions and sketch the quadratic function.

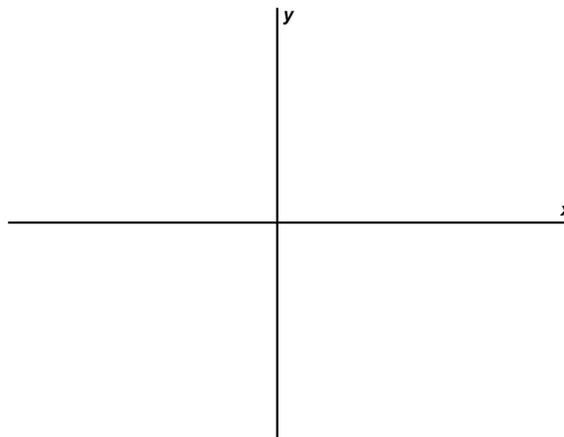
- a. The quadratic function has two different real roots and a positive leading coefficient.



- b. The quadratic function has a double real root and a negative leading coefficient.



- c. The quadratic function has two different complex roots and all the coefficients (in the $y = ax^2 + bx + c$ form) are real numbers.



Homework Questions 4.4

WORKING WITH COMPLEX NUMBERS AND POLYNOMIAL ROOTS

For each of the following, determine the key points that you need to graph the function and graph the function. Answer the given questions. To find the roots use any method—solving equations, factoring, graphing calculator or the quadratic formula. Carefully show and describe all of your work.

1. $y = x^3 + 8$. Hint: Factor first then use the quadratic formula on the quadratic component.
 - a. The x values of the roots of this function are:
 - b. The factored form of this function is:
 - c. This function has ____ # real roots and ____ # complex roots.
 - d. The graph of this function crosses the x -axis ____ times

2. $y = x^4 - 16$. Hint: Factor first then use the quadratic formula on the quadratic component.
 - a. The x values of the roots of this function are:
 - b. The factored form of this function is:
 - c. This function has ____ # real roots and ____ # complex roots.
 - d. The graph of this function crosses the x -axis ____ times

3. $y = x^5 + x^4 + 2x^3 + 2x^2 - 3x - 3$. Hint: Find one root using your calculator, use polynomial division, factor the resulting quintic into two quadratic factors and then deal with each of the quadratic factors.
 - a. The x values of the roots of this function are:
 - b. The factored form of this function is:
 - c. This function has ____ # real roots and ____ # complex roots.
 - d. The graph of this function crosses the x -axis ____ times

CHAPTER 4 VOCABULARY AND REVIEW TOPICS

VOCABULARY

Activity Set 4.1

1. Cubic Polynomial
2. Quartic Polynomial
3. Quintic Polynomial
4. Local Minimum
5. Local Maximum
6. Absolute Minimum
7. Absolute Maximum

Activity Set 4.2

8. Difference of Perfect Squares
9. Difference of Perfect Cubes

Activity Set 4.3

10. Polynomial Root
11. Real Polynomial Root
12. Complex Polynomial Root
13. Closure
14. Green and yellow tiles
15. Complex Conjugate

SKILLS AND CONCEPTS

Activity Set 4.1

- A. To learn to use graphing and algebra together to analyze higher degree polynomial functions.
- B. To use a graphing calculator to find local minimum and local maximum values of a polynomial function.
- C. To use a graphing calculator to find the x -intercepts of a polynomial function.
- D. To use x -intercepts to help factor polynomial functions.

Activity Set 4.2

- E. Working with the Difference of Perfect Squares
- F. Working with the Sum and Difference of Perfect Cubes
- G. Factoring special polynomials
- H. Polynomial Division
- I. Using polynomial division with a factor to find other factors of a polynomial function

Activity Set 4.3

- J. The special number $i = \sqrt{-1}$
- K. The set of Complex Numbers
- L. Modeling Complex Number Operations: $+$, $-$, \times and \div
- M. Using complex conjugates for Complex Number division

Activity Set 4.4

- N. Properties of square roots
- O. Using complex numbers to work with square roots of negative numbers
- P. Finding all polynomial roots of various quadratic, cubic, quartic and quintic polynomials
- Q. Relating polynomial roots to the polynomial factors and the graphs of the polynomial functions

CHAPTER 4 PRACTICE EXAM

1. Give the factored form of a polynomial function with three x -intercepts and a range of all real numbers. Sketch the function and label all key points.
2. Give the factored form of a polynomial function with one x -intercept, a range of $y \geq 0$ and a degree greater than 3. Sketch the function and label all key points.
3. Give the factored form of a polynomial function with two local minimums with different y - values; and three local maximums with different y - values. Sketch the function and label all key points.
4. Find the x -intercepts of $h(x) = 2x^5 + 6x^4 - 10x^3 - 30x^2 + 8x + 24$ using the CALC menu and factor the function. Graph the function, give the range and label all key points (intercepts, local and absolute minimums and maximums). Show your work.
5. Find the x -intercepts of $h(x) = -x^4 + x^3 + 7x^2 - x - 6$ using the CALC menu and factor the function. Graph the function, give the range and label all key points (intercepts, local and absolute minimums and maximums). Show your work.
6. Assume a is any real number. Consider $y = x^5 + a^5$. Find one x -intercept and then use polynomial division to create a Sum of Perfect Quintics Template.

For questions #7 - #10, factor each polynomial function and sketch a simple graph of each function. Label the x -intercepts of the functions with their x -values. Show your work.

7. $y = x^4 - 81$
8. $y = 27x^3 - 64$.
9. $y = x^{10} + 9x^5 + 18$
10. $y = x^6 - 5x^5 - 4x^4 + 21x^3 - 5x^2 - 4x + 20$. Hint: Use polynomial division multiple times.
11. Use black, red, green and yellow tiles to model the solution to $(4 + 2i) + (5 - 3i) = ?$ Sketch, label and explain your work. Be sure to carry out the whole operation; don't short cut by changing signs.
12. Use black, red, green and yellow tiles to model the solution to $(-6 + 2i) - (4 - 2i) = ?$ Sketch, label and explain your work. Be sure to carry out the whole operation; don't short cut by changing signs.
13. Use black, red, green and yellow tiles and edge pieces to model a sequence of minimal arrays showing the multiplication steps for $(-2 - 3i) \times (3 + 2i) = ?$ Sketch, label and explain your work.

14. Use black, red, green and yellow tiles and edge pieces to model a sequence of minimal arrays showing the division steps for $(8 - i) \div (2 + 3i) = ?$ Sketch, label and explain your work. Check your answer using the complex conjugate numerical division shortcut Show your work.
15. Describe the result of each of the following, give examples as needed:
- The sum of an all real number and an all imaginary number.
 - The sum of two complex numbers.
 - The difference of an all real number and an all imaginary number.
 - The difference of two complex numbers.
 - The product of an all real number and an all imaginary number.
 - The product of two all imaginary numbers.
 - The product of two complex numbers.
 - The quotient of an all real number and an all imaginary number.
 - The quotient of two all imaginary numbers.
 - The quotient of two complex numbers.
16. For $y = x^4 + 27x$, determine the key points that you need to graph the function and graph the function. Find the roots (real and complex) of the function and the factored form of the function. To find the roots use any method—solving equations, factoring, graphing calculator or the quadratic formula. Carefully show and describe all of your work.
17. For $y = x^6 - 1$, determine the key points that you need to graph the function and graph the function. Find the roots (real and complex) of the function and the factored form of the function. To find the roots use any method—solving equations, factoring, graphing calculator or the quadratic formula. Carefully show and describe all of your work.
18. Give an example of a cubic function (degree 3) with one real and two complex roots. Graph the function and give the x-values for the roots.
19. Give an example of a quartic function (degree 4) with one double real and two complex roots. Graph the function and give the x-values for the roots.
20. Give an example of a degree six polynomial with one triple real, one real and two complex roots. Graph the function and give the x-values for the roots.

BACK OF BOOK

APPENDIX A

SELECTED ACTIVITY SOLUTIONS

PRACTICE EXAM SOLUTIONS

Visual Algebra for College Students: Appendix A

ALTERNATING SEQUENCES TABLES

Patterns for Even Number Inputs

Index—Input	n	2	4	6	8
Total—Output	T	6	8	12	16
$T = 2n$	$2(n)$	$2(2) = 4$	$2(4) = 8$	$2(6) = 12$	$2(8) = 16$

Index—Input	n	2	4	6	8
Total—Output	T	5	9	13	17
$T = 2n + 1$	$2(n) + 1$	$2(2) + 1 = 5$	$2(4) + 1 = 9$	$2(6) + 1 = 13$	$2(8) + 1 = 17$

Index—Input	n	2	4	6	8
Total—Output	T	3	7	11	15
$T = 2n - 1$	$2(n) - 1$	$2(2) - 1 = 3$	$2(4) - 1 = 7$	$2(6) - 1 = 11$	$2(8) - 1 = 15$

Index—Input	n	2	4	6	8
Total—Output	T	1	2	3	4
$T = \frac{n}{2}$	$\frac{n}{2}$	$\frac{2}{2} = 1$	$\frac{4}{2} = 2$	$\frac{6}{2} = 3$	$\frac{8}{2} = 4$

Index—Input	n	2	4	6	8
Total—Output	T	2	3	4	5
$T = \frac{n+2}{2}$	$n + 2$	$2 + 2 = 4$	$4 + 2 = 6$	$6 + 2 = 8$	$8 + 2 = 10$
	$\frac{n+2}{2}$	$\frac{4}{2} = 2$	$\frac{6}{2} = 3$	$\frac{8}{2} = 4$	$\frac{10}{2} = 5$

Index—Input	n	2	4	6	8
Total—Output	T	0	1	2	3
$T = \frac{n-2}{2}$	$n - 2$	$2 - 2 = 0$	$4 - 2 = 2$	$6 - 2 = 4$	$8 - 2 = 6$
	$\frac{n-2}{2}$	$\frac{0}{2} = 0$	$\frac{2}{2} = 1$	$\frac{4}{2} = 2$	$\frac{6}{2} = 3$

Patterns for Odd Number Inputs

Index—Input	n	1	3	5	7
Total—Output	T	2	6	10	14
$T = 2n$	$2(n)$	$2(1) = 2$	$2(3) = 6$	$2(5) = 10$	$2(7) = 14$

Index—Input	n	1	3	5	7
Total—Output	T	3	7	11	15
$T = 2n + 1$	$2(n) + 1$	$2(1) + 1 = 3$	$2(3) + 1 = 7$	$2(5) + 1 = 11$	$2(7) + 1 = 15$

Index—Input	n	1	3	5	7
Total—Output	T	1	5	9	13
$T = 2n - 1$	$2(n) - 1$	$2(1) - 1 = 1$	$2(3) - 1 = 5$	$2(5) - 1 = 9$	$2(7) - 1 = 13$

Index—Input	n	1	3	5	7
Total—Output	T	1	2	3	4
$T = \frac{n+1}{2}$	$n + 1$	$1 + 1 = 2$	$3 + 1 = 4$	$5 + 1 = 6$	$7 + 1 = 8$
	$\frac{n+1}{2}$	$\frac{2}{2} = 1$	$\frac{4}{2} = 2$	$\frac{6}{2} = 3$	$\frac{8}{2} = 4$

Index—Input	n	1	3	5	7
Total—Output	T	0	1	2	3
$T = \frac{n-1}{2}$	$n - 1$	$1 - 1 = 0$	$3 - 1 = 2$	$5 - 1 = 4$	$7 - 1 = 6$
	$\frac{n-1}{2}$	$\frac{0}{2} = 0$	$\frac{2}{2} = 1$	$\frac{4}{2} = 2$	$\frac{6}{2} = 3$

Appendix A: Alternating Sequence Tables

Patterns for $n = 3k + 1$ Inputs ($k = 1, 2, 3 \dots; n = 1, 4, 7 \dots$)

Index—Input	n	1	4	7	10
Total—Output	T	0	1	2	3
	$n - 1$	$1 - 1 = 0$	$4 - 1 = 3$	$7 - 1 = 6$	$10 - 1 = 9$
$T = \frac{n-1}{3}$	$\frac{n-1}{3}$	$\frac{0}{3} = 0$	$\frac{3}{3} = 1$	$\frac{6}{3} = 2$	$\frac{9}{3} = 3$

Index—Input	n	1	4	7	10
Total—Output	T	1	2	3	4
	$n + 2$	$1 + 2 = 3$	$4 + 2 = 6$	$7 + 2 = 9$	$10 + 2 = 12$
$T = \frac{n+2}{3}$	$\frac{n+2}{3}$	$\frac{3}{3} = 1$	$\frac{6}{3} = 2$	$\frac{9}{3} = 3$	$\frac{12}{3} = 4$

Index—Input	n	1	4	7	10
Total—Output	T	1	3	5	7
	$2(n) + 1$	$2(1) + 1 = 3$	$2(4) + 1 = 9$	$2(7) + 1 = 15$	$2(10) + 1 = 21$
$T = \frac{2n+1}{3}$	$\frac{2n+1}{3}$	$\frac{3}{3} = 1$	$\frac{9}{3} = 3$	$\frac{15}{3} = 5$	$\frac{21}{3} = 7$

Index—Input	n	1	4	7	10
Total—Output	T	0	2	4	6
	$2(n) - 2$	$2(1) - 2 = 0$	$2(4) - 2 = 6$	$2(7) - 2 = 12$	$2(10) - 2 = 18$
$T = \frac{2n-2}{3}$	$\frac{2n-2}{3}$	$\frac{0}{3} = 0$	$\frac{6}{3} = 2$	$\frac{12}{3} = 4$	$\frac{18}{3} = 6$

Patterns for $n = 3k + 2$ Inputs ($k = 1, 2, 3 \dots; n = 2, 5, 8 \dots$)

Index—Input	n	2	5	8	11
Total—Output	T	0	1	2	3
	$n - 2$	$2 - 2 = 0$	$5 - 2 = 3$	$8 - 2 = 6$	$11 - 2 = 9$
$T = \frac{n-2}{3}$	$\frac{n-2}{3}$	$\frac{0}{3} = 0$	$\frac{3}{3} = 1$	$\frac{6}{3} = 2$	$\frac{9}{3} = 3$

Index—Input	n	2	5	8	11
Total—Output	T	1	2	3	4
	$n + 1$	$2 + 1 = 3$	$5 + 1 = 6$	$8 + 1 = 9$	$11 + 1 = 12$
$T = \frac{n+1}{3}$	$\frac{n+1}{3}$	$\frac{3}{3} = 1$	$\frac{6}{3} = 2$	$\frac{9}{3} = 3$	$\frac{12}{3} = 4$

Index—Input	n	2	5	8	11
Total—Output	T	2	4	6	8
	$2(n) + 2$	$2(2) + 2 = 6$	$2(5) + 2 = 12$	$2(8) + 2 = 18$	$2(11) + 2 = 24$
$T = \frac{2n+2}{3}$	$\frac{2n+2}{3}$	$\frac{6}{3} = 2$	$\frac{12}{3} = 4$	$\frac{18}{3} = 6$	$\frac{24}{3} = 8$

Index—Input	n	2	5	8	11
Total—Output	T	1	3	5	7
	$2(n) - 1$	$2(2) - 1 = 3$	$2(5) - 1 = 9$	$2(8) - 1 = 15$	$2(11) - 1 = 21$
$T = \frac{2n-1}{3}$	$\frac{2n-1}{3}$	$\frac{3}{3} = 1$	$\frac{9}{3} = 3$	$\frac{15}{3} = 5$	$\frac{21}{3} = 7$

Appendix A: Alternating Sequence Tables

Patterns for $n = 3k$ Inputs ($k = 1, 2, 3 \dots; n = 3, 6, 9 \dots$)

Index—Input	n	3	6	9	12
Total—Output	T	1	2	3	4
$T = \frac{n}{3}$	$\frac{n}{3}$	$\frac{3}{3} = 1$	$\frac{6}{3} = 2$	$\frac{9}{3} = 3$	$\frac{12}{3} = 4$

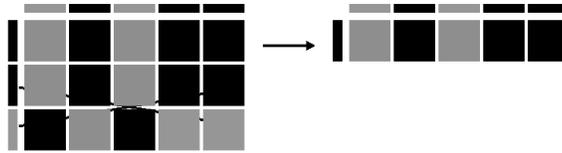
Index—Input	n	3	6	9	12
Total—Output	T	0	1	2	3
$T = \frac{n-3}{3}$	$n-3$	$3-3=0$	$6-3=3$	$9-3=6$	$12-3=9$
	$\frac{n-3}{3}$	$\frac{0}{3} = 0$	$\frac{3}{3} = 1$	$\frac{6}{3} = 2$	$\frac{9}{3} = 3$

Index—Input	n	3	6	9	12
Total—Output	T	2	3	4	5
$T = \frac{n+3}{3}$	$n+3$	$3+3=6$	$6+3=9$	$9+3=12$	$12+3=15$
	$\frac{n+3}{3}$	$\frac{6}{3} = 2$	$\frac{9}{3} = 3$	$\frac{12}{3} = 4$	$\frac{15}{3} = 5$

Index—Input	n	3	6	9	12
Total—Output	T	3	5	7	9
$T = \frac{2n+3}{3}$	$2(n)+3$	$2(3)+3=9$	$2(6)+3=15$	$2(9)+3=21$	$2(12)+3=27$
	$\frac{2n+3}{3}$	$\frac{9}{3} = 3$	$\frac{15}{3} = 5$	$\frac{21}{3} = 7$	$\frac{27}{3} = 9$

Index—Input	n	3	6	9	12
Total—Output	T	1	3	5	7
$T = \frac{2n-3}{3}$	$2(n)-3$	$2(3)-3=3$	$2(6)-3=9$	$2(9)-3=15$	$2(12)-3=21$
	$\frac{2n-3}{3}$	$\frac{3}{3} = 1$	$\frac{9}{3} = 3$	$\frac{15}{3} = 5$	$\frac{21}{3} = 7$

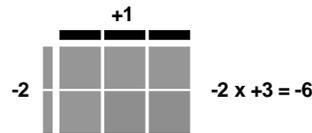
6. a.



12. Sample Observation: For the array to have net value -6 , there must be six red tiles once all of the zero pairs are removed

Activity Set 1.5

1.



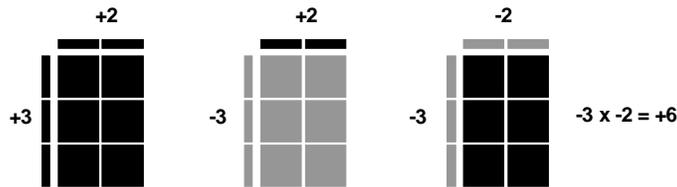
a. -6

b. -2 is one factor, $+3$ is the second factor and the product is -6

c. $-2 \times +3 = -6$

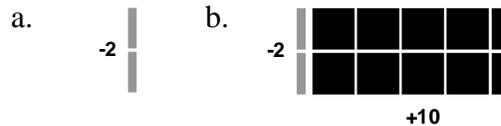
2.

d. Two negative factors, flip rows and edge for -3 and then flip rows and edge for -2 ; all black

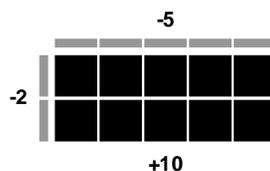


Activity Set 1.6

1.



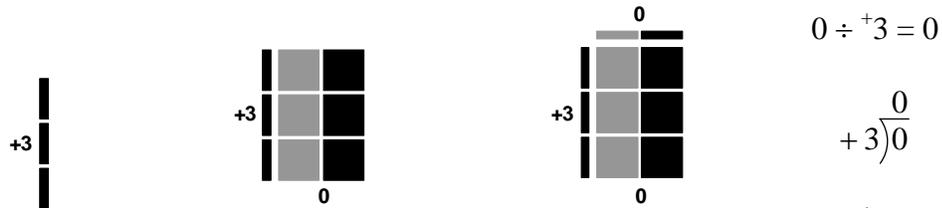
c. The net value has to be -5 . Since the left edge is red, the top edge must also be red (negative). Since there are 10 tiles and the left edge is 2 tiles tall, the top edge must be 5 tiles long.



- d. $+10 \div \bar{2} = \bar{5}$ which can also be thought of as $\bar{2} \overline{) +10}^{-5}$, which matches the visual set up of the array.
- e. The multiplication sentence would be: $\bar{2} \times \bar{5} = +10$.
- f. Multiplication and division are inverse operations. $+10 \div \bar{2}$ is $\bar{5}$ since $\bar{2} \times \bar{5} = +10$.

3.

a.



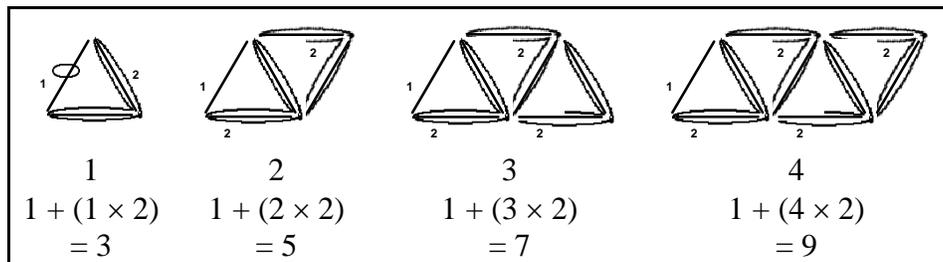
1. Lay out Edge I 2. Fill in the array 3. Determine Edge Set II
 Note: The final array could be Column 1—black and Column 2—red, the result would be the same.

4.

- a. POSITIVE. In the black and red model, if one edge is black and the array is black, the other edge must also be black (black edge \times black edge = black array); therefore the quotient must be black and positive.

Activity Set 2.1

1. Sample Numerical Solution



2. Sample: Words: 1 for first pick + (figure \times 2 for Vs)

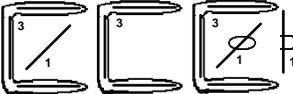
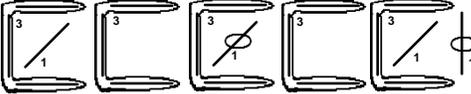
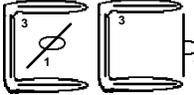
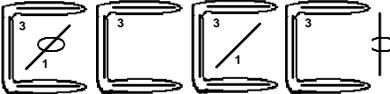
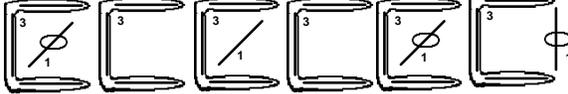
3. 1 for first pick + (5 \times 2 for Vs) = 1 + 2(5) = 11



6. Sample: Symbols: $T = 1 + 2n$

Activity Set 2.2

1. Sample Solution

<div style="text-align: center;">  <p>1</p> $3 + 1 + 1 = 5$ </div> <div style="text-align: center; margin-top: 20px;">  <p>3</p> $(3 \times 3) + (2 \times 1) + 1 = 12$ </div> <div style="text-align: center; margin-top: 20px;">  <p>5</p> $(5 \times 3) + (3 \times 1) + 1 = 19$ </div> <p>Odd Words Figure $\times 3$ for each C + (half of 1 more than the figure for the diagonals) + final 1</p> <p>Odd Symbols $T = (n \times 3) + ((n + 1)/2) + 1 = (7n + 3)/2$</p>	<div style="text-align: center;">  <p>2</p> $(2 \times 3) + 1 + 1 = 8$ </div> <div style="text-align: center; margin-top: 20px;">  <p>4</p> $(4 \times 3) + (2 \times 1) + 1 = 15$ </div> <div style="text-align: center; margin-top: 20px;">  <p>6</p> $(6 \times 3) + (3 \times 1) + 1 = 22$ </div> <p>Even Words Figure $\times 3$ for each C + (half of figure for the diagonals) + final 1</p> <p>Even Symbols $T = (n \times 3) + (n/2) + 1 = (7n + 2)/2$</p>
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2. $T(99) = (7(99) + 3)/2 = (696)/2 = 348$

$T(100) = (702)/2 = 351$

3.

$T = 108$

$(7n + 3)/2 = 54; 7n + 3 = 108; 7n = 105; n = 15$

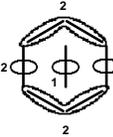
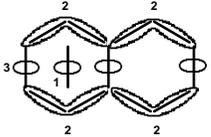
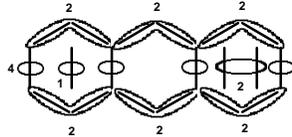
$(7n + 2)/2 = 54; 7n + 2 = 108; 7n = 106; n \notin \mathbb{N}$

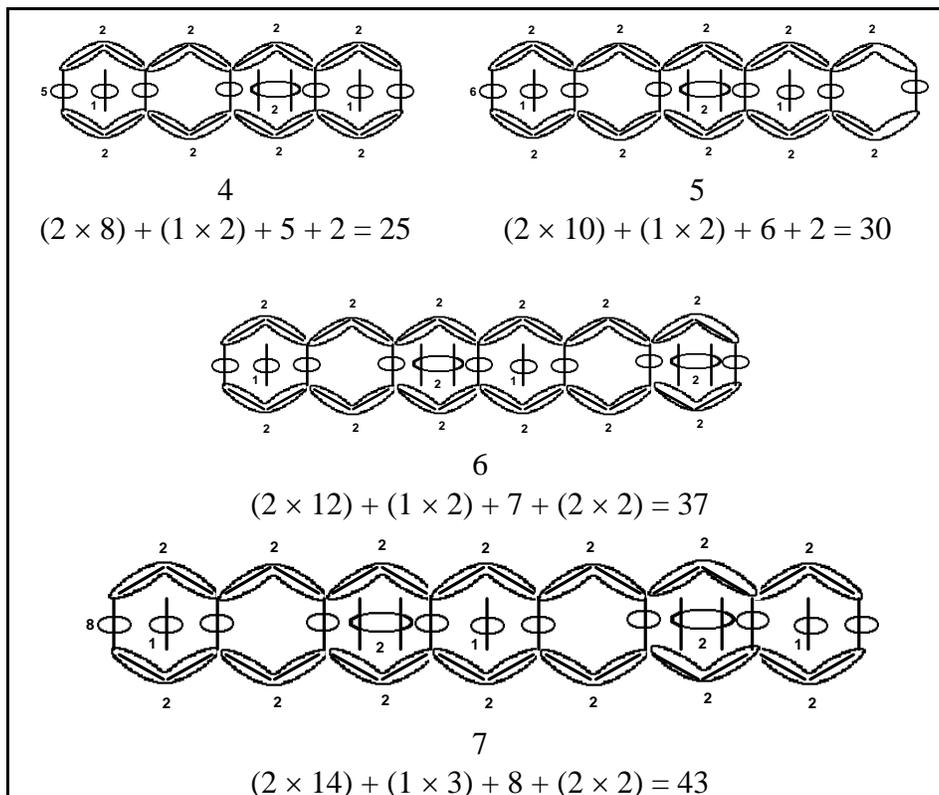
$T = 120$

$(7n + 3)/2 = 120; 7n + 3 = 240; 7n = 238; n \notin \mathbb{N}$

$(7n + 2)/2 = 120; 7n + 2 = 240; 7n = 238; n = 34$

7. Sample Solution

 <p>1</p> $(2 \times 2) + 1 + 2 = 7$	 <p>2</p> $(2 \times 4) + 1 + 3 = 12$	 <p>3</p> $(2 \times 6) + 1 + 4 + 2 = 19$
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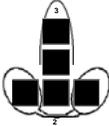


a. and b.

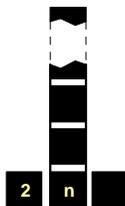
<p>$n = 3k + 1$ Words ($n = 1, 4, 7, \dots$) $(2 \times \text{twice figure for Vs}) + (1/3 \text{ of } 2 \text{ more than the figure for middle vertical picks}) + 1 \text{ more than the figure for vertical } 2 \text{ hexagon edges} + (2 \times 1/3 \text{ of } 1 \text{ less than the figure for double vertical picks})$</p>	<p>$n = 3k + 1$ Symbols $T = (2 \times 2n) + (1/3)(n + 2) + (n + 1) + (2 \times (1/3)(n - 1)) = 6n + 1$</p>
<p>$n = 3k + 2$ Words ($n = 2, 5, 8, \dots$) $(2 \times \text{twice figure for Vs}) + (1/3 \text{ of } 1 \text{ more than the figure for middle vertical picks}) + 1 \text{ more than the figure for vertical } 2 \text{ hexagon edges} + (2 \times 1/3 \text{ of } 2 \text{ less than the figure for double vertical picks})$</p>	<p>$n = 3k + 2$ Symbols $T = (2 \times 2n) + (1/3)(n + 1) + (n + 1) + (2 \times (1/3)(n - 2)) = 6n$</p>
<p>$n = 3k$ Words ($n = 3, 6, 9, \dots$) $(2 \times \text{twice figure for Vs}) + (1/3 \text{ of figure for middle vertical picks}) + 1 \text{ more than the figure for vertical } 2 \text{ hexagon edges} + (2 \times 1/3 \text{ of figure for double vertical picks})$</p>	<p>$n = 3k$ Symbols $T = (2 \times 2n) + (1/3)n + (n + 1) + (2 \times (1/3)n) = 6n + 1$</p>

Activity Set 2.3

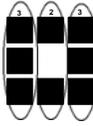
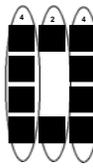
1. Sample Solution

				<p>Words 2 tiles on the left and right of the bottom row and the figure number of tiles in the middle column.</p> <p>Symbols $T = 2 + n$</p>
1	2	3	4	
$2 + 1$	$2 + 2$	$2 + 3$	$2 + 4$	

2. The figure has 3 black tiles in a row with 99 black tiles in a column extending up from the middle tile to make an upside down Tee.
3. $T = n + 2 = 2002$; $n = 2000$. It will be the 2000 figure which is 3 tiles wide with 1 tile in the 1st and 3rd column and 2000 tiles in the 2nd column.
4. The n th Tee has 1 black tile in the 1st column, n black tiles in the 2nd column and 1 black tile in the 3rd column. It has $n + 2$ tiles total.



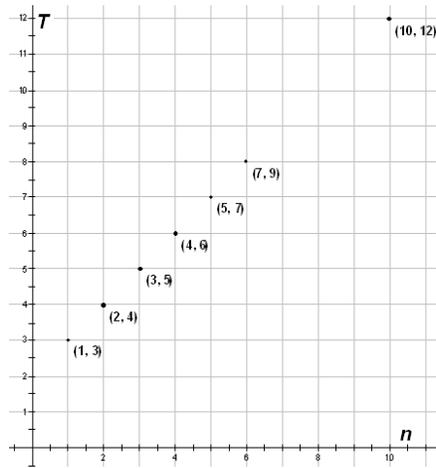
5. Sample Solution

				<p>Words: Twice the figure number + 2 tiles for the middle column</p> <p>Symbols: $T = (2 \times n) + 2 = 2n + 2$</p>
1	2	3	4	

$$(2 \times 1) + \frac{\quad}{2} \quad (2 \times 2) + \frac{\quad}{2} \quad (2 \times 3) + \frac{\quad}{2} \quad (2 \times 4) + \frac{\quad}{2}$$

6.

n	1	2	3	4	5	6	n
T	3	4	5	6	7	8	$n + 2$



- a. “By inspection” will vary; symbolically $T(10) = 10 + 2 = 12$
- b. Sample Observation: The points are in a straight line

Activity Set 2.4

5.

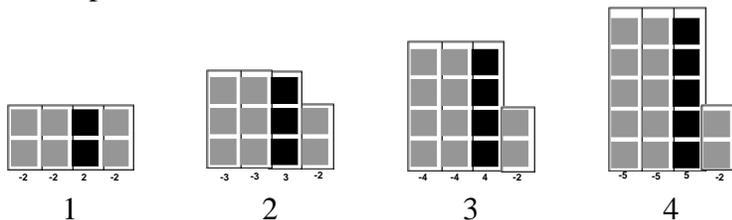
ALGEBRA PIECE WORK with notes	SYMBOLIC WORK
<p style="text-align: center;">Set up pieces</p> 	<p style="text-align: center;">Set up equation</p> $3 \times 2n + 10 = 160$
<p style="text-align: center;">Add 10 red tiles to each side</p> 	$3 \times 2n + 10 = 160$ $+10 +10$
<p style="text-align: center;">Simplify</p> 	$6n = 150$
<p>Divide each side into 6 groups, each group:  = 25</p>	$\frac{6n}{6} = \frac{150}{6} = 25$

6. The $(n + 1)$ st Rectangle is 2 tiles tall and $n + 1$ tiles wide. It has $2n + 2$ tiles.



Activity Set 2.5

1. Sample Solution

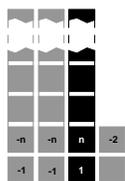


Words
 2 columns of 1 more than the figure in red, 1 column of one more than the figure in black and 2 red

$$\begin{matrix} (-2 \times 2) + 2 - & (-3 \times 2) + 3 - & (-4 \times 2) + 4 - & (-4 \times 2) + 4 - & \text{Symbols } C(n) = -2(n + 1) + n - 2 \\ 2 = -4 & 2 = -5 & 2 = -6 & 2 = -7 \end{matrix}$$

2. 2 red columns, each 101 tiles tall, 1 black column, 101 tiles tall followed by 2 red tiles.
 $L(100) = -103$.

3.

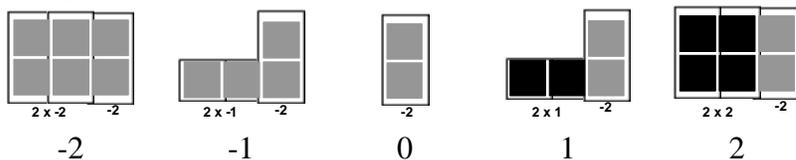


4. $n = 800$. 2 columns, each with 801 red tiles followed by 1 column with 801 black tiles followed by 2 red tiles for a total of 1604 red tiles and 801 black tiles. $C(800) = -803$.

5. $n = -2402$. 2 columns, each with 2403 red tiles followed by 1 column with 2403 black tiles followed by 2 red tiles for a total of 4808 red tiles and 2403 black tiles.

Activity Set 2.6

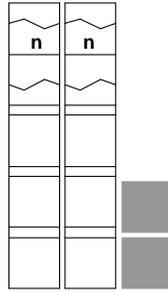
1. Sample Solution



Words
 Twice the figure + 2 reds

$$\begin{matrix} (-2 \times 2) - 2 & (-1 \times 2) - 2 & (0 \times 2) - 2 & (1 \times 2) - 2 & (2 \times 2) - 2 & \text{Symbolic:} \\ = -6 & = -4 & = -2 & = 0 & = 4 & A(n) = 2n - 2 \end{matrix}$$

2.

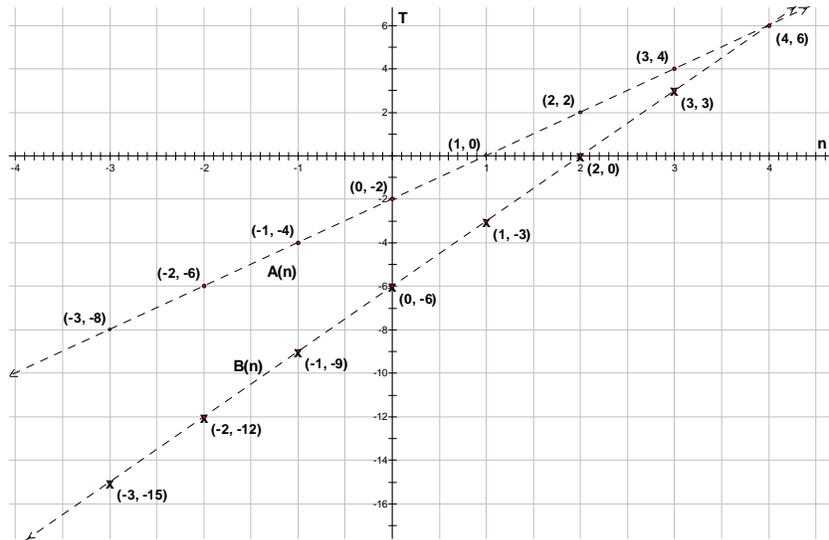


5.

n	-3	-2	-1	0	1	2	3	n
$A(n)$	-8	-6	-4	-2	0	2	4	$2n - 2$

n	-3	-2	-1	0	1	2	3	n
$B(n)$	-15	-12	-9	-6	-3	0	3	$3n - 6$

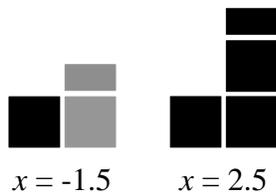
6.



Activity Set 3.1

2.

a.

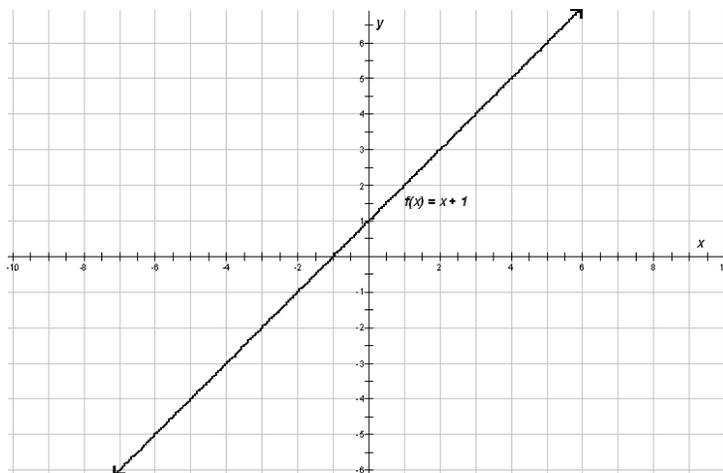


b.

x	-3	-2.2	-2	-1.5	-1	-.5	0	1	1.5	2	2.5	3	x
$y = f(x)$	-2	-1.2	-1	-.5	0	.5	1	2	2.5	4	3.5	4	$x + 1$

c. Range = \mathbf{R} . See Example 1 in the Introduction

d.



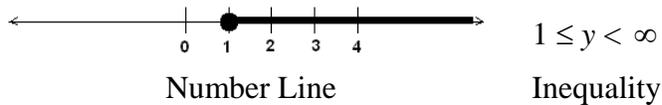
e. The function continues infinitely up and infinitely down, visually showing $-\infty < y < \infty$

9.

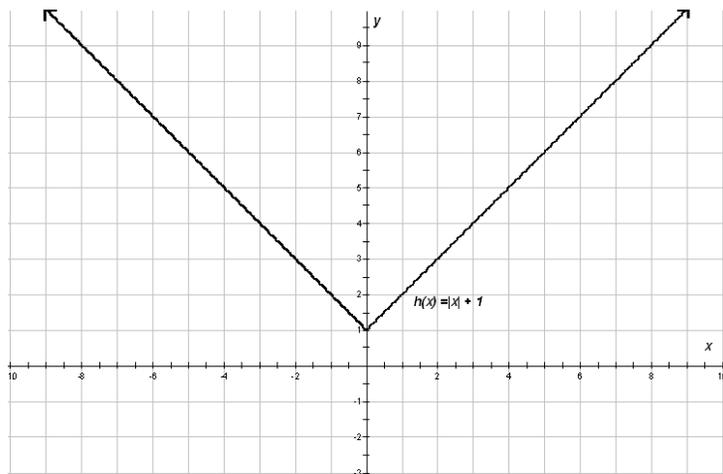
a.

x	-2.5	-2	-1	0	.5	1	2	$x \leq 0$	$x \geq 0$	x
$y = h(x)$	3.5	3	2	1	1.4	2	3	$-x + 1$	$x + 1$	$ x + 1$

b.



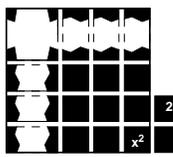
c.



Activity Set 3.2

2.

a. Sample Feature: The difference between the net values of consecutive tile figures is not constant



c. The 10th $y = f(x)$ figure will be a 100 by 100 square of black tiles followed by a column of 2 black tiles. $f(100) = 10002$

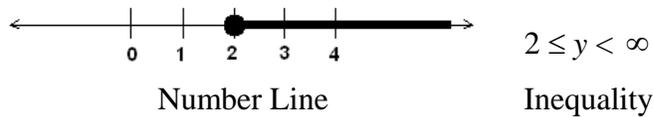
d. $x = \pm 50$

e.

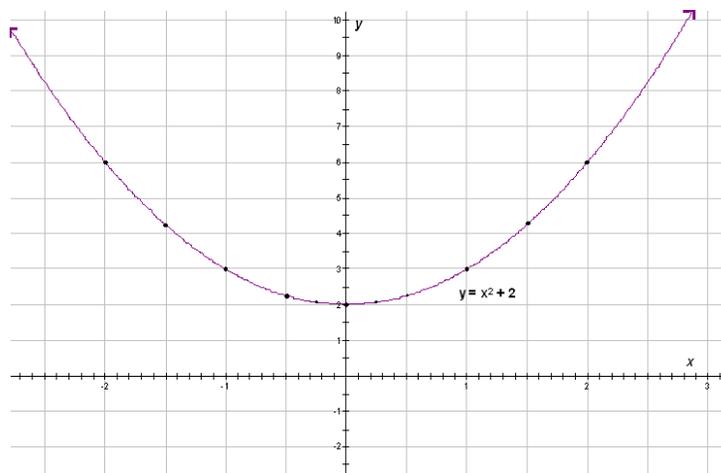
e.

x	-2	-1.5	-1	-.5	-.25	0	.25	.5	1	1.5	2	x
$y = f(x)$	6	4.25	3	2.25	2.07	2	2.07	2.25	3	4.25	6	$x^2 + 2$

f.



g.

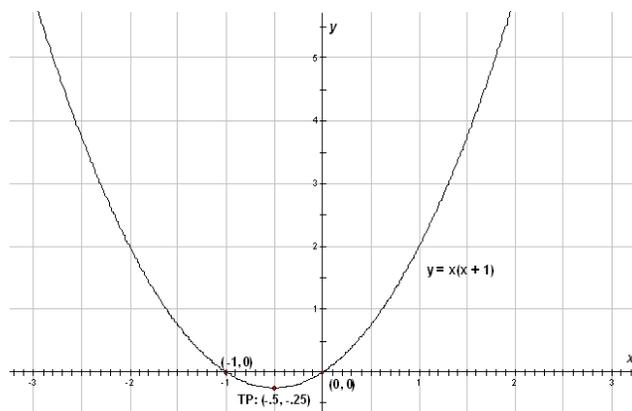


h. The function goes up from 2 on the left where $x \leq 0$ and also goes up from 0 on the right where $x \geq 0$. The function graph is always above or at the same level as $y = 2$.

i. Because the graph is symmetric about the y -axis, if $f(a) = 2502$, then so does $-a$. There are two x values for each output $y > 2$. In general $f(-a) = f(a)$ on this graph.

j. (0, 2) The y value of the turning point is the smallest value in the range of the function.

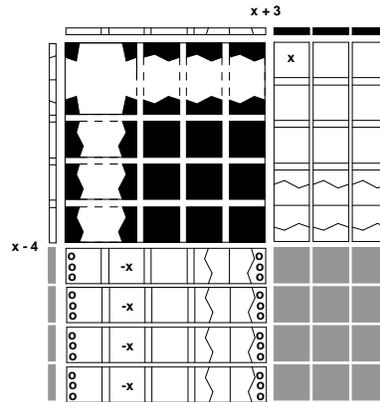
6h.



- i. Turning point is $(-0.5, -0.25)$, range is $y \geq -0.25$. Range is $y \geq y$ value of turning point.
- j. $g(x) < 0$ when $-1 < x < 0$ and there are no tile figures for this portion of the domain.
- k. $(0, 0)$
- l. $(-1, 0)$ and $(0, 0)$

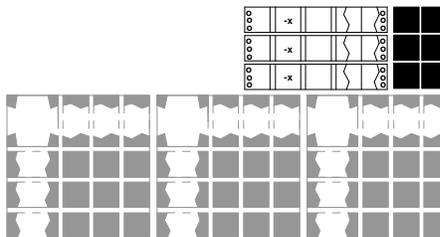
Activity Set 3.3

- 1.
 - c. Add three zero pairs of white and opposite white x -strips.

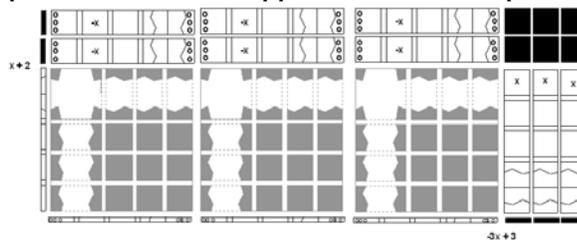


- d. $y = f(x) = (x - 4)(x + 3)$
- e. $(x - 4) = 0; x = 4$ or $(x + 3) = 0; x = -3$. $(4, 0)$ and $(-3, 0)$
- f. Halfway between the x -intercepts; $(0.5, -12.25)$

- 3.
 - a. $g(x) = -3x^2 - 3x + 6$



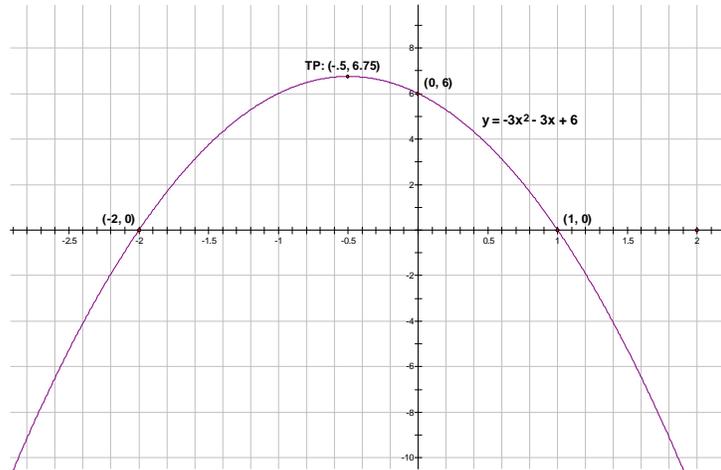
Add three zero pairs of white and opposite white x -strips to make a rectangle



b.

y-intercept	x-intercepts	Turning Point	Range	Factored Form
(0, 6)	(1, 0) and (-2, 0)	(-0.5, 6.75)	$y \leq 6.75$	$y = -3(x-1)(x+2)$

c.



Activity Set 3.4

3.

a. $y = x^2 + 2$ is $y = x^2$ vertically shifted 2 up.b. $y = x^2 - 2$ is $y = x^2$ vertically shifted 2 down.c. $y = x^2 + c$ is $y = x^2$ vertically shifted c up if $c > 0$ and $|c|$ down if $c < 0$.d. TP: $(0, c)$, range: $y \geq c$ e. $(0, c)$ f. If $c > 0$ there are no x -intercepts. If $c < 0$, the x -intercepts are $(\sqrt{|c|}, 0)$ and $(-\sqrt{|c|}, 0)$

4.

a. $y = (x+2)^2$ is $y = x^2$ horizontally shifted 2 left.b. $y = (x-2)^2$ is $y = x^2$ horizontally shifted 2 right.c. $y = (x-h)^2$ is $y = x^2$ horizontally shifted h right if $h > 0$ and horizontally shifted $|h|$ left if $h < 0$.d. TP: $(h, 0)$, range: $y \geq 0$ e. $(0, h^2)$

f. There is just one: $(h, 0)$

5.

a. $y = 2x^2$ is $y = x^2$ stretched thinner.

b. $y = -2x^2$ is $y = x^2$ stretched thinner and flipped upside down.

c. $y = ax^2$ is $y = x^2$ stretched thinner if $a > 1$ and stretched fatter if $0 < a < 1$. It is also flipped upside down if $a < 0$.

d. TP: $(0, 0)$, range: $y \geq 0$

e. $(0, 0)$

f. There is just one: $(0, 0)$

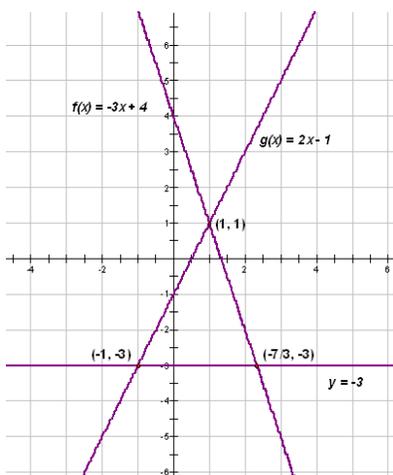
Activity Set 3.5

1.

a.

Points of Intersections

$f(x) = -3$	$g(x) = -3$	$f(x) = g(x)$
$(-\frac{7}{3}, -3)$	$(-1, -3)$	$(1, 1)$



b. $f(x) < -3$ when: $x < -\frac{7}{3}$

$f(x) > -3$ when: $x < -\frac{7}{3}$

c. $g(x) < -3$ when: $x < -1$

$g(x) > -3$ when: $x > -1$

d. $f(x) < g(x)$ when: $x > 1$

$f(x) > g(x)$ when: $x < 1$

Activity Set 4.1

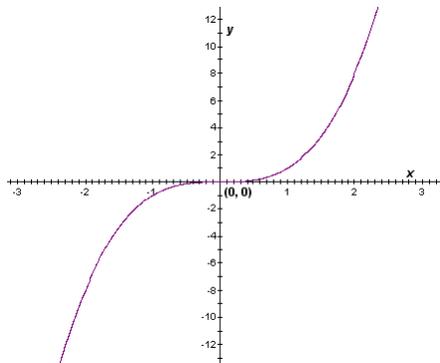
1.

a.

x	-2	-1.5	-1	-.5	0	1	1.5	2	2.5
$y = x^3$	-8	-3.375	-1	-.124	0	1	3.375	8	15.625

b. All real numbers

c.



d. The graph continues up to the right and down to the left.

e. $f(0) = 0$ f. $(0, 0)$

g. No

h. No

Activity Set 4.2

2. $y = x^2 + a^2$ does not cross the x -axis, therefore it does not have any real roots. Additionally, using the quadratic formula, $\sqrt{0 - 4a^2} = \sqrt{-4a^2}$ which has no real number solution.

$$3. y = x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

$$7. y = x^5 - a^5 = (x - a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4)$$

$$14. y = x^6 + 7x^3 + 12 = (x^3 + 3)(x^3 + 4) = (x + \sqrt[3]{3})(x^2 - \sqrt[3]{3}x + \sqrt[3]{9})(x + \sqrt[3]{4})(x^2 - \sqrt[3]{4}x + \sqrt[3]{16}).$$

Two x -intercepts— $(-\sqrt[3]{3}, 0)$ and $(-\sqrt[3]{4}, 0)$

Activity Set 4.3

2.

a. $-1 - i$

b. $-3 + 7i$

c. -4

d. $6i$

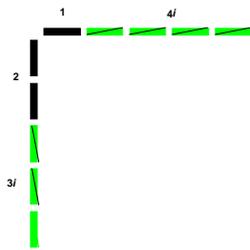
6.

×	B	R	G	Y
B	B	R	G	Y
R	R	B	Y	G
G	G	Y	R	B
Y	Y	G	B	R

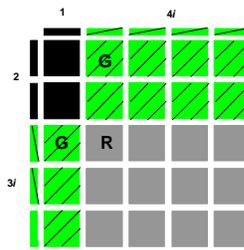
×	1	-1	i	i
1	1	-1	i	i
-1	-1	1	i	I
i	i	i	-1	1
i	i	I	1	-1

7a.

Set up factor edges



Fill in tiles



Determine Product

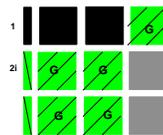
$$(2 + 3i) \times (1 + 4i) = -10s + 11i$$

10f.

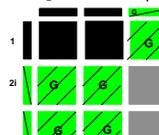
Set up divisor edge



Fill in dividend tiles



Set up quotient edge



Determine Quotient

$$5i \div (1 + 2i) = 2 + i$$

Corresponding multiplication sentence

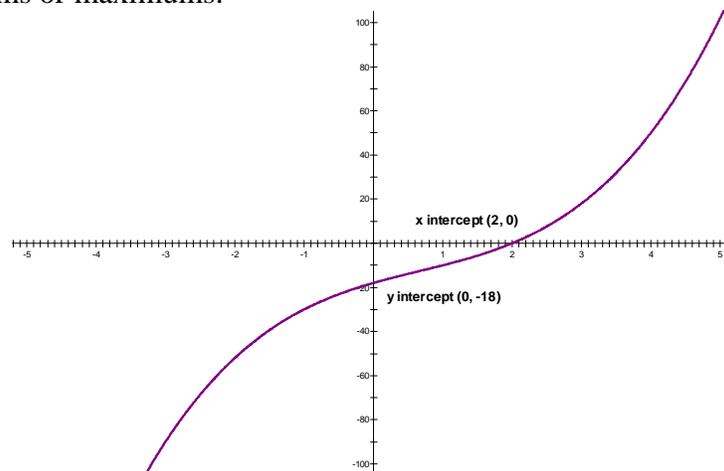
$$(1 + 2i) \times (2 + i) = 5i$$

Activity Set 4.4

2.

a. $p(x) = (x - 2)(x^2 + 9) = x^3 + 9x - 2x^2 - 18$

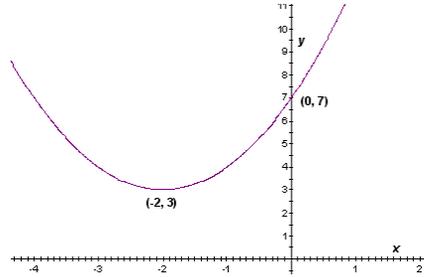
b. No minimums or maximums.



c. There is one real root which corresponds to one x-intercept.

5. 2 complex roots 0 real roots. Graph does not cross the x -axis.

$$x = \frac{-4 \pm \sqrt{16 - 28}}{2} = \frac{-4 \pm \sqrt{-12}}{2} = \frac{-4 \pm 2\sqrt{-3}}{2} = -2 \pm i\sqrt{3}$$



SOLUTIONS—Chapter 1 Practice Exam

1.

Collection	Total # of tiles	# Black Tiles	# Red Tiles	Net Value
	12	7	5	+2
Opposite Collection	Total # of tiles	# Black Tiles	# Red Tiles	(Opposite) Net Value
	12	5	7	-2

2. No such collection exists; 6 black tiles in the original collection and 3 black tiles in the opposite collection means 3 red tiles and 6 black tiles in the original collection which has net value +3 and opposite net value -3, not opposite net value +2.

3.

Collection	Total # of tiles	# Black Tiles	# Red Tiles	Net Value
	19	7	12	-5
Opposite Collection	Total # of tiles	# Black Tiles	# Red Tiles	(Opposite) Net Value
	19	12	7	+5

4. No such collection exists; the absolute value of the net value cannot exceed the total number of tiles.

5.

a. To reduce a collection to a minimal collection, you must remove an even number of tiles. An even number – an even number = even number, not an odd number. Such a collection cannot exist.

b. Similarly, an odd number – an even number = an odd number, not an even number. Such a collection cannot exist.

6. +5; combine and count the two addends of all black tiles.

7. +1, combine the two addends, remove the zero pair and count the remaining collection of tiles.

8. -5; combine and count the two addends of all red tiles.

9. Think of matching 400 red tiles to 400 black tiles. This is equivalent to taking 400 black tiles away from the original 1000 tiles.

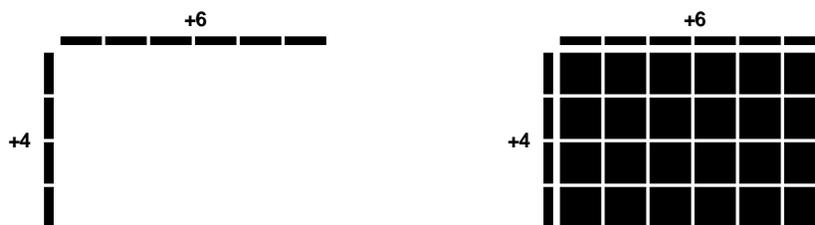
10. Sample solution: Count the number of tiles in the usual way; $1400 + 2345 = 3845$. Since the 2845 tiles are red, the sum is negative.

11. -3; add in at least 3 zero pairs, remove 4 black tiles, determine net value of final collection.

12. +5; add in at least 4 zero pairs, remove 4 red tiles, determine net value of final collection.

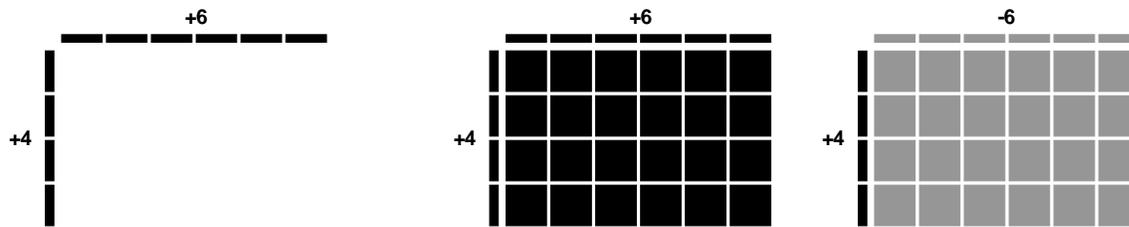
13. -5; add in at least 4 zero pairs, remove 4 black tiles, determine net value of final collection.

14. $+3$; add in at least 4 zero pairs, remove 4 red tiles, determine net value of final collection.
15. Sample solution: $+2 - 6 = ?$ can be converted to $2 + 6 = ?$ because, to take away 6 red tiles, you add in 6 zero pairs and take away the 6 red tiles. This is equivalent to just adding in the 6 black tiles ($2 + 6$). However $+2 - 6 = ?$ cannot be converted to $2 + 6 = ?$ as you must add in at least 4 zero pairs to take out 6 black tiles. This leaves 4 red tiles, not 8 black tiles which is the correct sum of $2 + 6$.
16. a. The minimal arrays with this net value are: $+1 \times -4$, $-1 \times +4$ and $+2 \times -2$.
- b. The minimal arrays with this net value are: $+1 \times +5$ and -1×-5 .
- c. The minimal arrays with this net value are: $+1 \times -16$, $-1 \times +16$, $+2 \times -8$, $-2 \times +8$ and $+4 \times -4$.
- d. There are infinitely many nonempty arrays with net value 0; at least one edge set must have net value 0 (half black and half red edge pieces) and the resulting array will also have half black and half red tiles. Any such nonempty array will not be minimal as it will have matching zero pair rows or columns. There is only one minimal array with net value zero: The Empty Array.
17. Since the edge net value is a factor of the net value of the array, the net value of the array will be a multiple of -3 . It can be 0×-3 , $\pm 1 \times -3$, $\pm 2 \times -3$, etc. The net value of the array can be: 0 ± 3 , ± 6 , etc.
18. Since one factor of the array is 0, the net value of the array must be 0. the array will have half black and half red tiles.
19. Yes, if Edge Set II is also all red.
20. No, the array will have net value 0 which is neither negative nor positive.
- 21.



1. Lay out and label all black edges
2. Fill in all black tiles
3. Use the array to determine the final product: $+4 \times +6 = +24$

22.



1. Lay out and label all black edges
2. Fill in all black tiles
3. Flip column and edge pieces for $\bar{6}$
4. Use the array to determine the final product: ${}^+4 \times \bar{6} = \bar{24}$

23.

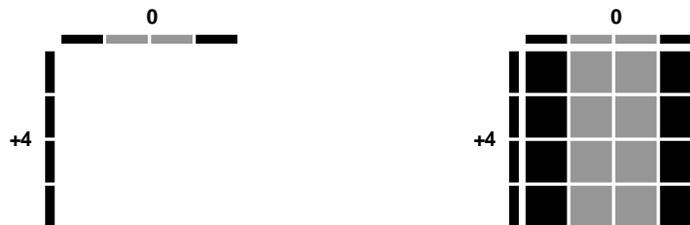


1. Lay out and label all black edges
2. Fill in all black tiles
3. Flip column and edge pieces for $\bar{6}$
4. Flip row and edge pieces for $\bar{4}$

5. Use the array to determine the final product: $\bar{4} \times \bar{6} = {}^+24$

Note the order of the last two steps does not matter.

24.



1. Lay out and label black edge for ${}^+4$ and an equal number of black and red edge pieces for 0.
2. Fill in matching tiles

3. Use the array to determine the final product: ${}^+4 \times 0 = 0$

Note: There are many such arrays, this is just one example

25. Sample solution: If 0 is a factor but the other factor is nonzero, then one edge set is half black and half red tiles and the other edge set can be reduced to all black or all red. Thus, the array will be half red and half black and the product will be 0. If two edges (two factors) are both 0, both will be half black and half red. Since black \times black = black, black \times red = red, red \times black = red and red \times red = red, the array again will be half black and half red and have net value (the product) 0.

26.

$+12 \div +6 = +2$
 $+6 \overline{) +12}$
 $+2 \times +6 = +12$

1. Lay out Edge I 2. Fill in the array 3. Determine Edge Set II

27.

$+12 \div -6 = -2$
 $-6 \overline{) +12}$
 $-2 \times -6 = +12$

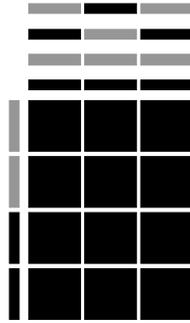
1. Lay out Edge I 2. Fill in the array 3. Determine Edge Set II

28.

$-12 \div +6 = -2$
 $+6 \overline{) -12}$
 $-2 \times +6 = -12$

1. Lay out Edge I 2. Fill in the array 3. Determine Edge Set II

29. This is not possible. If the first edge is 0 and non minimal, it is half black and half red. If you lay in 12 black tiles in the array, no second edge makes sense as illustrated here:



30. 0 can be a dividend or a quotient as shown in this division sentence: $0 \div \text{any nonzero divisor} = 0$. 0 can never be a divisor; you cannot make an array that represents: a nonzero dividend $\div 0 = ?$ and you can make infinitely many arrays that show $0 \div 0 = ?$; the last division question does not have a unique solution.

SOLUTIONS—Chapter 2 Practice Exam

1. Looping and words will vary. Symbolic equations should reduce to $T = 6n + 1$.
2. Figures alternate even / odd
 n even: Looping and words will vary, $T = 3n + 1 + 5\left(\frac{n}{2}\right)$
 n odd: Looping and words will vary, $T = 3n + 1 + 2\left(\frac{n+1}{2}\right) + 3\left(\frac{n-1}{2}\right)$
3.
 - a. 413
 - b. 419
 - c. 424
4.
 - a. Figure 366
 - b. Figure 367
 - c. Figure 370
5. Looping and words will vary. $T = 5n + 1$
6.
 - a. 376
 - b. 381
 - c. 386
7.
 - a. Figure 400
 - b. Figure 404
 - c. Figure 407
8. Figures alternate for $n = 3k + 1$, $n = 3k + 2$ and $n = 3k$; $k = 1, 2, 3 \dots$
 $n = 3k + 1$: Looping and words will vary; (unsimplified) $T = 3n + 1 + 2\left(\frac{n+2}{3}\right) + 3\left(\frac{2n-2}{3}\right)$
 $n = 3k + 2$: Looping and words will vary; (unsimplified) $T = 3n + 1 + 2\left(\frac{n+1}{3}\right) + 3\left(\frac{2n-1}{3}\right)$
 $n = 3k$: Looping and words will vary; (unsimplified) $T = 3n + 1 + 2\left(\frac{n}{3}\right) + 3\left(\frac{2n}{3}\right)$

9.

- a. 431
- b. 437
- c. 426

10.

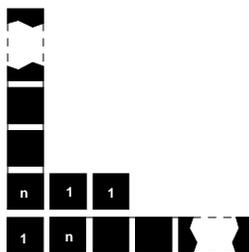
- a. Figure 354; $n = 3k$
- b. Figure 359; $n = 3k + 2$
- c. Figure 361; $n = 3k + 1$

11. Looping and words will vary. $T = 2n + 3$.

12. The 100th figure will be an L shape with 1 black tile in the corner, 100 black tiles in a row to the right of that, 100 black tiles above that and 2 more black tiles above the first 2 of the horizontal tiles. There will be 203 tiles total.

13. The 1000 figure will be an L shape with 1 black tile in the corner, 1000 black tiles in a row to the right of that, 1000 black tiles above that and 2 more black tiles above the first 2 of the horizontal tiles.

14. L shape with 1 black tile in the corner, n black tiles in a row to the right of that, n black tiles above that and 2 more black tiles above the first 2 of the horizontal tiles.

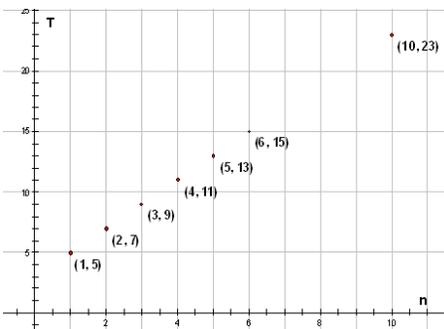


15.

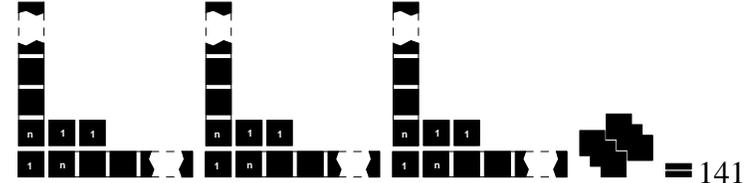
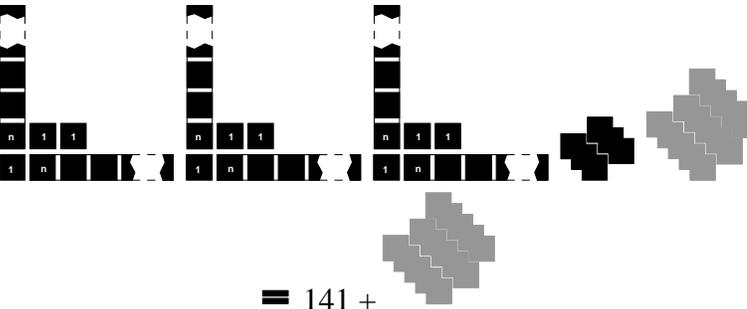
a.

n	1	2	3	4	5	6	n
T	5	7	9	11	13	15	$2n + 3$

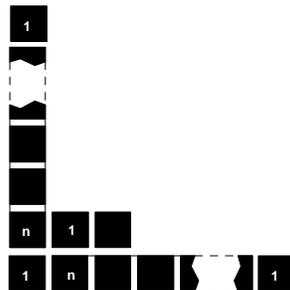
b. and c. The ordered pair is (10, 23)



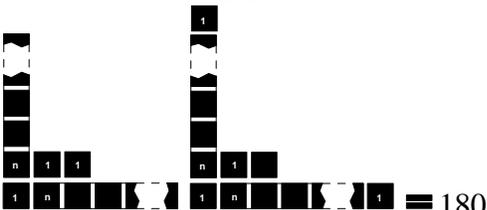
16. The 21st figure.

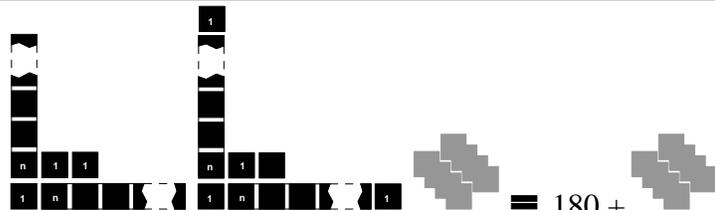
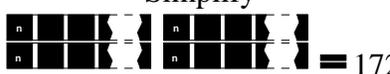
ALGEBRA PIECE WORK with notes	SYMBOLIC WORK
<p style="text-align: center;">Set up pieces</p>  <p style="text-align: right;">$= 141$</p>	<p style="text-align: center;">Set up equation</p> $3(2n + 3) + 6 = 141$
<p style="text-align: center;">Add 15 red tiles to each side</p>  <p style="text-align: right;">$= 141 +$</p>	$3(2n + 3) + 6 = 141$ $6n + 9 + 6 = 141$ $+15 +15$
<p style="text-align: center;">Simplify</p>  <p style="text-align: right;">$= 126$</p>	$6n = 126$
<p style="text-align: center;">Divide each side into 6 groups, each group:</p>  <p style="text-align: right;">$= 21$</p>	$\frac{6n}{6} = \frac{126}{6} = 21$

17.



18. The 43rd and 44th figures.

ALGEBRA PIECE WORK with notes	SYMBOLIC WORK
<p style="text-align: center;">Set up pieces</p>  <p style="text-align: right;">$= 180$</p>	<p style="text-align: center;">Set up equation</p> $2n + 3 + 2(n + 1) + 3 = 180$
<p style="text-align: center;">Add 8 red tiles to each side</p>	$2n + 3 + 2(n + 1) + 3 = 180$

	$4n + 8 = 180$ $+ 8 \quad + 8$
<p>Simplify</p> 	$4n = 172$
<p>Divide each side into 4 groups, each group:</p> 	$\frac{4n}{4} = \frac{172}{4} = 43$

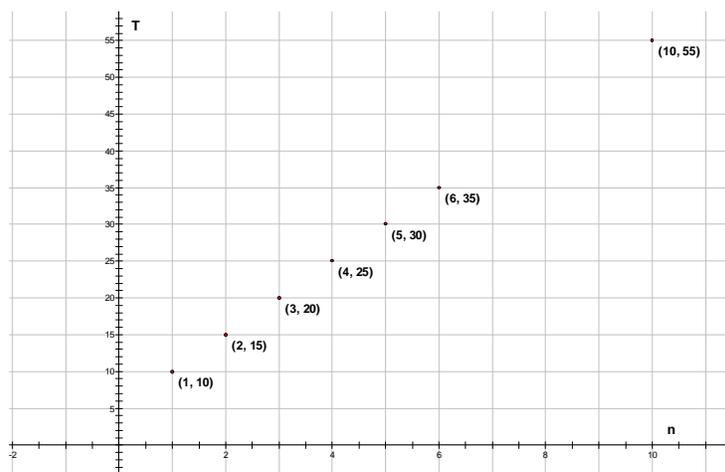
19. Figures will vary but should have 10, 15, 20 and 25 black tiles for $n = 1, 2, 3$ and 4 respectively. The n th figure should have 5 black n -strips and 5 black tiles.

20.

a.

n	1	2	3	4	5	6	n
T	10	15	20	25	30	35	$5n + 5$

b. and c. The ordered pair is (10, 55)

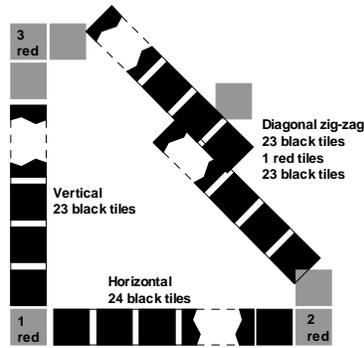


21. $-n - 3 + -(n + 3) - 3 = -109$; $n = 50$. The 50th and 53rd figures.

22. $5(-n - 3) - (-100) = 0$; $n = 17$. The 17th figure.

23. $4n - 6 + 4(n + 5) - 6 = 336$; $n = 41$. The 41st and 46th figures.

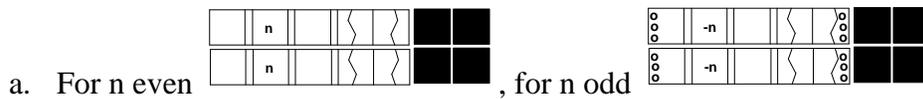
24. There are a total of $4n + 8$ tiles in each figure. $4n + 8 + 4(n + 1) + 8 = 204$; $n = 23$. The 23rd figure has $23 + 24 + 23 + 23 = 93$ black tiles and 7 red tiles as shown here:



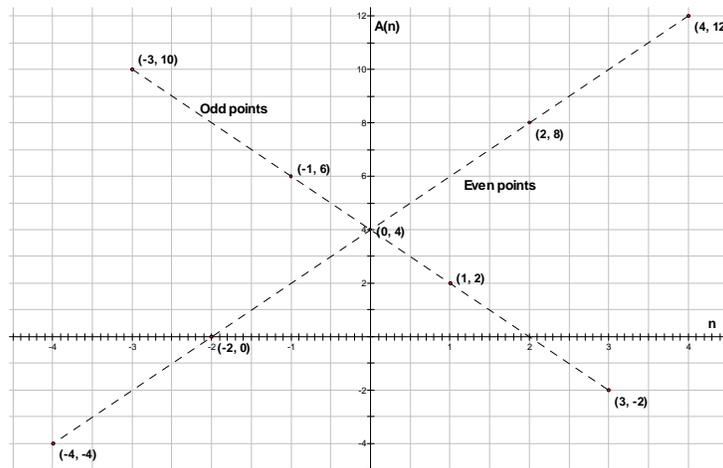
The 24th figure has $24 + 25 + 24 + 24 = 96$ black tiles and 7 red tiles.

25. $3(-4(n - 1)) + 96 = 96$; $n = 1$; which is a valid solution.
 $3(-4(n - 1)) + 96 = -96$; $n = 17$; which is a valid solution.

26. For n even $A(n) = 2n + 4$, for n odd $A(n) = -2n + 4$



b.



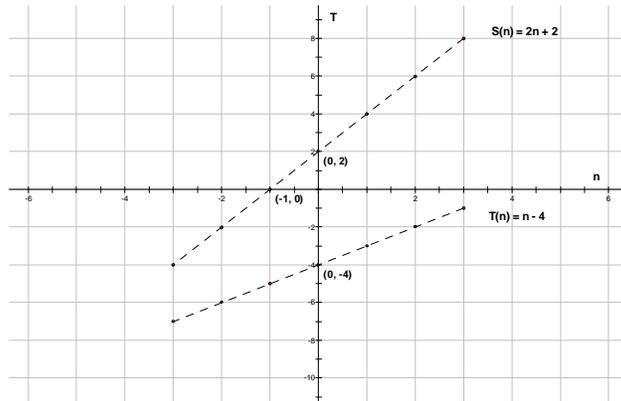
27. $3(-2n + 4) + 15 = 15$, $n = -2$, not valid since -2 is even
 $3(-2n + 4) + 15 = -15$, $n = 7$, valid since 7 is odd

$3(2n + 4) + 15 = 15$, $n = -2$, valid since -2 is even
 $3(2n + 4) + 15 = -15$, $n = -7$, n is not even, not valid

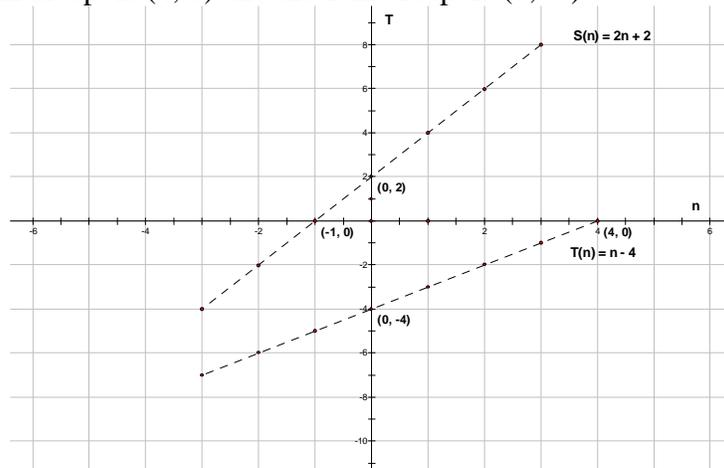
28.

n	-3	-2	-1	0	1	2	3	n
$S(n)$	-4	-2	0	2	4	6	8	$2n + 2$

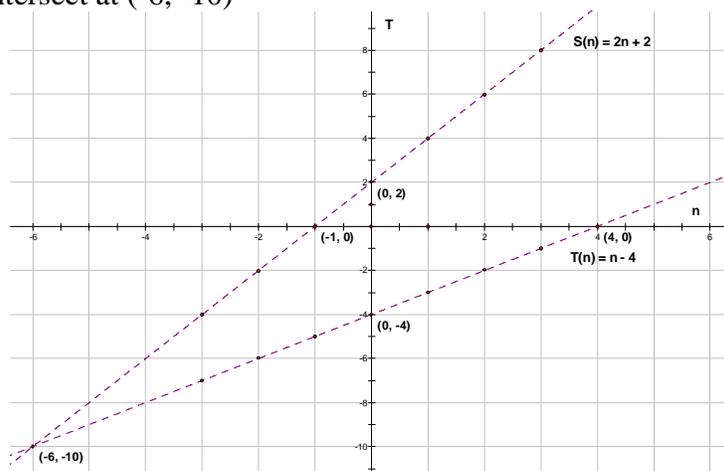
n	-3	-2	-1	0	1	2	3	n
T(n)	-7	-6	-5	-4	3	2	1	n + 4



29. For $S(n)$, the n -intercept is $(-1, 0)$ and the T -intercept is $(0, 2)$
 For $T(n)$, the n -intercept is $(4, 0)$ and the T -intercept is $(0, -4)$

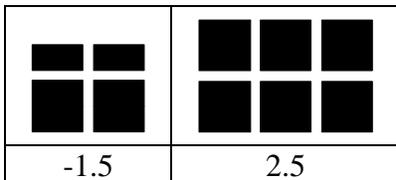


30. $S(n)$ and $T(n)$ intersect at $(-6, -10)$

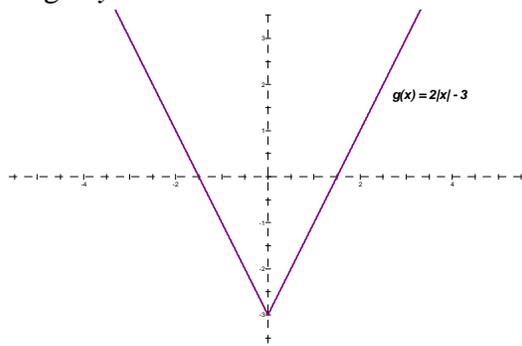


SOLUTIONS—Chapter 3 Practice Exam

1. $f(x) = |2x + 1|$

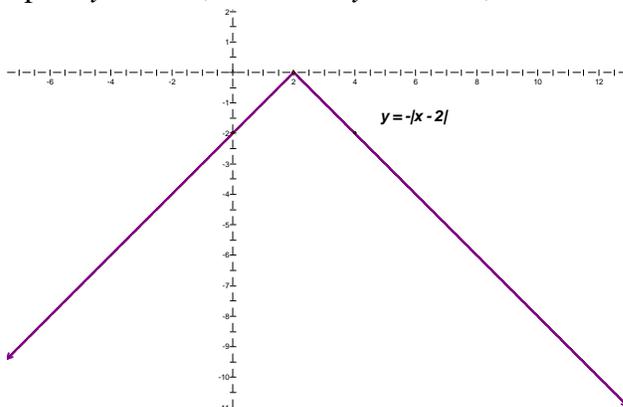


2. Range: $y \geq -3$



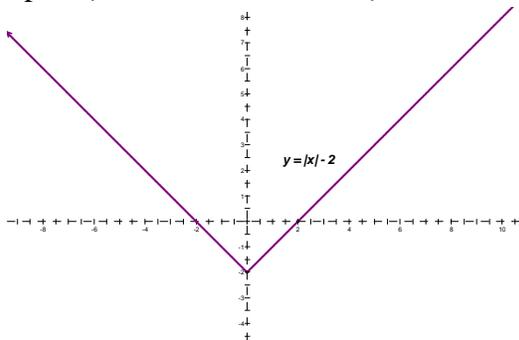
3. One solution: $(2, 0)$ is the top of a “v” point. $y = -|x - 2|$.

Split: $y = x - 2, x \leq 2$ and $y = -x + 2, x \geq 2$

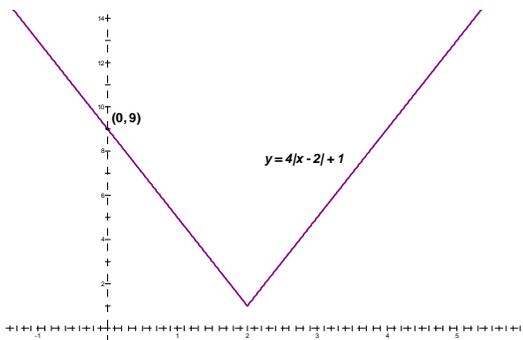


Second solution: $(0, -2)$ is the bottom of a “v” point. $y = |x| - 2$.

Split: $y = -x - 2, x \leq 0$ and $y = x - 2, x \geq 0$



4.



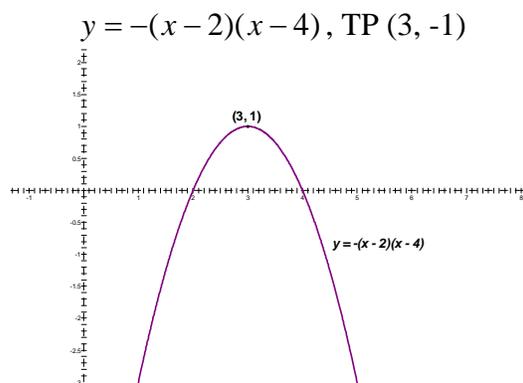
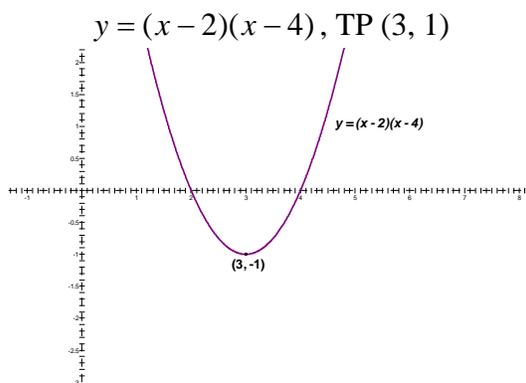
a. Range: $y \geq 1$ (2, 1) is the lowest point on the graph.

b. (0, 9)

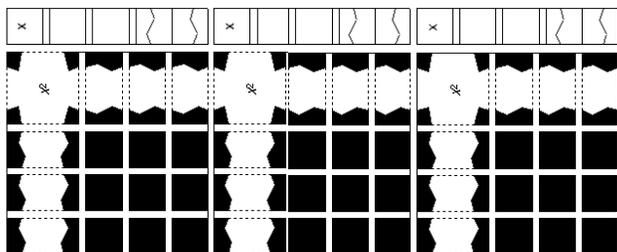
c. There are none

5. $y = -5x + 19, x \leq 2$ and $y = 5x - 1, x > 2$

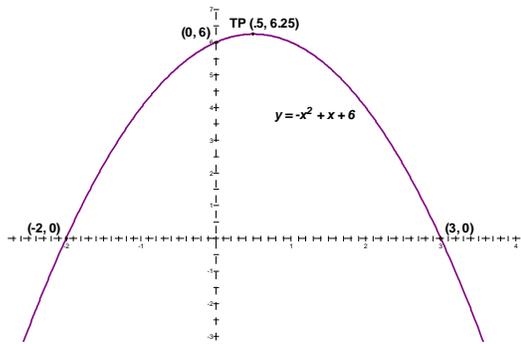
6. There are many parabolas that satisfy these conditions. The x -value for the turning point will always be 1, but the y -value for the turning point can change. Here are two examples, but the general form is $y = a(x - 2)(x - 4)$ where a is any real number.



7. $y = 3x(x + 1)$, figure rotated 90° left.



8. $y = -(x - 3)(x + 2) = -x^2 + x + 6$. Range: $y \leq 6.25$

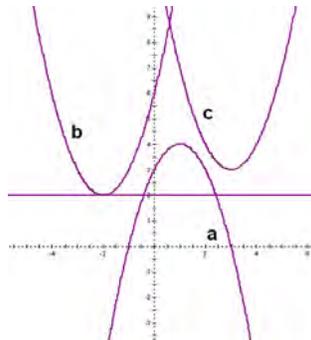


9. There are many solutions. Here are three possibilities.

a. $y = -(x + 1)(x - 3)$

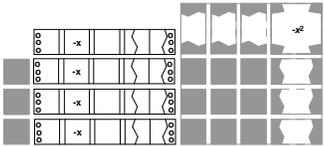
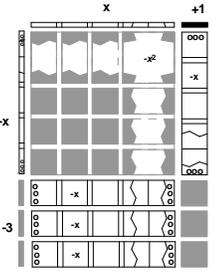
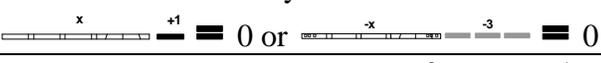
b. $y = (x + 2)^2 + 2$

c. $y = (x - 2)^2 + 3$

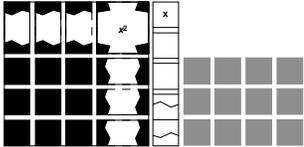
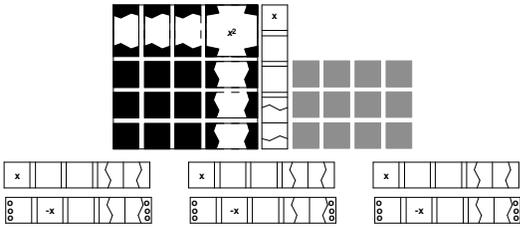
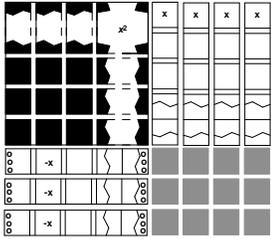
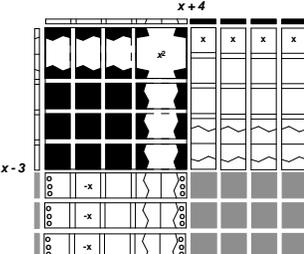


10. Solutions will vary but a) and b) should look like rectangular figures with dimensions $y = (x + 3)(x - 1)$. c) The minimal collection is $x^2 + 2x - 3$, one black x -square, 2 white x -strips and 3 red tiles.

11. x -intercepts $(-1, 0)$ and $(3, 0)$. x -intercept $(0, -3)$, turning point $(-2, 1)$

ALGEBRA PIECE WORK with notes	SYMBOLIC WORK
Set up pieces  = 0	Set up equation $-x^2 - 4x - 3 = 0$
Rearrange pieces;; set up edge sets  = 0	$(-x - 3)(x + 1) = 0$
Array = 0 if  = 0 or $x = -3$ or $x = -1$	$-x - 3 = 0$ or $x + 1 = 0$

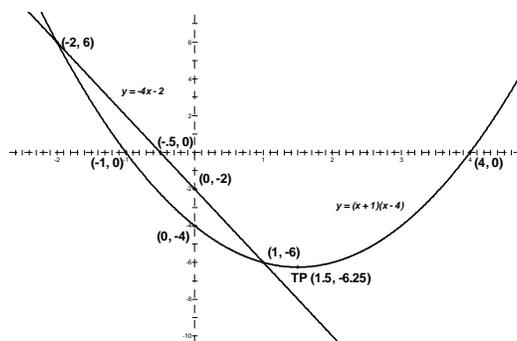
12.

	<p>Start with the algebra pieces representing $y = x^2 + x - 12$</p>
	<p>Since an array of 12 red tiles can be 1×12, 2×6 or 3×4 (in size), compare this to the number of x-strips. Since there is only 1 x-strip, we can add zero pairs to form a collection with 4 x-strips and 3 opposite x-strip (but not 1 and 12 or 2 and 6). Add 3 zero pairs of x-strips and opposite x-strips.</p>
	<p>Rearrange all of the pieces into a rectangular array</p>
	<p>Form and measure edge sets</p> $x^2 + x - 12 = (x - 3)(x + 4)$

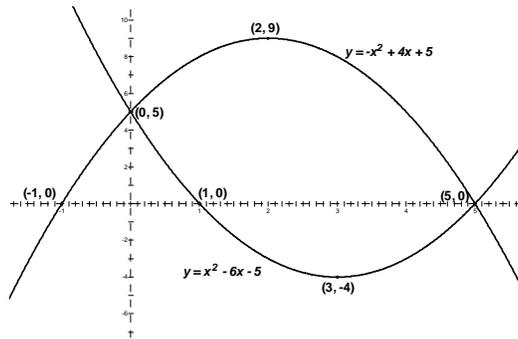
13.

- $y = (x + 5)(x + 1)$
- $y = (x + 5)(x - 5)$
- $y = 2(x - 5)(x + 3)$
- $y = (x + 4)(x + 4)$

14. Symbolic work for points of intersection: $x^2 - 3x - 4 = -4x - 2 \implies x^2 + x - 2 = 0 \implies (x + 2)(x - 1) = 0$, $x = -2, x = 1$



15. Symbolic work for points of intersection: $x^2 - 6x + 5 = -x^2 + 4x + 5 \implies 2x^2 - 10x = 0$
 $\implies 2x(x - 5) = 0, x = 0, x = 5$



16.

a. Turning Point: $(\frac{1}{2}, 12.5)$

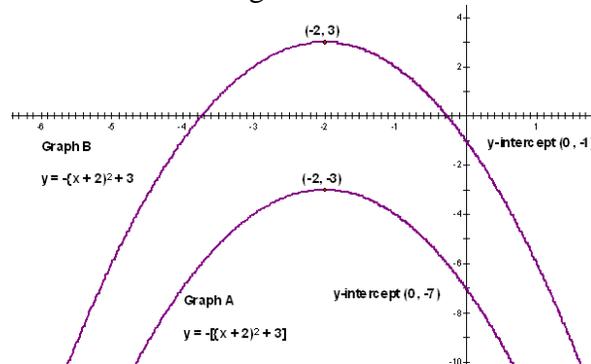
b. Range, $y \leq 12.5$

c. $y = -2(x + 2)(x - 3)$

d. $y = -2(x - \frac{1}{2})^2 + 12.5$

17. Graph A is $y = -(x + 2)^2 + 3 = -x^2 - 4x - 1$. Using the quadratic formula, the x -intercepts are: $(-3.7, 0)$ and $(-.27, 0)$. This is $y = x^2$ shifted left 2, flipped upside down and then shifted up 3 (or flipped upside down, then left 2 and up 3)

Graph B is $y = -[(x + 2)^2 + 3] = -x^2 - 4x - 7$. There are no x -intercepts. This is $y = x^2$ shifted left 2, up 3 and then the whole thing is reflected across the x -axis.

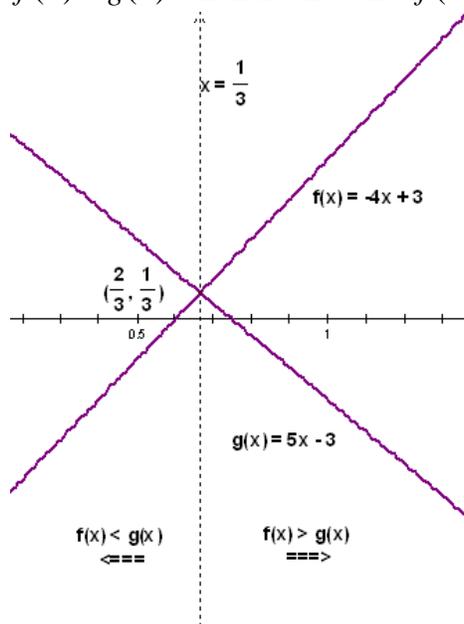


18. a. $(0, -9)$ b. $(-3, 0)$ and $(-1, 0)$ c. $(-2, 3)$ d. $y \leq 3$ e. $y = -3(x + 3)(x + 1)$
 f. $y = -3(x + 2)^2 + 3$

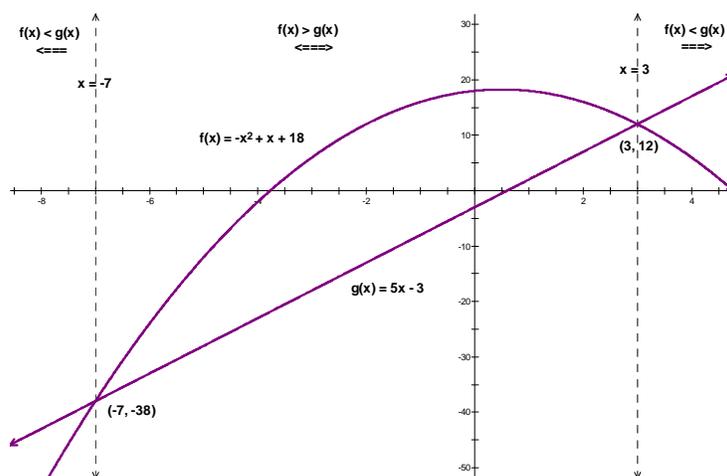
19. Linear, $y = 2x - 5$

20. Quadratic, 2nd difference is $6 = 2a$. c is given, $c = 1$. $y = 3x^2 + 2x + 1$

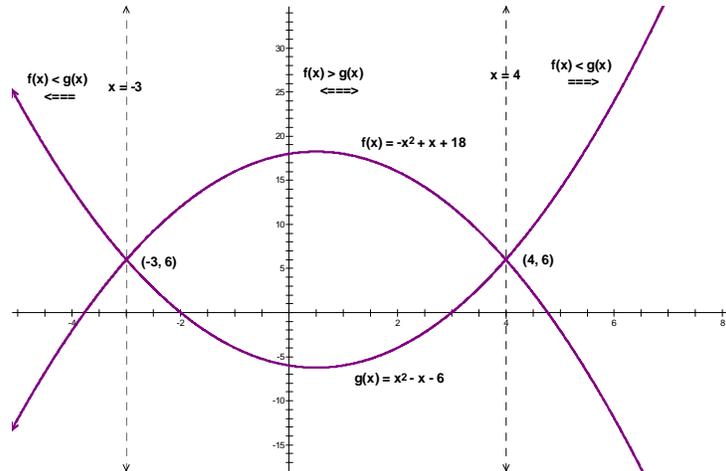
21. $f(x) = g(x)$ at $(\frac{2}{3}, \frac{1}{3})$. $f(x) < g(x)$ when $x < \frac{2}{3}$ and $f(x) > g(x)$ $x > \frac{2}{3}$.



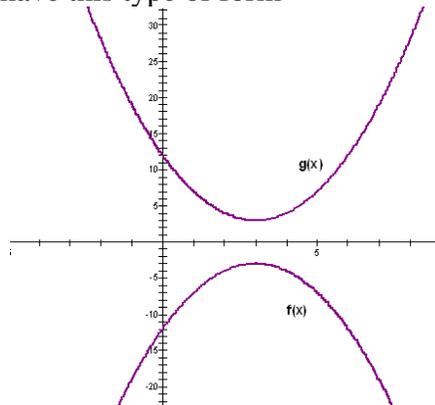
22. $f(x) = g(x)$ at $(-7, -38)$ and $(3, 12)$, $f(x) < g(x)$ when $x < -7$ or $x > 3$ and $f(x) > g(x)$ when $-7 < x < 3$.



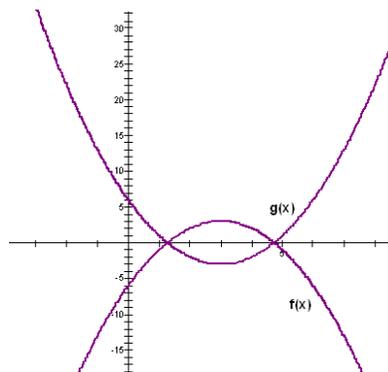
23. $f(x) = g(x)$ at $(-3, 6)$ and $(4, 6)$, $f(x) < g(x)$ when $x < -3$ or $x > 4$ and $f(x) > g(x)$ when $-3 < x < 4$.



24. Answers will vary but will have this type of form



25. Answers will vary but will have this type of form



SOLUTIONS—Chapter 4 Practice Exam

1. Answers will vary but a reasonable answer is a cubic polynomial such as $h(x) = (x+1)(x+2)(x+3)$
2. Answers will vary but a reasonable answer is an even degree polynomial such as $y = x^4$, $y = x^6$, etc.
3. Answers will vary but a reasonable answer is degree six polynomial such as $y = -(x+4)(x-2)(x+3)(x-3)(x+1)(x-1)$.
4. $h(x) = -(x+2)(x+1)(x-1)(x-3)$. Range $y \leq 12.95$
5. $h(x) = 2(x+2)(x-2)(z+1)(x-1)(x+3)$. Range **R**
6. $y = x^5 + a^5 = (x+a)(x^4 - ax^3 + a^2x^2 - a^3x + a^4)$
7. $y = x^4 - 81 = (x^2 - 9)(x^2 + 9) = (x-3)(x+3)(x^2 + 9)$. Two x -intercepts: $(\pm 3, 0)$
8. $y = 27x^3 - 64 = (3x-4)(9x^2 + 12x + 16)$. One x -intercept: $(\frac{4}{3}, 0)$
9. $y = x^{10} + 9x^5 + 18 = (x^5 + 3)(x^5 + 6) = (x + \sqrt[5]{3})(x^4 - \sqrt[5]{3}x^3 + \sqrt[5]{9}x^2 - \sqrt[5]{27}x + \sqrt[5]{81})(x + \sqrt[5]{6})(x^4 - \sqrt[5]{6}x^3 + \sqrt[5]{36}x^2 - \sqrt[5]{216}x + \sqrt[5]{1296})$
Two x -intercepts: $(-\sqrt[5]{3}, 0)$ and $(-\sqrt[5]{6}, 0)$
10. $y = x^6 - 5x^5 - 4x^4 + 21x^3 - 5x^2 - 4x + 20 = (x^2 - 4)(x^3 + 1)(x - 5) = (x - 2)(x + 2)(x - 1)(x^2 + x + 1)(x - 5)$
Four x -intercepts: $(\pm 2, 0)$, $(1, 0)$ and $(5, 0)$.
11. $(4 + 2i) + (5 - 3i) = 9 - i$.
There are no zero black and red pairs.
There are two zero green and yellow pairs.
12. $(\bar{6} + 2i) - (4 - 2i) = \bar{10}$.
 $\bar{6} + 4i$ must be converted to 10 red, 4 black, 4 green and 2 yellow tiles before taking away 4 black and 2 yellow tiles.

13.

Set up factor edges Fill in tiles Determine Product

$(-2 - 3i) \times (3 + 2i) = -13i$

14.

Set up divisor edge Fill in dividend tiles Set up quotient edge Determine Quotient

$(8 - i) \div (2 + 3i) = 1 - 2i$
Corresponding multiplication sentence
 $(2 + 3i) \times (1 - 2i) = 8 - i$

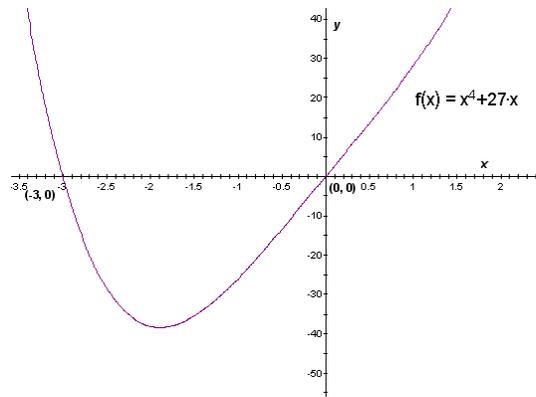
Check: $\frac{8 - i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i} = \frac{13 - 26i}{13} = 1 - 2i$

15.

- A complex number with both nonzero real and nonzero imaginary components.
- The sum can be: All real (if imaginary components cancel); all imaginary (if real components cancel) or a complex number with both nonzero real and nonzero imaginary components.
- A complex number with both nonzero real and nonzero imaginary components.
- The difference can be: All real (if imaginary components cancel); all imaginary (if real components cancel) or a complex number with both nonzero real and nonzero imaginary components.
- A positive or negative all imaginary number.
- A positive or negative all real number.
- Can be all real (Ex: $(1 + i) \times (2 - 2i)$), all imaginary (Ex: $(1 + i) \times (2 + 2i)$) or a complex number with both nonzero real and nonzero imaginary components. (Ex: $(1 + 2i) \times (1 + 2i)$)
- An all imaginary number.
- An all real number.
- Can be all real (Ex: $(1 + i) \div (1 + i)$), all imaginary (Ex: $(1 + i) \div (1 - i)$), or a complex number with both nonzero real and nonzero imaginary components (see #14).

16. $y = x^4 + 27x = x(x+3)(x^2 - 3x + 9)$

Real roots: $x = 0$ and $x = -3$. Complex roots $x = \frac{3 \pm \sqrt{9 - 36}}{2} = \frac{3 \pm \sqrt{-27}}{2} = \frac{3 \pm 3i\sqrt{3}}{2}$

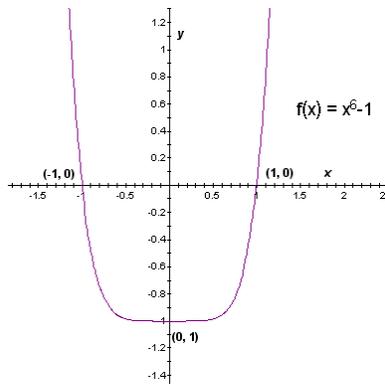


17. $y = x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$

Two Real roots: $x = \pm 1$.

Four Complex roots:

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} \text{ and } x = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$



18. Answers will vary, but in the $y = (ax^2 + bx + c)(x + d)$ form with $\sqrt{b^2 - 4ac} = 0$. The graph should just cross the x -axis once.

19. Answers will vary, but in the $y = (ax^2 + bx + c)(x + d)^2$ form with $\sqrt{b^2 - 4ac} = 0$. The graph should just bounce off the x -axis at $x = -d$.

20. Answers will vary, but in the $y = (ax^2 + bx + c)(x + d)(x + e)^3$ form with $\sqrt{b^2 - 4ac} = 0$. The graph should just cross the x -axis at $x = -d$ and at $x = -e$.

Visual Algebra for College Students

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Many of the ideas in this book were inspired by the ideas set forth in the original *Math in the Mind's Eye* materials published by the *Math Learning Center* (<http://www.mathlearningcenter.org/>) and are used with the explicit permission of the *Math Learning Center*.

The original Math in the Mind's Eye units that included these ideas are:

“Modeling Integers,” Albert Bennett, Eugene Maier and Ted Nelson

“Picturing Algebra,” Michael Arcidiocano and Eugene Maier

“Graphing Algebraic Relationships,” Eugene Maier and Michael Shaughnessey

“Modeling Real and Complex Numbers,” Eugene Maier and Ted Nelson

“Sketching Solutions to Algebraic Equations,” Eugene Maier

